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On the hyperstability of a radical functional equation related to Drygas mappings in ultrametric 2-Banach spaces

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Abstract The aim of this paper is to introduce and solve the following p-radical functional equation related to Drygas mappings

$$f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)=2f(x)+f(y)+f(-y),\quad x,y\in\mathbb{R},$$

where f is a mapping from $\mathbb{R}$ into a vector space X and $p\geq 3$ is an odd natural number. Using an analogue version of Brzdęk's fixed point theorem [14], we establish some hyperstability results for the considered equation in ultrametric 2-Banach spaces. Also, we give some hyperstability results for the inhomogeneous p-radical functional equation related to Drygas mappings

$$f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)=2f(x)+f(y)+f(-y)+G(x,y).$$

1 Introduction

A classical question in the theory of functional equation is the following:
" Is it true that a function which approximately satisfies a functional equation must be close to an exact solution of this equation."
If the answer is affirmative, then we say that equation is stable. In 1940, S. M. Ulam [45]) asked the following question concerning the stability of group homomorphisms
Let $(G_1,*_1)$ be a group and let $(G_2,*_2)$ be a metric group with a metric $d(.,.)$. Given $\varepsilon>0$, does there exists a $\delta>0$ such that if a mapping $h:G_1\to G_2$ satisfies the inequality

$$d\big(h(x*_1 y),h(x)*_2 h(y)\big)<\delta$$

for all $x,y\in G_1$, then there exists a homomorphism $H:G_1\to G_2$ with

$$d\big(h(x),H(x)\big)<\varepsilon$$

for all $x\in G_1$?
This question seems to be the starting point of studying the stability of functional equations. Since then, this question has attracted the attention of many researchers. The first partial answer was raised by D. H. Hyers [30] in 1941 under the assumption that G_1 and G_2 are Banach spaces for the the additive functional equation as follows:

Theorem 1.1. *[30] Let E_1 and E_2 be two Banach spaces and $f:E_1\to E_2$ be a mapping such that*

$$\|f(x+y)-f(x)-f(y)\|\leq\delta$$

for some $\delta > 0$ *and for all* $x, y \in E_1$. *Then the limit*

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for each $x \in E_1$, *and* $A : E_1 \to E_2$ *is the unique additive mapping such that*

$$\|f(x) - A(x)\| \leq \delta$$

for all $x \in E_1$.

Later, T. Aoki [10] and D. G. Bourgin [11] considered the problem of stability with unbounded Cauchy differences. In 1978, Th. M. Rassias [39] attempted to weaken the condition for the bound of the norm of Cauchy difference $\|f(x+y) - f(x) - f(y)\|$ and proved a generalization of Theorem 1.1 by using a direct method (cf. Theorem 1.2):

Theorem 1.2. *[39] Let* E_1 *and* E_2 *be two Banach spaces. If* $f : E_1 \to E_2$ *satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta\big(\|x\|^p + \|y\|^p\big)$$

for some $\theta \geq 0$, *for some* $p \in \mathbb{R}$ *with* $0 \leq p < 1$, *and for all* $x, y \in E_1$, *then there exists a unique additive mapping* $A : E_1 \to E_2$ *such that*

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p$$

for each $x \in E_1$. *If, in addition,* $f(tx)$ *is continuous in t for each fixed* $x \in E_1$, *then the mapping* A *is linear.*

After then, Th. M. Rassias [40],[41] motivated Theorem 1.2 as follows:

Theorem 1.3. *[40],[41] Let* E_1 *be a normed space,* E_2 *be a Banach space, and* $f : E_1 \to E_2$ *be a mapping. If f satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta\big(\|x\|^p + \|y\|^p\big) \tag{1.1}$$

for some $\theta \geq 0$, *for some* $p \in \mathbb{R}$ *with* $p \neq 1$, *and for all* $x, y \in E_1 - \{0_{E_1}\}$, *then there exists a unique additive mapping* $A : E_1 \to E_2$ *such that*

$$\|f(x) - A(x)\| \leq \frac{2\theta}{|2 - 2^p|}\|x\|^p \tag{1.2}$$

for each $x \in E_1 - \{0_{E_1}\}$.

Note that Theorem 1.3 reduces to Theorem 1.1 when $p = 0$. For $p = 1$, the analogous result is not valid. Also, J. Brzdęk [13] showed the estimation (1.2) is optimal for $p \geq 0$ in the general case.

