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FABER POLYNOMIAL COEFFICIENT ESTIMATES FOR CERTAIN SUBCLASS OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH (p, q)-RUSCHEWEYH DERIVATIVE

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Abstract In the present work, we first introduce a new subclass of analytic and bi-univalent functions associated with (p, q)-Rusheweyh derivative operator and we obtain upper bounds for the coefficients of functions belonging to these subclass by using faber polynomial expansion.

1 Introduction

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are regular in the open unit disc $\mathcal{D} = \{z \in \mathcal{C}; |z| < 1\}$. By $\mathcal{S}$ we mean the class of all functions $\mathcal{A}$ which are univalent in $\mathcal{D}$.

For any two regular functions f and g in $\mathcal{D}$, we say that f is subordinate to g, if there exists a schwarz function w in $\mathcal{D}$ with $|w(z)| < 1$ such that $f(z) = g(w(z)), \quad z \in \mathcal{D}$.
From the Koebe's one quarter theorem [2], it is well known that every univalent function $f \in \mathcal{A}$ has an inverse f^{-1}, which is defined as $f^{-1}(f(z)) = z$, $z \in \mathcal{D}$ and $f(f^{-1}(w)) = w, (|w| < r_0(f)), r_0(f) \geq \frac{1}{4}$, where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - a_2 a_3 + a_4)w^4 + ... \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathcal{D}$ if both f and f^{-1} are univalent in $\mathcal{D}$. Let Σ denote the class of regular and bi-univalent functions in $\mathcal{D}$ given by the Taylor's-Macluarin series expansion (1.1). Few examples for the functions in the class Σ are given by

$$\frac{z}{1-z}, \quad -log(1-z), \quad \frac{1}{2} log\left(\frac{1+z}{1-z}\right).$$

Faber polynomials introduced by Faber play a vital role in different areas of Mathematical Sciences, especially in Geometric Function Theory. Grunsky succeeded in defining a set of necessary and sufficient conditions for the univalence for a given function and in these conditions, the coefficients of the Faber polynomials play a significant role for more details see [7, 1].
By using Faber polynomial expansion of functions $f \in \mathcal{A}$ of the form (1.1), the coefficients of its inverse map $g = f^{-1}$ may be expressed as

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, ...)w^n, \tag{1.3}$$

where

$$K_{n-1}^{-n} = \frac{(-n)!}{(-2n+1)!(n-1)!}a_2^{n-1} + \frac{(-n)!}{[2(-n+1)]!(n-3)!}a_2^{n-3}a_3 + \frac{(-n)!}{(-2n+3)!(n-4)!}a_2^{n-4}a_4 + \frac{(-n)!}{[2(-n+2)]!(n-5)!}a_2^{n-5}(a_5+(-n+2)a_3^2) + \frac{(-n)!}{(-2n+5)!(n-6)!}a_2^{n-6}[a_6+(-2n+5)a_3a_4] + \sum_{j\geq 7} a_2^{n-j}V_j, \quad (1.4)$$

such that V_j with $7 \leq j \leq n$ is a homogeneous polynomial in the variables $|a_2|, |a_3|, ...|a_n|$ [5]. In particular, the first three terms of K_{n-1}^{-n} are given below

$$\frac{1}{2}K_1^{-2} = -a_2,$$
$$\frac{1}{3}K_2^{-3} = 2a_2^2 - a_3, \quad (1.5)$$
$$\frac{1}{4}K_3^{-4} = -(5a_2^3 - 5a_2a_3 + a_4).$$

In general, for any $p \in \mathcal{N}$ and $n \geq 2$, an expansion of K_{n-1}^p is given by[4],

$$K_{n-1}^p = pa_n + \frac{p(p-1)}{2}E_{n-1}^2 + \frac{p!}{(p-3)!3!}E_{n-1}^3 + ... + \frac{p!}{(p-n+1)!(n-1)!}E_{n-1}^{n-1},$$

where $E_{n-1}^p = E_{n-1}^p(a_2, a_3, ...)$ and given by [6]

$$E_{n-1}^m(a_2, ..., a_n) = \sum_{n=2}^{\infty} \frac{m_!(a_2)^{\mu_1}...(a_n)_{n-1}^{\mu}}{\mu_1!...\mu_{n-1}!}, \quad for \quad m \leq n.$$

While $a_1 = 1$ and the sum is taken over all non-negative integers $\mu_1, ...\mu_n$ satisfying

$$\mu_1 + \mu_2 + ... + \mu_{n-1} = m$$

$$\mu_1 + 2\mu_2 + ... + (n-1)\mu_{n-1} = n-1.$$

Evidently, $E_{n-1}^{n-1}(a_2, a_3, ...a_n) = a_2^{n-1}$, (see [6]), while $a_1 = 1$, and the sum is taken over all non-negative integers $\mu_1, ... \mu_n$ satisfying

$$\mu_1 + \mu_2 + ... + \mu_n = m,$$
$$\mu_1 + 2\mu_2 + ... + n\mu_n = n.$$

It is clear that $E_n^n(a_1, a_2, ..a_n) = a_1^n$. The first and the last polynomials are :
$E_n^1 = a_n, \quad E_n^n = a_1^n$.
Let $\phi(z)$ be regular functions with positive real part in $\mathcal{D}$, with $\phi(0) = 1$ and $\phi'(0) > 0$. Also let $\phi(E)$ be starlike with respect to 1 and symmetric with respect to the real axis. Thus ϕ has the Taylor's series expansion

$$\phi(z) = 1 + B_1(z) + B_2z^2 + B_3z^3 + ...(B_1 > 0). \quad (1.6)$$

Suppose that $u(z)$ and $v(w)$ are regular in the unit disc $\mathcal{D}$ with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(w)| < 1$, and suppose that

$$u(z) = c_1(z) + \sum_{n=2}^{\infty} c_nz^n, \quad v(w) = d_1w + \sum_{n=2}^{\infty} d_nw^n, \quad (|z| < 1). \quad (1.7)$$

It is cleared that

$$|c_1| \leq 1, \quad |c_2| \leq 1 - |c_1|^2, \quad |d_1| \leq 1, \quad |d_2| \leq 1 - |d_1|^2. \tag{1.8}$$