In 1994, P. Găvruţă [29] provided a further generalization of Rassias theorem in which he replaced the bound $\theta\,(\|x\|^p + \|y\|^p)$ in (1.1) by a general control function $\varphi(x, y)$ for the existence of a unique linear mapping.

Recently, J. Brzdęk [17] showed that Theorem 1.3 can be significantly improved. Namely, in the case $p < 0$, each $f : E_1 \to E_2$ satisfying (1.1) must actually be additive. This result is called the hyperstability of Cauchy functional equation. However, the term of hyperstability was introduced for the first time probably in [35] and it was developed with the fixed point theorem of Brzdęk in [14]. There after, the hyperstability of a several functional equations have been studied by many authors (see, for example, [5, 7, 2, 17, 35]).

In 2013, Brzdęk [16] improved, extended and complemented several earlier classical stability results concerning the additive Cauchy equation (in particular Theorem 1.3). Over the last few years, many mathematicians have investigated various generalizations, extensions and applications of the Hyers-Ulam stability of a number of functional equations (see, for instance, [18], [19] and references therein).

Characterizing quasi-inner product spaces, H. Drygas considers in [22] the functional equation

$$f(x)+f(y)=f(x-y)+\left\{f(\frac{x+y}{2})-f(\frac{x-y}{2})\right\}, \quad x,y\in\mathbb{R}, \tag{1.3}$$

which can be reduced to the following equation [42, Remark 9.2, p. 131]

$$f(x+y)+f(x-y)=2f(x)+f(y)+f(-y), \quad x,y\in\mathbb{R}. \tag{1.4}$$

This equation is known in the literature as Drygas equation and is a generalization of the quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y), \quad x,y\in\mathbb{R}.$$

The general solution of Drygas equation was given by B. R. Ebanks, P. L. Kannappan and P. K. Sahoo in [23]. It has the form

$$f(x)=A(x)+Q(x), \quad x\in\mathbb{R},$$

where $A\colon\mathbb{R}\to\mathbb{R}$ is an additive function and $Q\colon\mathbb{R}\to\mathbb{R}$ is a quadratic function, see also [32]. A set-valued version of Drygas equation was considered by W. Smajdor in [44].
Recently, the hyperstability of the Drygas functional equation has been studied in [38], [43] and [6].

During the 16^{th} International Conference on Functional Equations and Inequalities (Będlewo, Poland, May 17-23, 2015), W. Sintunavarat presented a talk concerning the Ulam type stability (for information and further references concerning this notion see, e.g., [12]) of the so-called radical functional equation

$$f\left(\sqrt{x^2+y^2}\right)=f(x)+f(y)$$

in the class of real functions. A question of J. Schwaiger about the general solution of the equation was answered a bit later by the author of this paper (see [36], p. 196). In this regard, many papers concerning the solutions and stability of radical functional equations have been established (the reader can refer, for example, to [1, 2, 3, 25, 24, 33, 34]).

We need to recall some basic notion concerning the non-Archimedean 2-normed spaces. Indeed, the notion of linear 2-normed spaces was introduced by Gähler [27],[28] in the middle of 1960s. We need to recall some basic facts concerning 2-normed spaces and some preliminary results.

Definition 1.4. [26] Let X be a real linear space with $dimX>1$ and $\|.,.\|:X\times X\longrightarrow[0,\infty)$ be a function satisfying the following properties:

(i) $\|x,y\|=0$ if and only if x and y are linearly dependent,

(ii) $\|x,y\|=\|y,x\|$,

(iii) $\|\lambda x,y\|=|\lambda|\|x,y\|$,

(iv) $\|x,y+z\|\le\|x,y\|+\|x,z\|$,

for all $x,y,z\in X$ and $\lambda\in\mathbb{R}$. Then the function $\|.,.\|$ is called *a 2-norm* on X and the pair $(X,\|.,.\|)$ is called a *linear 2-normed space*. Sometimes the condition (4) called the *triangle inequality*.

Example 1.5. For $x=(x_1,x_2)$, $y=(y_1,y_2)\in X=\mathbb{R}^2$, the Euclidean 2-norm $\|x,y\|_{\mathbb{R}^2}$ is defined by

$$\|x,y\|_{\mathbb{R}^2}=|x_1y_2-x_2y_1|\,.$$

Lemma 1.6. *Let* $(X,\|.,.\|)$ *be a 2-normed space. If* $x\in X$ *and* $\|x,y\|=0$*, for all* $y\in X$*, then* $x=0$.

Definition 1.7. A sequence $\{x_k\}$ in a 2-normed space X is called a *convergent sequence* if there is an $x \in X$ such that

$$\lim_{k\to\infty} \|x_k - x, y\| = 0,$$

for all $y \in X$. If $\{x_k\}$ converges to x, write $x_k \longrightarrow x$ with $k \longrightarrow \infty$ and call x the limit of $\{x_k\}$. In this case, we also write $\lim_{k\to\infty} x_k = x$.

Definition 1.8. A sequence $\{x_k\}$ in a 2-normed space X is said to be a *Cauchy sequence* with respect to the 2-norm if

$$\lim_{k,l\to\infty} \|x_k - x_l, y\| = 0,$$

for all $y \in X$. If every Cauchy sequence in X converges to some $x \in X$, then X is said to be *complete* with respect to the 2-norm. Any complete 2-normed space is said to be a *2-Banach space*.

Now, we state the following results as lemmas (See [37] for the details).

Lemma 1.9. *Let X be a 2-normed space. Then,*

(i) $\big|\|x, z\| - \|y, z\|\big| \leq \|x - y, z\|$ *for all* $x, y, z \in X$,

(ii) if $\|x, z\| = 0$ *for all* $z \in X$, *then* $x = 0$,

(iii) for a convergent sequence x_n *in* X,

$$\lim_{n\longrightarrow\infty} \|x_n, z\| = \left\|\lim_{n\longrightarrow\infty} x_n, z\right\|$$

for all $z \in X$.

Lemma 1.10. *Let X be a linear 2-normed space and let $x, y_1, y_2 \in X$ such that y_1, y_2 are linearly independent.*
If

$$\big\|x, y_1\big\| = 0 = \big\|x, y_2\big\|,$$

then $x = 0$.

In the following, we recall some basic facts and definitions concerning to the ultrametric spaces and ultrametric 2-Banach spaces.

Definition 1.11. [33] By a *non-Archimedean* field, we mean a field $\mathbb{K}$ equipped with a function (*valuation*) $|\cdot| : \mathbb{K} \to [0, \infty)$ such that for all $r, s \in \mathbb{K}$, the following conditions hold:

(i) $|r| = 0$ if and only if $r = 0$,

(ii) $|rs| = |r||s|$,

(iii) $|r + s| \leq \max\big\{|r|, |s|\big\}$.

The pair $(\mathbb{K}, |.|)$ is called a *valued field*.

Remark 1.12. In any non-Archimedean field, we have $|1| = |-1| = 1$ and $|n| \leq 1$ for $n \in \mathbb{N}$.

Definition 1.13. [33] Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $||\cdot||_* : X \to \mathbb{R}$ is a *non-Archimedean norm (valuation)* if it satisfies the following conditions:

(i) $\|x\|_* = 0$ if and only if $x = 0$,

(ii) $\|rx\|_* = |r|\ \|x\|_*\ \ (r \in \mathbb{K}, x \in X)$,

(iii) The strong triangle inequality (ultrametric); namely

$$\|x + y\|_* \leq \max\big\{\|x\|_*, \|y\|_*\big\}\ \ x, y \in X.$$

Then $(X, \|\cdot\|_*)$ is called *a non-Archimedean normed space* or *an ultrametric normed space*.

At the end of this section, we give the definition of a ultrametric 2-normed space which has been introduced in [26].

Definition 1.14. [26] Let X be a vector space (with $dimX > 1$) over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $||.,.||_* : X^2 \to \mathbb{R}_+$ is called a *ultrametric 2-norm (valuation)* if it satisfies the following conditions, for each $x, y, z \in X$ and each $r \in \mathbb{K}$, :

(i) $\|x, y\|_* = 0$ if and only if x and y are linearly independent,

(ii) $\|x, y\|_* = \|y, x\|_*$,

(iii) $\|rx, y\|_* = |r| \, \|x, y\|_*$,

(iv) $\|x, y + z\|_* \leq \max\big\{\|x, y\|_*\ ,\ \|x, z\|_*\big\}$.

Then $(X, \|\cdot, \cdot\|_*)$ is called *a non-Archimedean 2-normed space* or *an ultrametric 2-normed space*.