So the equations (1.6) and (1.7) lead to

$$\phi(u(z)) = 1 + B_1 u(z) + B_2(u(z))^2 + B_3(u(z))^3 + ...$$

$$\begin{aligned} &= 1 + B_1 c_1 z + (B_1 c_2 + B_2 c_1^2) z^2 + ... \\ &= 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} B_k E_n^k(c_1, c_2, ...c_n) z^n, \end{aligned} \tag{1.9}$$

and

$$\phi(v(w)) = 1 + B_1 v(w) + B_2(v(w))^2 + ...$$

$$\begin{aligned} &= 1 + B_1 d_1 w + (B_1 d_2 + B_2 d_1^2) w^2 + ... \\ &= 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} B_K E_n^k(d_1, d_2, ..d_n) w^n. \end{aligned} \tag{1.10}$$

For $0 < q < p \leq 1$, we define (p, q)-integer number by[8]

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}$$

and (p, q)-factorial of integer number n is given by [8]

$$[n]_{p,q}! = \begin{cases} [n]_{p,q}[n-1]_{p,q}...[1]_{p,q}, & n = 1, 2, .. \\ 1 \quad n = 0. \end{cases}$$

The (p, q)-analogue of Jackson derivative of a function f [8] defined as

$$D_{p,q}(f(z)) = \frac{f(pz) - f(qz)}{(p - q)z}, \quad z \neq 0, p \neq q, 0 < q < 1.$$

Thus

$$D_{p,q}(g(w)) = \frac{g(pw) - f(qw)}{(p - q)w}$$

$$= 1 - [2]_{p,q} a_2 w + [3]_{p,q}(2a_2^2 - a_3) w^2 - [4]_{p,q}(5a_2^3 - 5a_2 a_3 + a_4) w^3 + ...$$

For $\delta \geq 0$, the Ruscheweyh type (p, q)- differential operator $\mathcal{R}_{p,q}^{\delta} : \mathcal{A} \to \mathcal{A}$ is given by

$$\mathcal{R}_{p,q}^{\delta} f(z) = z + \sum_{n=2}^{\infty} \frac{[n + \delta - 1]_{p,q}!}{[\delta]_{p,q}![n-1]_{p,q}!} a_n z^n. \tag{1.11}$$

We note that

$$\mathcal{R}_{p,q}^{0} f(z) = f(z)$$

$$\mathcal{R}_{p,q}^{1} f(z) = z D_{p,q} f(z)$$

$$\mathcal{R}_{p,q}^{\delta} f(z) = \frac{z D_{p,q}^{\delta}(z^{\delta-1} f(z))}{[\delta]_{p,q}!}.$$

We observe that, for $p = 1$ and $q \to 1$, the (p, q)-integer number $[n]_{p,q}$ reduces to the ordinary number n and (p, q)-Ruscheweyh differential operator reduces to the Ruscheweyh differential operator defined by Ruscheweyh in [11].

Now using the differential operator $\mathcal{R}_{p,q}^{\delta} f(z)$ and the concept of subordination, we define a new subclass of Σ as follows.

Definition 1.1. For $b \in \frac{\mathcal{C}}{\{0\}}$, $0 < q < p \leq 1$ and $\delta \geq 0$. Then $f \in \Sigma$ is said to be in the class $\mathcal{R}_\Sigma(p, q, \delta, b, \phi)$ if and only if

$$\left[1 + \frac{1}{b}\left(D_{p,q}(R^{\delta}_{p,q} f(z)) - 1\right)\right] \prec \phi(z) \tag{1.12}$$

and

$$\left[1 + \frac{1}{b}\left(D_{p,q}(R^{\delta}_{p,q} g(w)) - 1\right)\right] \prec \phi(w) \tag{1.13}$$

where $g(w) = f^{-1}(w)$ and $\mathcal{R}^{\delta}_{p,q}$ are given by (1.2) and (1.11) respectively.

By suitably specializing the parameters δ, b and p we have the following subclasses of bi-univalent functions $\mathcal{R}^{p,q}_{\Sigma,b}(\phi)$, $\mathcal{R}^{q}_{\Sigma,b}(\phi)$, $\mathcal{R}^{p,q}_{\Sigma}(\phi)$ and $\mathcal{R}^{q}_{\Sigma}(\phi)$ defined by Sahsene Altinkaya and Sibel Yalcin [10] . In the main results, the Faber polynomial expansion is used to obtain bounds for the general coefficients $|a_n|$ of bi-univalent functions in $\mathcal{R}_\Sigma(p, q, \delta, b, \phi)$ as well as we provide estimates for the initial coefficients of these functions.

2 Main results

Theorem 2.1. *For $\delta \geq 0$ and $b \in \frac{\mathcal{C}}{\{0\}}$. Let $f \in \mathcal{R}_\Sigma(p, q, \delta, b, \phi)$. If $a_m = 0$; $2 \leq m \leq n-1$, then*

$$|a_n| \leq \frac{B_1|b|}{[n]_{p,q}\left|\frac{[n+\delta-1]_{p,q}!}{[\delta]_{p,q}![n-1]_{p,q}!}\right|}, \quad n \geq 3.$$

Proof. Let f be given by (1.1), we have

$$\left[1 + \frac{1}{b}\left(D_{p,q}(R^{\delta}_{p,q} f(z)) - 1\right)\right] = 1 + \frac{1}{b}\left[1 + \sum_{n=2}^{\infty} \frac{[n+\delta-1]_{p,q}!}{[\delta]_{p,q}![n-1]_{p,q}!}[n]_{p,q} a_n z^{n-1}\right] \tag{2.1}$$

and for $g = f^{-1}$, we have

$$\left[1 + \frac{1}{b}\left(D_{p,q}(R^{\delta}_{p,q} g(w)) - 1\right)\right] = 1 + \frac{1}{b}\left[1 + \sum_{n=2}^{\infty} \frac{[n+\delta-1]_{p,q}!}{[\delta]_{p,q}![n-1]_{p,q}!}[n]_{p,q} b_n w^{n-1}\right]. \tag{2.2}$$

From (1.12) and (1.13)

$$\left[1 + \frac{1}{b}\left(D_{p,q}(R^{\delta}_{p,q} f(z)) - 1\right)\right] = \phi(u(z)) \tag{2.3}$$

$$\left[1 + \frac{1}{b}\left(D_{p,q}(R^{\delta}_{p,q} g(w)) - 1\right)\right] = \phi(v(w)). \tag{2.4}$$

On comparing the corresponding coefficients of (2.3) and (2.4), we get

$$\frac{1}{b}\frac{[n+\delta-1]_{p,q}!}{[\delta]_{p,q}![n-1]_{p,q}!}[n]_{p,q} a_n = B_1 c_{n-1}, \tag{2.5}$$

and

$$\frac{1}{b}\frac{[n+\delta-1]_{p,q}!}{[\delta]_{p,q}![n-1]_{p,q}!}[n]_{p,q} b_n = B_1 d_{n-1}. \tag{2.6}$$

Note that for $a_m = 0$; $2 \leq m \leq n-1$ we have $b_n = -a_n$ and so

$$\frac{1}{b}\frac{[n+\delta-1]_{p,q}!}{[\delta]_{p,q}![n-1]_{p,q}!}[n]_{p,q} a_n = B_1 c_{n-1},$$

$$-\frac{1}{b}\frac{[n+\delta-1]_{p,q}!}{[\delta]_{p,q}![n-1]_{p,q}!}[n]_{p,q} a_n = B_1 d_{n-1}.$$

Now taking the absolute values of either of the above two equations we get

$$|a_n| = \frac{B_1|b|}{[n]_{p,q}\left|\frac{[n+\delta-1]_{p,q}!}{[\delta]_{p,q}![n-1]_{p,q}!}\right|}|c_{n-1}| = \frac{B_1|b|}{[n]_{p,q}\left|\frac{[n+\delta-1]_{p,q}!}{[\delta]_{p,q}![n-1]_{p,q}!}\right|}|d_{n-1}|.$$

From (1.8), we have

$$|a_n| \leq \frac{B_1|b|}{[n]_{p,q}\left|\frac{[n+\delta-1]_{p,q}!}{[\delta]_{p,q}![n-1]_{p,q}!}\right|}.$$

This completes the proof of Theorem 2.1. □

If we take $\delta = 0$ in the above Theorem, we get the following corollary (see Theorem 1, [10]).

Corollary 2.2. *For* $b \in \frac{\mathcal{C}}{\{0\}}$. *Let* $f \in \mathcal{R}^{p,q}_{\Sigma,b}(\phi)$. *If* $a_m = 0$; $2 \leq m \leq n-1$, *then*

$$|a_n| \leq \frac{B_1|b|}{[n]_{p,q}}, \quad n \geq 3.$$

Put $p = 1$ in Corollary 2.2, we have the following result (see Theorem 2, [10]).

Corollary 2.3. *For* $b \in \frac{\mathcal{C}}{\{0\}}$. *Let* $f \in \mathcal{R}^{q}_{\Sigma,b}(\phi)$. *If* $a_m = 0$; $2 \leq m \leq n-1$, *then*

$$|a_n| \leq \frac{B_1|b|(1-q)}{(1-q^n)}, \quad n \geq 3.$$

If we take $b = 1$ in Corollary 2.2, we obtain the following result (see Theorem 3, [10]).

Corollary 2.4. *Let* $f \in \mathcal{R}^{p,q}_{\Sigma}(\phi)$. *If* $a_m = 0; 2 \leq m \leq n-1$, *then*

$$|a_n| \leq \frac{B_1}{[n]_{p,q}}, \quad n \geq 3.$$

Taking $p = 1$ in Corollary 2.4, we have the following result (see Theorem 4, [10]).

Corollary 2.5. *Let* $f \in \mathcal{R}^{q}_{\Sigma}(\phi)$. *If* $a_m = 0, 2 \leq m \leq n-1$, *then*

$$|a_n| \leq \frac{B_1(1-q)}{(1-q^n)}, \quad n \geq 3.$$

3 coefficient estimates

In this section, we investigate the coefficient estimates for the functions belonging to the class $\mathcal{R}_{\Sigma}(p, q, \delta, b, \phi)$.

Theorem 3.1. *For* $b \in \frac{\mathcal{C}}{\{0\}}$ *and let* $f \in \mathcal{R}_{\Sigma}(p, q, \delta, b, \phi)$. *Then*

$$|a_2| \leq \min\left\{F(p,q), \frac{|b|B_1\sqrt{B_1}}{\sqrt{\left|\frac{[\delta+1]_{p,q}[\delta+2]_{p,q}}{[2]_{p,q}}[3]_{p,q}bB_1^2 - [\delta+1]^2_{p,q}[2]^2_{p,q}B_2\right| + [\delta+1]^2_{p,q}[2]^2_{p,q}B_1}}\right\}$$

and

$$|a_3| \leq min\{G(p,q), H(p,q))\}$$

where

$$F(p,q) = \begin{cases} \sqrt{\dfrac{[2]_{p,q}!|b|B_1}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}}}, & |B_2| \leq B_1 \\ \sqrt{\dfrac{[2]_{p,q}!|bB_2|}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}}}, & |B_2| > B_1 \end{cases}$$

and

$$G(p,q)=\begin{cases}\dfrac{[2]_{p,q}|b|B_1}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}}, & |B_2|\leq B_1\\ \dfrac{[2]_{p,q}|bB_2|}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}}, & |B_2|>B_1\end{cases}$$

$$H(p,q)=\begin{cases}\dfrac{[2]_{p,q}|b|B_1}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}}, \quad B_1\leq \dfrac{[2]^3_{p,q}[\delta+1]_{p,q}}{[\delta+2]_{p,q}[3]_{p,q}|b|}\\ \dfrac{[2]_{p,q}!|b|B_1}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}}\left[\dfrac{\left|\frac{[\delta+2]_{p,q}}{[2]_{p,q}!}[3]_{p,q}bB_1^2-[\delta+1]_{p,q}[2]^2_{p,q}B_2\right|+\frac{[\delta+2]_{p,q}|b|B_1^2}{[2]_{p,q}!}[3]_{p,q}}{\left|\frac{[\delta+2]_{p,q}}{[2]_{p,q}!}[3]_{p,q}bB_1^2-[\delta+1]_{p,q}[2]^2_{p,q}B_2\right|+[\delta+1]_{p,q}[2]^2_{p,q}B_1}\right],\\ \qquad B_1>\dfrac{[2]^3_{p,q}[\delta+1]_{p,q}}{[\delta+2]_{p,q}[3]_{p,q}|b|}.\end{cases}$$