Example 1.15. Let p be a fixed prime number. For $x = (x_1, x_2)$ and $y = (y_1, y_2)$ we define the ultrametric 2-norm in ${\mathbb{Q}_p}^2$ by $\|x, y\|_p = |x_1y_2 - x_2y_1|_p$.

Throughout this paper, we will denote the set of natural numbers by $\mathbb{N}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, the set of real numbers by $\mathbb{R}$, $\mathbb{R}_+ = [0, \infty)$ the set of non negative real numbers and $\mathbb{R}_0 = \mathbb{R}\backslash\{0\}$. By $\mathbb{N}_{m_0}$, $m_0 \in \mathbb{N}$, we will denote the set of all natural numbers greater than or equal to m_0. Let X be a linear space and let $p \in \mathbb{N}_3$ be an odd natural number. M. E. Hryrou et al. [31] proved the general solution of the following functional equation

$$f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) = 2f(x) + f(y) + f(-y) \quad x, y \in \mathbb{R}, \tag{1.5}$$

where $f : \mathbb{R} \to X$ which is called p-radical functional equation related to Drygas equation (1.4).

The main purpose of this paper is to achieve the general solution of the functional equation (1.5) and establish some hyperstability results for the considered equation in ultrametric 2-Banach space. We also provide some corollaries and outcomes concerning the hyperstability results for the inhomogeneous of p-radical functional equation.
Before proceeding to the main results, we state Theorem 1.16 which is useful for our purpose. To present it, we introduce the following three hypotheses:

(H1) X is a nonempty set, Y is an ultrametric Banach space over a non-Archimedean field, $g : X \to Y$ be a mapping such that the set $g(X) \subseteq Y$ containing two linearly independent vectors, $f_1, ..., f_k : X \longrightarrow X$ and $L_1, ..., L_k : X \longrightarrow \mathbb{R}_+$ are given.

(H2) $\mathcal{T} : Y^X \longrightarrow Y^X$ is an operator satisfying the inequality

$$\Big\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\ ,\ g(z)\Big\|_* \leq \max_{1\leq i\leq k}\left\{L_i(x)\Big\|\xi\Big(f_i(x)\Big) - \mu\Big(f_i(x)\Big)\ ,\ g(z)\Big\|_*\right\}, \qquad \xi, \mu \in Y^X, x, z \in X.$$

(H3) $\Lambda : \mathbb{R}_+^{X\times X} \longrightarrow \mathbb{R}_+^{X\times X}$ is a linear operator defined by

$$\Lambda\delta(x, z) := \max_{1\leq i\leq k}\left\{L_i(x)\delta\Big(f_i(x)\ ,\ z\Big)\right\}, \qquad \delta \in \mathbb{R}_+^{X\times X}, \qquad x, z \in X.$$

Thanks to a result due to J. Brzdęk and K. Ciepliński [15, Remark 2] and M. Almahalebi and A. Chahbi [9], we state an analogue of the fixed point theorem [15, Theorem 1] in ultrametric 2-Banach space. We use it to assert the existence of a unique fixed point of operator $\mathcal{T} : Y^X \longrightarrow Y^X$.

Theorem 1.16. *Let hypotheses **(H1)**-(**H3**) be valid and functions* $\varepsilon : X \times X \longrightarrow \mathbb{R}_+$ *and* $\varphi : X \longrightarrow Y$ *fulfill the following two conditions*

$$\|\mathcal{T}\varphi(x) - \varphi(x)\ ,\ g(z)\|_* \leq \varepsilon(x, z), \qquad x, z \in X,$$

$$\lim_{n\to\infty} \Lambda^n\varepsilon(x, z) = 0, \qquad x, z \in X.$$

Then there exists a unique fixed point $\psi \in Y^X$ *of* $\mathcal{T}$ *with*

$$\|\varphi(x) - \psi(x)\ ,\ g(z)\|_* \leq \sup_{n\in\mathbb{N}_0} \Lambda^n \varepsilon(x,z), \qquad x,z \in X.$$

Moreover

$$\psi(x) := \lim_{n\to\infty} \mathcal{T}^n \varphi(x), \qquad x \in X.$$

2 Main results

In this section, we examine the hyperstability of the equation (1.5) in ultrametric 2-Banach space by using, as a basic tool, Theorem 1.16.