Proof. Replacing n by 2 and 3 in (2.5) and (2.6) respectively, we find that

$$\frac{1}{b}[\delta+1]_{p,q}[2]_{p,q}a_2=B_1c_1, \tag{3.1}$$

$$\frac{1}{b}\frac{[\delta+1]_{p,q}[\delta+2]_{p,q}}{[2]_{p,q}!}[3]_{p,q}a_3=B_1c_2+B_2c_1^2, \tag{3.2}$$

$$-\frac{1}{b}[\delta]_{p,q}[2]_{p,q}a_2=B_1d_1, \tag{3.3}$$

$$\frac{1}{b}\frac{[\delta+1]_{p,q}[\delta+2]_{p,q}}{[2]_{p,q}!}[3]_{p,q}(2a_2^2-a_3)=B_1d_2+B_2d_1^2. \tag{3.4}$$

From (3.1) and (3.3), we get

$$c_1=-d_1. \tag{3.5}$$

By adding (3.4) to (3.2), further computations using (3.5) lead to

$$\frac{2}{b}\frac{[\delta+1]_{p,q}[\delta+2]_{p,q}}{[2]_{p,q}!}[3]_{p,q}a_2^2=B_1(c_2+d_2)+2B_2c_1^2. \tag{3.6}$$

Making use of (3.1) in the above equality (3.6), we get

$$2[\delta+1]_{p,q}\left(\frac{[\delta+2]_{p,q}}{[2]_{p,q}!}[3]_{p,q}bB_1^2-[\delta+1]_{p,q}[2]^2_{p,q}B_2\right)a_2^2=b^2B_1^3(c_2+d_2) \tag{3.7}$$

Combining (3.7) and (1.8), we get

$$\begin{aligned}2[\delta+1]_{p,q}\left|\frac{[\delta+2]_{p,q}}{[2]_{p,q}!}[3]_{p,q}bB_1^2-[\delta+1]_{p,q}[2]^2_{p,q}B_2\right||a_2|^2&=|b|^2B_1^3(|c_2|+|d_2|)\\ &\leq 2|b|^2B_1^3(1-|c_1|^2) \\ &=2|b|^2B_1^3-2|b|^2B_1^3|c_1|^2.\end{aligned} \tag{3.8}$$

It follows from (3.1) that

$$|a_2|^2\leq \frac{|b|B_1\sqrt{B_1}}{\sqrt{\left|\frac{[\delta+1]_{p,q}[\delta+2]_{p,q}}{[2]_{p,q}!}[3]_{p,q}bB_1^2-[\delta+1]^2_{p,q}[2]^2_{p,q}B_2\right|+[\delta+1]^2_{p,q}[2]^2_{p,q}B_1}}. \tag{3.9}$$

Moreover, by (1.8) and (3.6)

$$\frac{2}{b}\frac{[\delta+1]_{p,q}[\delta+2]_{p,q}}{[2]_{p,q}!}[3]_{p,q}|a_2|^2\leq B_1(|c_2|+|d_2|)+2|B_2||c_1|^2$$

$$\leq 2B_1(1-|c_1|^2)+2|B_2||c_1|^2$$
$$= 2B_1+2|c_1|^2(|B_2|-B_1)$$
$$\frac{1}{|b|}\frac{[\delta+1]_{p,q}[\delta+2]_{p,q}}{[2]_{p,q}!}[3]_{p,q}|a_2|^2 \leq \begin{cases} B_1, & |B_2|\leq B_1 \\ |B_2|, & |B_2|>B_1. \end{cases}$$

Clearly, we can see that

$$|a_2| \leq \begin{cases} \sqrt{\dfrac{[2]_{p,q}!|b|B_1}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}}}, & |B_2|\leq B_1 \\ \sqrt{\dfrac{[2]_{p,q}!|bB_2|}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}}}, & |B_2|>B_1. \end{cases}$$

Next, in order to find the bound on $|a_3|$, subtract (3.4) from (3.2), we get

$$\frac{2}{b}\frac{[\delta+1]_{p,q}[\delta+2]_{p,q}}{[2]_{p,q}!}[3]_{p,q}a_3 = \frac{2}{b}\frac{[\delta+1]_{p,q}[\delta+2]_{p,q}}{[2]_{p,q}!}[3]_{p,q}a_2^2+B_1(c_2-d_2). \tag{3.10}$$

Clearly from (3.6) we obtain

$$a_3 = \frac{[2]_{p,q}!}{2[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}}\left[b(B_1(c_2+d_2)+2B_2c_1^2)+bB_1(c_2-d_2)\right]$$
$$= \frac{[2]_{p,q}!b}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}}(B_1c_2+B_2c_1^2)$$

and consequently

$$|a_3| \leq \frac{[2]_{p,q}!|b|}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}}(|B_1||c_2|+|B_2||C_1|^2)$$
$$\leq \frac{[2]_{p,q}!|b|}{[\delta+1]_{p,q}[2]_{p,q}[3]_{p,q}}(B_1(1-|c_1|^2)+|B_2||c_1|^2)$$
$$\leq \frac{[2]_{p,q}!|b|B_1+[2]_{p,q}!|b|(|B_2|-B_1)|c_1|^2}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}}$$
$$= \frac{[2]_{p,q}!|b|}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}}\left(B_1+|c_1|^2(|B_2|-B_1)\right).$$

Hence we write

$$|a_3| \leq \begin{cases} \dfrac{[2]_{p,q}!|b|B_1}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}}, & |B_2|\leq B_1 \\ \dfrac{[2]_{p,q}!|b||B_2|}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}}, & |B_2|>B_1. \end{cases}$$

On the other hand, by using (1.8) and (3.10), we have

$$\frac{2}{|b|}\frac{[\delta+1]_{p,q}[\delta+2]_{p,q}}{[2]_{p,q}!}[3]_{p,q}|a_3| \leq \frac{2}{|b|}\frac{[\delta+1]_{p,q}[\delta+2]_{p,q}}{[2]_{p,q}!}[3]_{p,q}|a_2|^2+B_1(|c_2|+|d_2|)$$
$$\leq \frac{2}{|b|}\frac{[\delta+1]_{p,q}[\delta+2]_{p,q}}{[2]_{p,q}!}[3]_{p,q}|a_2|^2+2B_1(1-|c_1|^2).$$

From (3.1) we have

$$B_1|b|\frac{[\delta+1]_{p,q}[\delta+2]_{p,q}}{[2]_{p,q}!}[3]_{p,q}|a_3| \leq \left(\frac{[\delta+1]_{p,q}[\delta+2]_{p,q}|b|B_1}{[2]_{p,q}!}[3]_{p,q}-[\delta+1]_{p,q}^2[2]_{p,q}^2\right)|a_2|^2+B_1^2|b|^2.$$

Now from(3.9), we get

$$|a_3| \leq \frac{[2]_{p,q}!|b|B_1}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}}\left[1+\frac{B_1\left(\frac{[\delta+2]_{p,q}}{[2]_{p,q}!}[3]_{p,q}|b|B_1-[\delta+1]_{p,q}[2]_{p,q}^2\right)}{\left|\frac{[\delta+2]_{p,q}}{[2]_{p,q}!}[3]_{p,q}bB_1^2-[\delta+1]_{p,q}[2]_{p,q}^2B_2\right|+[\delta+1]_{p,q}[2]_{p,q}^2B_1}\right].$$

This completes the proof. □

For $\delta = 0$ in Theorem 3.1, we get the following result (see Theorem 5, [10]).

Corollary 3.2. *Let* $f \in \mathcal{R}^{p,q}_{\Sigma,b}(\phi)$, $\left(b \in \frac{\mathcal{C}}{\{0\}}\right)$. *Then*

$$|a_2| \leq \min\left\{K(p,q), \frac{|b|B_1\sqrt{B_1}}{\sqrt{\left|(B_1^2b - B_2)(p^2+q^2) + (B_1^2b - 2B_2)pq\right| + B_1(p^2+2pq+q^2)}}\right\}$$

and

$$|a_3| \leq min\{L(p,q), M(p,q)\}$$

where

$$K(p,q) = \begin{cases} \sqrt{\dfrac{|b|B_1}{p^2+pq+q^2}}, & |B_2| \leq B_1 \\ \sqrt{\dfrac{|bB_2|}{p^2+pq+q^2}}, & |B_2| > B_1 \end{cases}$$

$$L(p,q) = \begin{cases} \dfrac{|b|B_1}{p^2+pq+q^2}, & |B_2| \leq B_1 \\ \dfrac{|bB_2|}{p^2+pq+q^2}, & |B_2| > B_1 \end{cases}$$

and

$$M(p,q) = \begin{cases} \dfrac{|b|B_1}{p^2+pq+q^2}, & B_1 \leq \dfrac{p^2+2pq+q^2}{(p^2+pq+q^2)|b|} \\ \dfrac{|b|B_1[|(B_1^2b-B_2)(p^2+q^2)+(B_1^2b-2B_2)pq| + B_1^2|b|(p^2+pq+q^2)]}{[|(B_1^2b-B_2)(p^2+q^2)+(B_1^2b-2B_2)pq| + B_1(p^2+2pq+q^2)(p^2+pq+q^2)]}, & \\ B_1 > \dfrac{p^2+2pq+q^2}{(p^2+pq+q^2)|b|}. \end{cases}$$

For $p = 1$ in the above Corollary, we have the following result (see Theorem 6 in [10]).

Corollary 3.3. *Let* $f \in \mathcal{R}^{q}_{\Sigma,b}(\phi)$ $\left(b \in \frac{\mathcal{C}}{\{0\}}\right)$. *Then*

$$|a_2| \leq \min\left\{K(q), \frac{|b|B_1\sqrt{B_1}}{\sqrt{\left|(B_1^2b - B_2)(1+q^2) + (B_1^2b - 2B_2)q\right| + B_1(1+2q+q^2)}}\right\}$$

and

$$|a_3| \leq min\{L(q), M(q)\}$$

where

$$K(q) = \begin{cases} \sqrt{\dfrac{|b|B_1}{1+q+q^2}}, & |B_2| \leq B_1 \\ \sqrt{\dfrac{|bB_2|}{1+q+q^2}}, & |B_2| > B_1 \end{cases}$$

$$L(q) = \begin{cases} \dfrac{|b|B_1}{1+q+q^2}, & |B_2| \leq B_1 \\ \dfrac{|bB_2|}{1+q+q^2}, & |B_2| > B_1 \end{cases}$$

and

$$M(q) = \begin{cases} \dfrac{|b|B_1}{1+q+q^2}, & B_1 \leq \dfrac{1+2q+q^2}{(1+q+q^2)|b|} \\ \dfrac{|b|B_1[|(B_1^2b-B_2)(1+q^2)+(B_1^2b-2B_2)q| + B_1^2|b|(1+q+q^2)]}{[|(B_1^2b-B_2)(1+q^2)+(B_1^2b-2B_2)q| + B_1(1+2q+q^2)(1+q+q^2)]}, & \\ B_1 > \dfrac{1+2q+q^2}{(1+q+q^2)|b|}. \end{cases}$$

For $b = 1$ in Corollary 3.2, we obtain the following result (see Theorem 7, [10]).

Corollary 3.4. *Let* $f \in \mathcal{R}_{\Sigma}^{p,q}(\phi)$. *Then*

$$|a_2| \leq \min\left\{K(p,q), \frac{B_1\sqrt{B_1}}{\sqrt{|(B_1^2 - B_2)(p^2+q^2) + (B_1^2 - 2B_2)pq| + B_1(p^2+2pq+q^2)}}\right\}$$

and

$$|a_3| \leq min\{L(p,q), M(p,q)\}$$

where

$$K(p,q) = \begin{cases} \sqrt{\dfrac{B_1}{p^2+pq+q^2}}, & |B_2| \leq B_1 \\ \sqrt{\dfrac{|B_2|}{p^2+pq+q^2}}, & |B_2| > B_1 \end{cases}$$

$$L(p,q) = \begin{cases} \dfrac{B_1}{p^2+pq+q^2}, & |B_2| \leq B_1 \\ \dfrac{|B_2|}{p^2+pq+q^2}, & |B_2| > B_1 \end{cases}$$

and

$$M(p,q) = \begin{cases} \dfrac{B_1}{p^2+pq+q^2}, & B_1 \leq \dfrac{p^2+2pq+q^2}{(p^2+pq+q^2)} \\ \dfrac{B_1[|(B_1^2-B_2)(p^2+q^2)+(B_1^2-2B_2)pq| + B_1^2(p^2+pq+q^2)]}{[|(B_1^2-B_2)(p^2+q^2)+(B_1^2-2B_2)pq| + B_1(p^2+2pq+q^2)(p^2+pq+q^2)]}, \\ B_1 > \dfrac{p^2+2pq+q^2}{(p^2+pq+q^2)}. \end{cases}$$

For $p = 1$ and $b = 1$ in Corollary 3.2, we obtain the following result (see Theorem 8 in [10]).