Theorem 2.1. *Let* p *be an odd natural number,* $(X\ ,\ \|\cdot,\cdot\|_*)$ *be an ultrametric 2-Banach space,* $g : \mathbb{R}_0 \to X$ *be a mapping such that the set* $g(\mathbb{R}_0) \subseteq X$ *containing two linearly independent vectors and let* $h_1, h_2 : \mathbb{R}_0 \times \mathbb{R}_0 \to \mathbb{R}_+$ *be two functions such that*

$$\mathcal{U} := \big\{n \in \mathbb{N} : \alpha_n = \max\{\lambda_1(n+1)\lambda_2(n+1)\ ,\ \lambda_1(2n+1)\lambda_2(2n+1)\ ,\ \lambda_1(n)\lambda_2(n)\ ,\ \lambda_1(-n)\lambda_2(-n)\ \} < 1\big\} \neq \phi,$$

where

$$\lambda_i(m) := \inf\{t \in \mathbb{R}_+ : h_i(mx,z) \leq t\, h_i(x,z),\ \ x,z \in \mathbb{R}_0\},$$

for all $m \in \mathbb{N}$*, where* $i = 1,2$ *such that*

$$\lim_{m\to\infty} \lambda_1(m+1)\lambda_2(m) = 0. \tag{2.1}$$

Assume that $f : \mathbb{R} \to X$ *satisfies the inequality*

$$\left\|f\left(\sqrt[p]{x^p+y^p}\right) + f\left(\sqrt[p]{x^p-y^p}\right) - 2f(x) - f(y) - f(-y)\ ,\ g(z)\right\|_* \leq h_1(x^p,z)h_2(y^p,z), \tag{2.2}$$

for all $x,y,z \in \mathbb{R}_0$ *such that* $x \neq y$ *and* $x \neq -y$*. Then* f *is a solution of the equation (1.5) on* $\mathbb{R}_0$*.*

Proof. Replacing x by $\sqrt[p]{m+1}\ x$ and y by $\sqrt[p]{m}\ x$ in the inequality (2.2), we get

$$\begin{aligned}&\Big\|2f\left(\sqrt[p]{m+1}\ x\right) - f\left(\sqrt[p]{2m+1}\ x\right) + f\left(\sqrt[p]{m}\ x\right)\\ &+ f\left(\sqrt[p]{-m}\ x\right) - f(x)\ ,\ g(z)\Big\|_* \leq h_1((m+1)x^p,z)h_2((m)x^p,z),\end{aligned} \tag{2.3}$$

for all $x,z \in \mathbb{R}_0$. For each $m \in \mathbb{N}$, we define the operator $\mathcal{T}_m : X^{\mathbb{R}_0} \to X^{\mathbb{R}_0}$ by

$$\mathcal{T}_m\xi(x) := 2\xi\left(\sqrt[p]{m+1}\ x\right) - \xi\left(\sqrt[p]{2m+1}\ x\right) + \xi\left(\sqrt[p]{m}\ x\right) + \xi\left(\sqrt[p]{-m}\ x\right),$$

for all $\xi \in X^{\mathbb{R}_0}$, $x \in \mathbb{R}_0$ and the function $\varepsilon_m : \mathbb{R}_0 \times \mathbb{R}_0 \to \mathbb{R}_+$ by

$$\varepsilon_m(x,z) := h_1((m+1)x^p,z)h_2((m)x^p,z), \ \ m \in \mathbb{N}, \ \ x,z \in \mathbb{R}_0.$$

We observe that

$$\varepsilon_m(x,z) \leq \lambda_1(m+1)\lambda_2(m)h_1(x^p,z)h_2(x^p,z), \tag{2.4}$$

for all $x,z \in \mathbb{R}_0$ and all $m \in \mathcal{U}$. Then the inequality (2.3) become as

$$\left\|\mathcal{T}_m f(x) - f(x)\ ,\ g(z)\right\|_* \leq \varepsilon_m(x,z), \quad x,z \in \mathbb{R}_0.$$

Furthermore, the operator $\Lambda_m : \mathbb{R}_+^{\mathbb{R}_0\times\mathbb{R}_0} \to \mathbb{R}_+^{\mathbb{R}_0\times\mathbb{R}_0}$ is defined by

$$\Lambda_m\delta(x,z) := \max_{1\leq i\leq 4}\big\{L_i(x)\delta\big(f_i(x),z\big)\big\},$$

for all $x \in \mathbb{R}_0$ and all $\delta \in \mathbb{R}_+^{\mathbb{R}_0}$ where $f_1(x) = \sqrt[p]{m+1}\ x$, $f_2(x) = \sqrt[p]{2m+1}\ x$, $f_3(x) = \sqrt[p]{m}\ x$, $f_4(x) = \sqrt[p]{-m}\ x$, and $L_1(x) = L_2(x) = L_3(x) = L_4(x) = 1$.

Moreover, for every $x \in \mathbb{R}_0$, $\xi, \mu \in X^{\mathbb{R}_0}$, we obtain

$$\left\|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x)\ ,\ g(z)\right\|_* \tag{2.5}$$

$$= \left\|2\left(\xi\left(\sqrt[p]{m+1}\ x\right) - \mu\left(\sqrt[p]{m+1}\ x\right)\right) - \left(\xi\left(\sqrt[p]{2m+1}\ x\right) - \mu\left(\sqrt[p]{2m+1}\ x\right)\right)\right. \tag{2.6}$$

$$\left. + \left(\xi\left(\sqrt[p]{m}\ x\right) - \mu\left(\sqrt[p]{m}\ x\right)\right) + \left(\xi\left(\sqrt[p]{-m}\ x\right) - \mu\left(\sqrt[p]{-m}\ x\right)\right)\ ,\ g(z)\right\|_* \tag{2.7}$$

$$\leq \max\left\{2\left\|\xi\left(\sqrt[p]{2m+1}\ x\right) - \mu\left(\sqrt[p]{2m+1}\ x\right)\ ,\ g(z)\right\|_*\ ,\ \left\|\xi\left(\sqrt[p]{m+1}\ x\right) - \mu\left(\sqrt[p]{m+1}\ x\right)\ ,\ g(z)\right\|_*\ ,\right. \tag{2.8}$$

$$\left.\left\|\xi\left(\sqrt[p]{m}\ x\right) - \mu\left(\sqrt[p]{m}\ x\right)\ ,\ g(z)\right\|_*\ ,\ \left\|\xi\left(\sqrt[p]{-m}\ x\right) - \mu\left(\sqrt[p]{-m}\ x\right)\ ,\ g(z)\right\|_*\right\} \tag{2.9}$$

$$\leq \max\left\{\left\|\xi\left(\sqrt[p]{2m+1}\ x\right) - \mu\left(\sqrt[p]{2m+1}\ x\right)\ ,\ g(z)\right\|_*\ ,\ \left\|\xi\left(\sqrt[p]{m+1}\ x\right) - \mu\left(\sqrt[p]{m+1}\ x\right)\ ,\ g(z)\right\|_*\ ,\right. \tag{2.10}$$

$$\left.\left\|\xi\left(\sqrt[p]{m}\ x\right) - \mu\left(\sqrt[p]{m}\ x\right)\ ,\ g(z)\right\|_*\ ,\ \left\|\xi\left(\sqrt[p]{-m}\ x\right) - \mu\left(\sqrt[p]{-m}\ x\right)\ ,\ g(z)\right\|_*\right\} \tag{2.11}$$

$$= \max_{1\leq i\leq 4}\left\{L_i(x)\left\|\xi(f_i(x)) - \mu(f_i(x))\ ,\ g(z)\right\|_*\right\}, \tag{2.12}$$

which means that (**H2**) is valid. Now we will show, by induction on $n \in \mathbb{N}_0$, that

$$\Lambda^n\varepsilon_m(x,z) \leq \lambda_1(m+1)\lambda_2(m)\alpha_m^n h_1(x^p,z)h_2(x^p,z). \tag{2.13}$$

for all $x, z \in \mathbb{R}_0$ and all $m \in \mathcal{U}$ where

$$\alpha_m = \max\left\{\lambda_1(m+1)\lambda_2(m+1)\ ,\ \lambda_1(2m+1)\lambda_2(2m+1)\ ,\ \lambda_1(m)\lambda_2(m)\ ,\ \lambda_1(-m)\lambda_2(-m)\ \right\}.$$