Corollary 3.5. *Let* $f \in \mathcal{R}_{\Sigma}^{q}(\phi)$. *Then*

$$|a_2| \leq \min\left\{K(q), \frac{B_1\sqrt{B_1}}{\sqrt{|(B_1^2 - B_2)(1+q^2) + (B_1^2 - 2B_2)q| + B_1(1+2q+q^2)}}\right\}$$

and

$$|a_3| \leq min\{L(q), M(q)\},$$

where

$$K(q) = \begin{cases} \sqrt{\dfrac{B_1}{1+q+q^2}}, & |B_2| \leq B_1 \\ \sqrt{\dfrac{|B_2|}{1+q+q^2}}, & |B_2| > B_1 \end{cases}$$

$$L(q) = \begin{cases} \dfrac{B_1}{1+q+q^2}, & |B_2| \leq B_1 \\ \dfrac{|B_2|}{1+q+q^2}, & |B_2| > B_1 \end{cases}$$

and

$$M(q) = \begin{cases} \dfrac{B_1}{1+q+q^2}, & B_1 \leq \dfrac{1+2q+q^2}{(1+q+q^2)} \\ \dfrac{B_1[|(B_1^2-B_2)(1+q^2)+(B_1^2-2B_2)q| + B_1^2(1+q+q^2)]}{[|(B_1^2 b-B_2)(1+q^2)+(B_1^2-2B_2)q| + B_1(1+2q+q^2)(1+q+q^2)]}, \\ B_1 > \dfrac{1+2q+q^2}{(1+q+q^2)}. \end{cases}$$

Corollary 3.6. *If we put* $\phi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + ...(0 < \alpha \leq 1)$ *in Theorem 2.1 we have*

$$|a_n| \leq \frac{2\alpha|b|}{[n]_{p,q}\left|\frac{[n+\delta-1]_{p,q}!}{[\delta]_{p,q}![n-1]_{p,q}!}\right|}, \quad n \geq 3.$$

Remark 3.7. Setting $\phi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha}$ in Corollary 2.2 and Corollary 2.3 respectively we have (see Corollary 1 in [10]).

$$|a_n| \leq \frac{2\alpha|b|}{[n]_{p,q}}, \quad n \geq 3,$$

and

$$|a_n| \leq \frac{2\alpha|b|(1-q)}{(1-q^n)}, \quad n \geq 3.$$

Remark 3.8. Letting $\phi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha}$ in Corollary 2.4 and Corollary 2.5, respectively we obtain (see remark 1, 2 in [10])

$$|a_n| \leq \frac{2\alpha}{[n]_{p,q}}, \quad n \geq 3,$$

and

$$|a_n| \leq \frac{2\alpha(1-q)}{(1-q^n)}, \quad n \geq 3.$$

Corollary 3.9. *If we take* $\phi(z) = \frac{1+(1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + ...$ $(0 \leq \beta < 1)$, *in Theorem 2.1 we have*

$$|a_n| \leq \frac{2(1-\beta)|b|}{[n]_{p,q}\left|\frac{[n+\delta-1]_{p,q}!}{[\delta]_{p,q}![n-1]_{p,q}!}\right|}, \quad n \geq 3.$$

Remark 3.10. Setting $\phi(z) = \frac{1+(1-2\beta)z}{1-z}$ in Corollary 2.2 and Corollary 2.3 respectively, we have (see Corollary 2 in [10])

$$|a_n| \leq \frac{2(1-\beta)|b|}{[n]_{p,q}}, \quad n \geq 3,$$

and

$$|a_n| \leq \frac{2(1-\beta)|b|(1-q)}{(1-q^n)}, \quad n \geq 3.$$

Remark 3.11. Letting $\phi(z) = \frac{1+(1-2\beta)z}{1-z}$ in Corollary 2.4 and Corollary 2.5 respectively, we obtain (see remark 3, 4 in [10])

$$|a_n| \leq \frac{2(1-\beta)}{[n]_{p,q}}, \quad n \geq 3,$$

and

$$|a_n| \leq \frac{2(1-\beta)(1-q)}{(1-q^n}, \quad n \geq 3.$$

Taking $\phi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha}$ in Theorem 3.1, we have the following result.

Corollary 3.12. *Let* $f \in \mathcal{R}_\Sigma\left(p, q, \delta, b, \left(\frac{1+z}{1-z}\right)^\alpha\right), \left(b \in \frac{\mathcal{C}}{\{0\}}\right)$. *Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{2\alpha[2]_{p,q}!|b|}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}}}, \frac{|b|2\alpha}{\sqrt{\alpha\left(\left|\frac{2[\delta+1]_{p,q}[\delta+2]_{p,q}}{[2]_{p,q}!}[3]_{p,q}b-[\delta+1]^2_{p,q}[2]^2_{p,q}\right|\right)+[\delta+1]^2_{p,q}[2]^2_{p,q}}}\right\}$$

and

$$|a_3| \leq min\left\{\frac{[2]_{p,q}!|b|2\alpha}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}}, L(p,q)\right\}$$

where

$$L(p,q) = \begin{cases} \frac{[2]_{p,q}!2\alpha|b|}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}}, & 0<\alpha\leq\frac{[2]^3_{p,q}[\delta+1]_{p,q}}{2[\delta+2]_{p,q}[3]_{p,q}|b|} \\ \frac{[2]_{p,q}!|b|2\alpha^2}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}}\left[\frac{\left|\frac{2[\delta+2]_{p,q}}{[2]_{p,q}!}[3]_{p,q}b-[\delta+1]_{p,q}[2]^2_{p,q}\right|+\frac{2[\delta+2]_{p,q}|b|}{[2]_{p,q}!}[3]_{p,q}}{\alpha\left(\left|\frac{2[\delta+2]_{p,q}}{[2]_{p,q}!}[3]_{p,q}b-[\delta+1]_{p,q}[2]^2_{p,q}\right|\right)+[\delta+1]_{p,q}[2]^2_{p,q}}\right], & \frac{[2]^3_{p,q}[\delta+1]_{p,q}}{2[\delta+2]_{p,q}[3]_{p,q}|b|}<\alpha\leq 1. \end{cases}$$

Taking $\phi(z) = \left(\frac{1+z}{1-z}\right)^\alpha$ in Corollaries 3.2, 3.3, 3.4 and 3.5 respectively we have the following results.