For $n = 0$, the inequality (2.13) is exactly (2.4). Next we will assume that (2.13) holds for $n = k$, where $k \in \mathbb{N}$. Then

$$\Lambda_m^{k+1}\varepsilon_m(x,z) = \Lambda_m\left(\Lambda_m^k\varepsilon_m(x,z)\right)$$

$$= \max\left\{\Lambda_m^k\varepsilon_m\left(\sqrt[p]{m+1}\ x, z\right)\ ,\ \Lambda_m^k\varepsilon_m\left(\sqrt[p]{2m+1}\ x, z\right)\ ,\ \Lambda_m^k\varepsilon_m\left(\sqrt[p]{m}\ x, z\right)\ ,\ \Lambda_m^k\varepsilon_m\left(\sqrt[p]{-m}\ x, z\right)\ \right\}$$

$$\leq \lambda_1(m+1)\lambda_2(m)\alpha_m^k \max\Big\{h_1\big((m+1)x^p,z\big)h_2\big((m+1)x^p,z\big)\ ,\ h_1\big((2m+1)x^p,z\big)h_2\big((2m+1)x^p,z\big)\ ,$$

$$h_1\big((m)x^p,z\big)h_2\big((m)x^p,z\big)\ ,\ h_1\big((-m)x^p,z\big)h_2\big((-m)x^p,z\big)\ \Big\}$$

$$\leq \lambda_1(m+1)\lambda_2(m)\alpha_m^k \max\Big\{\lambda_1(m+1)\lambda_2(m+1)\ ,\ \lambda_1(2m+1)\lambda_2(2m+1)\ ,\ \lambda_1(m)\lambda_2(m)\ ,$$

$$\lambda_1(-m)\lambda_2(-m)\ \Big\}h_1(x^p,z)h_2(x^p,z)$$

$$= \lambda_1(m+1)\lambda_2(m)\alpha_m^{k+1}h_1(x^p,z)h_2(x^p,z),$$

for all $x, z \in \mathbb{R}_0$ and all $m \in \mathcal{U}$. It shows that (2.13) holds for $n = k+1$. We conclude that the inequality (2.13) holds for all $n \in \mathbb{N}_0$.
Since $\alpha_m < 1$ for all $m \in \mathcal{U}$, we get

$$\lim_{n\to\infty}\Lambda^n\varepsilon_m(x,z) = 0,$$

for all $x, z \in \mathbb{R}_0$. According to Theorem 1.16, there exists, for each $m \in \mathcal{U}$, a fixed point $\mathcal{F}_m : \mathbb{R}_0 \to X$ of the operator $\mathcal{T}_m$ such that

$$\begin{aligned}\left\|f(x) - \mathcal{F}_m(x)\ ,\ g(z)\right\|_* &\leq \sup_{n\in\mathbb{N}}\left\{\Lambda_m^n\varepsilon_m(x,z)\right\} \\ &\leq \sup_{n\in\mathbb{N}}\left\{\lambda_1(m+1)\lambda_2(m)\alpha_m^n h_1(x^p,z)h_2(x^p,z)\right\}, \quad x, z \in \mathbb{R}_0.\end{aligned} \tag{2.14}$$

Moreover,

$$\mathcal{F}_m(x) = \lim_{n\to\infty} \left(\mathcal{T}_m^n f\right)(x), \qquad x \in \mathbb{R}_0.$$

Next, we should prove the following inequality

$$\begin{aligned}
&\Big\| \mathcal{T}_m^n f\left(\sqrt[p]{x^p+y^p}\right) + \mathcal{T}_m^n f\left(\sqrt[p]{x^p-y^p}\right) - 2\mathcal{T}_m^n f(x) - \mathcal{T}_m^n f(y) \\
&- \mathcal{T}_m^n f(-y)\ ,\ g(z)\Big\|_* \le \alpha_m^n h_1(x^p,z) h_2(y^p,z), \qquad (2.15)
\end{aligned}$$

for all $m \in \mathcal{U}$, all $x,y,z \in \mathbb{R}_0$ such that $x \neq y\ ,\ x \neq -y$ and all $n \in \mathbb{N}$.
We proceed by induction that the case $n = 0$ gives us (2.2). Assume that (2.15) holds for $n = k$ where $k \in \mathbb{N}$. Then for each $m \in \mathcal{U}$ and every $x,y,z \in \mathbb{R}_0$ such that $x \neq y$ and $x \neq -y$, we have