Remark 3.13. Let $f \in \mathcal{R}^{p,q}_{\Sigma,b}\left(\left(\frac{1+z}{1-z}\right)^\alpha\right), \left(b \in \frac{\mathcal{C}}{\{0\}}\right)$(Corollary 3, [10]). Then

$$|a_2| \leq \min\left\{\sqrt{\frac{2\alpha|b|}{p^2+pq+q^2}}, \frac{2\alpha|b|}{\sqrt{\alpha\left|(2b-1)(p^2+q^2)+2(b-1)pq\right|+(p^2+2pq+q^2)}}\right\}$$

and

$$|a_3| \leq min\left\{\frac{2\alpha|b|}{p^2+pq+q^2}, H(p,q)\right\}$$

where

$$H(p,q) = \begin{cases} \frac{2\alpha|b|}{p^2+pq+q^2}, & 0<\alpha\leq\frac{p^2+2pq+q^2}{2(p^2+pq+q^2)|b|} \\ \frac{2\alpha^2|b|[|(2b-1)(p^2+q^2)+2(b-1)pq|+2|b|(p^2+pq+q^2)]}{[\alpha|(2b-1)(p^2+q^2)+2(b-1)pq|+(p^2+2pq+q^2)](p^2+pq+q^2)}, & \frac{p^2+2pq+q^2}{2(p^2+pq+q^2)|b|}<\alpha\leq 1. \end{cases}$$

Corollary 3.14. *Let* $f \in \mathcal{R}^q_{\Sigma,b}(\left(\frac{1+z}{1-z}\right)^\alpha), \left(b \in \frac{\mathcal{C}}{\{0\}}\right)$ *(Corollary 4, [10]). Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{2\alpha|b|}{1+q+q^2}}, \frac{2\alpha|b|}{\sqrt{\alpha|(2b-1)(1+q^2)+2(b-1)q|+1+2q+q^2}}\right\}$$

and

$$|a_3| \leq min\left\{\frac{|b|2\alpha}{1+q+q^2}, H(q)\right\}$$

where

$$H(q) = \begin{cases} \frac{2\alpha|b|}{1+q+q^2}, & 0<\alpha\leq\frac{1+2q+q^2}{2(1+q+q^2)|b|} \\ \frac{2\alpha^2|b|[|(2b-1)(1+q^2)+2(b-1)q|+2|b|(1+q+q^2)]}{[\alpha|(2b-1)(1+q^2)+2(b-1)q|+(1+2q+q^2)](1+q+q^2)}, & \frac{1+2q+q^2}{2(1+q+q^2)|b|}<\alpha\leq 1. \end{cases}$$

Corollary 3.15. *Let* $f \in \mathcal{R}_{\Sigma}^{p,q}\left(\left(\frac{1+z}{1-z}\right)^{\alpha}\right)$ *(Corollary 5 , [10]). Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{(\alpha+1)(p^2+q^2)+2pq}},$$

and

$$|a_3| \leq \min\left\{\frac{2\alpha}{p^2+pq+q^2}, H(p,q)\right\}$$

where

$$H(p,q) = \begin{cases} \dfrac{2\alpha}{p^2+pq+q^2}, & 0 < \alpha \leq \dfrac{p^2+2pq+q^2}{2(p^2+pq+q^2)} \\ \dfrac{2\alpha^2[3(p^2+q^2)+2pq]}{[(\alpha+1)(p^2+q^2)+2pq](p^2+pq+q^2)}, & \dfrac{p^2+2pq+q^2}{2(p^2+pq+q^2)} < \alpha \leq 1. \end{cases}$$

Corollary 3.16. *Let* $f \in \mathcal{R}_{\Sigma}^{q}\left(\left(\frac{1+z}{1-z}\right)^{\alpha}\right)$ *(Corollary 6, [10]). Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{(\alpha+1)(1+q^2)+2q}},$$

and

$$|a_3| \leq \min\left\{\frac{2\alpha}{1+q+q^2}, H(q)\right\}$$

where

$$H(q) = \begin{cases} \dfrac{2\alpha}{1+q+q^2}, & 0 < \alpha \leq \dfrac{1+2q+q^2}{2(1+q+q^2)} \\ \dfrac{2\alpha^2[3(1+q^2)+2q]}{[(\alpha+1)(1+q^2)+2q](1+q+q^2)}, & \dfrac{1+2q+q^2}{2(1+q+q^2)} < \alpha \leq 1. \end{cases}$$

Taking $\phi(z) = \dfrac{1+(1-2\beta)z}{1-z}$ in Theorem 3.1, we have the following results.

Corollary 3.17. *Let* $f \in \mathcal{R}_{\Sigma}\left(p,q,\delta,b,\frac{1+(1-2\beta)z}{1-z}\right), \left(b \in \frac{\mathcal{C}}{\{0\}}\right)$. *Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{2(1-\beta)[2]_{p,q}!|b|}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}}}, \frac{2(1-\beta)|b|}{\sqrt{\left(\left|\frac{2(1-\beta)[\delta+1]_{p,q}[\delta+2]_{p,q}}{[2]_{p,q}!}[3]_{p,q}b-[\delta+1]_{p,q}^2[2]_{p,q}^2\right|\right)+[\delta+1]_{p,q}^2[2]_{p,q}^2}}\right\}$$

and

$$|a_3| \leq min\left\{\frac{2(1-\beta)[2]_{p,q}!|b|}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}}, M(p,q)\right\}$$

where

$$M(p,q) = \begin{cases} \dfrac{2(1-\beta)[2]_{p,q}!|b|}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}}, & \dfrac{2[\delta+2]_{p,q}[3]_{p,q}|b|-[2]_{p,q}^3}{2[\delta+2]_{p,q}[3]_{p,q}|b|} \leq \beta < 1. \\ \dfrac{2(1-\beta)[2]_{p,q}!|b|}{[\delta+1]_{p,q}[\delta+2]_{p,q}[3]_{p,q}} \left[\dfrac{\left|\frac{2(1-\beta)[\delta+2]_{p,q}}{[2]_{p,q}!}[3]_{p,q}b-[\delta+1]_{p,q}[2]_{p,q}^2\right|+\frac{2(1-\beta)[\delta+2]_{p,q}|b|}{[2]_{p,q}!}[3]_{p,q}}{\left(\left|\frac{2(1-\beta)[\delta+2]_{p,q}}{[2]_{p,q}!}[3]_{p,q}b-[\delta+1]_{p,q}[2]_{p,q}^2\right|\right)+[\delta+1]_{p,q}[2]_{p,q}^2}\right], & \\ & 0 \leq \beta < \dfrac{2[\delta+2]_{p,q}[3]_{p,q}|b|-[2]_{p,q}^3}{2[\delta+2]_{p,q}[3]_{p,q}|b|}. \end{cases}$$

By choosing $\phi(z) = \dfrac{1+(1-2\beta)z}{1-z}$ in Corollaries 3.2, 3.3, 3.4 and 3.5 respectively, we have the following results.

Corollary 3.18. *Let* $f \in \mathcal{R}^{p,q}_{\Sigma,b}\left(\frac{1+(1-2\beta)z}{1-z}\right)$, $\left(b \in \frac{\mathcal{C}}{\{0\}}\right)$ *(Corollary 7, [10]).Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{2(1-\beta)|b|}{p^2+pq+q^2}}, \frac{2(1-\beta)|b|}{\sqrt{|2(1-\beta)b[3]_{p,q}-[2]^2_{p,q}|+[2]^2_{p,q}}}\right\}$$

and

$$|a_3| \leq \min\left\{\frac{2(1-\beta)|b|}{p^2+pq+q^2}, R(p,q)\right\}$$

where

$$R(p,q)=\begin{cases}\frac{2(1-\beta)|b|}{p^2+pq+q^2}, & \frac{2(p^2+pq+q^2)|b|-(p^2+2pq+q^2)}{2(p^2+pq+q^2)|b|}\leq\beta<1\\ \frac{2(1-\beta)|b|[|2(1-\beta)b(p^2+pq+q^2)-(p^2+2pq+q^2)|+2(1-\beta)|b|(p^2+pq+q^2)]}{[|2(1-\beta)b(p^2+pq+q^2)-(p^2+2pq+q^2)|+(p^2+2pq+q^2)](p^2+pq+q^2)}, & \\ & 0\leq\beta<\frac{2(p^2+pq+q^2)|b|-(p^2+2pq+q^2)}{2(p^2+pq+q^2)|b|}.\end{cases}$$

Remark 3.19. Let $f \in \mathcal{R}^{q}_{\Sigma,b}\left(\frac{1+(1-2\beta)z}{1-z}\right)$, $\left(b \in \frac{\mathcal{C}}{\{0\}}\right)$ (Corollary 8, [10]). Then

$$|a_2| \leq \min\left\{\sqrt{\frac{2(1-\beta)|b|}{1+q+q^2}}, \frac{2(1-\beta)|b|}{\sqrt{|2(1-\beta)b(1+q+q^2)-(1+2q+q^2)|+1+2q+q^2}}\right\}$$

and

$$|a_3| \leq \min\left\{\frac{2(1-\beta)|b|}{1+q+q^2}, R(q)\right\}$$

where

$$R(q)=\begin{cases}\frac{2(1-\beta)|b|}{1+q+q^2}, & \frac{2(1+q+q^2)|b|-(1+2q+q^2)}{2(1+q+q^2)|b|}\leq\beta<1\\ \frac{2(1-\beta)|b|[|2(1-\beta)b(1+q+q^2)-(1+2q+q^2)|+2(1-\beta)|b|(1+q+q^2)]}{[|2(1-\beta)b(1+q+q^2)-(1+2q+q^2)|+(1+2q+q^2)](1+q+q^2)}, & \\ & 0\leq\beta<\frac{2(1+q+q^2)|b|-(1+2q+q^2)}{2(1+q+q^2)|b|}.\end{cases}$$

Corollary 3.20. *Let* $f \in \mathcal{R}^{p,q}_{\Sigma}\left(\frac{1+(1-2\beta)z}{1-z}\right)$*(Corollary 9, [10]). Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{2(1-\beta)}{p^2+pq+q^2}}, \frac{2(1-\beta)}{\sqrt{|(1-2\beta)(p^2+q^2)-2\beta pq|+p^2+2pq+q^2}}\right\}$$

and

$$|a_3| \leq \min\left\{\frac{2(1-\beta)}{p^2+pq+q^2}, R(p,q)\right\}$$

where

$$R(p,q)=\begin{cases}\frac{2(1-\beta)}{p^2+pq+q^2}, & \frac{(p^2+q^2)}{2(p^2+pq+q^2)}\leq\beta<1\\ \frac{(3-4\beta)(p^2+q^2)+2(1-2\beta)pq}{(p^2+pq+q^2)^2}, & 0\leq\beta<\frac{(p^2+q^2)}{2(p^2+pq+q^2)}.\end{cases}$$

Remark 3.21. Let $f \in \mathcal{R}^{q}_{\Sigma}\left(\frac{1+(1-2\beta)z}{1-z}\right)$ (Corollary 10, [10]). Then

$$|a_2| \leq \min\left\{\sqrt{\frac{2(1-\beta)}{1+q+q^2}}, \frac{2(1-\beta)}{\sqrt{|(1-2\beta)(1+q^2)-2\beta q|+1+2q+q^2}}\right\}$$

and

$$|a_3| \leq \min\left\{\frac{2(1-\beta)}{1+q+q^2}, R(q)\right\}$$

where

$$R(q) = \begin{cases} \dfrac{2(1-\beta)}{1+q+q^2}, & \dfrac{(1+q^2)}{2(1+q+q^2)} \leq \beta < 1 \\ \dfrac{(3-4\beta)(1+q^2)+2(1-2\beta)q}{(1+q+q^2)^2}, & 0 \leq \beta < \dfrac{1+q^2}{2(1+q+q^2)}. \end{cases}$$

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