$$\begin{aligned}
&\Big\| \mathcal{T}_m^{k+1} f\left(\sqrt[p]{x^p+y^p}\right) + \mathcal{T}_m^{k+1} f\left(\sqrt[p]{x^p-y^p}\right) - 2\mathcal{T}_m^{k+1} f(x) - \mathcal{T}_m^{k+1} f(y) - \mathcal{T}_m^{k+1} f(-y), g(z)\Big\|_* \\
&= \Big\| 2\mathcal{T}_m^k f\left(\sqrt[p]{m+1}\sqrt[p]{x^p+y^p}\right) - \mathcal{T}_m^k f\left(\sqrt[p]{2m+1}\sqrt[p]{x^p+y^p}\right) + \mathcal{T}_m^k f\left(\sqrt[p]{m}\sqrt[p]{x^p+y^p}\right) \\
&+ \mathcal{T}_m^k f\left(\sqrt[p]{-m}\sqrt[p]{x^p+y^p}\right) + 2\mathcal{T}_m^k f\left(\sqrt[p]{m+1}\sqrt[p]{x^p-y^p}\right) - \mathcal{T}_m^k f\left(\sqrt[p]{2m+1}\sqrt[p]{x^p-y^p}\right) \\
&+ \mathcal{T}_m^k f\left(\sqrt[p]{m}\sqrt[p]{x^p-y^p}\right) + \mathcal{T}_m^k f\left(\sqrt[p]{-m}\sqrt[p]{x^p-y^p}\right) - 4\mathcal{T}_m^k f\left(\sqrt[p]{m+1}\ x\right) \\
&+ 2\mathcal{T}_m^k f\left(\sqrt[p]{2m+1}\ x\right) - 2\mathcal{T}_m^k f\left(\sqrt[p]{m}\ x\right) - 2\mathcal{T}_m^k f\left(\sqrt[p]{-m}\ x\right) - 2\mathcal{T}_m^k f\left(\sqrt[p]{m+1}\ y\right) \\
&+ \mathcal{T}_m^k f\left(\sqrt[p]{2m+1}\ y\right) - \mathcal{T}_m^k f\left(\sqrt[p]{m}\ y\right) - \mathcal{T}_m^k f\left(\sqrt[p]{-m}\ y\right) - 2\mathcal{T}_m^k f\left(\sqrt[p]{m+1}\ (-y)\right) \\
&+ \mathcal{T}_m^k f\left(\sqrt[p]{2m+1}\ (-y)\right) - \mathcal{T}_m^k f\left(\sqrt[p]{m}\ (-y)\right) - \mathcal{T}_m^k f\left(\sqrt[p]{-m}\ (-y)\right), g(z)\Big\|_* \\
&\le \max\Big\{ \Big\| \mathcal{T}_m^k f\left(\sqrt[p]{m+1}\sqrt[p]{x^p+y^p}\right) + \mathcal{T}_m^k f\left(\sqrt[p]{m+1}\sqrt[p]{x^p-y^p}\right) - 2\mathcal{T}_m^k f\left(\sqrt[p]{m+1}\ x\right) \\
&- \mathcal{T}_m^k f\left(\sqrt[p]{m+1}\ y\right) - \mathcal{T}_m^k f\left(\sqrt[p]{m+1}\ (-y)\right), g(z)\Big\|_*\ , \\
&\Big\| \mathcal{T}_m^k f\left(\sqrt[p]{2m+1}\sqrt[p]{x^p+y^p}\right) + \mathcal{T}_m^k f\left(\sqrt[p]{2m+1}\sqrt[p]{x^p-y^p}\right) - 2\mathcal{T}_m^k f\left(\sqrt[p]{2m+1}\ x\right) \\
&- \mathcal{T}_m^k f\left(\sqrt[p]{2m+1}\ y\right) - \mathcal{T}_m^k f\left(\sqrt[p]{2m+1}\ (-y)\right), g(z)\Big\|_*\ , \\
&\Big\| \mathcal{T}_m^k f\left(\sqrt[p]{m}\sqrt[p]{x^p+y^p}\right) + \mathcal{T}_m^k f\left(\sqrt[p]{m}\sqrt[p]{x^p-y^p}\right) - 2\mathcal{T}_m^k f\left(\sqrt[p]{m}\ x\right) \\
&- \mathcal{T}_m^k f\left(\sqrt[p]{m}\ y\right) - \mathcal{T}_m^k f\left(\sqrt[p]{m}\ (-y)\right), g(z)\Big\|_*\ , \\
&\Big\| \mathcal{T}_m^k f\left(\sqrt[p]{-m}\sqrt[p]{x^p+y^p}\right) + \mathcal{T}_m^k f\left(\sqrt[p]{-m}\sqrt[p]{x^p-y^p}\right) - 2\mathcal{T}_m^k f\left(\sqrt[p]{-m}\ x\right) \\
&- \mathcal{T}_m^k f\left(\sqrt[p]{-m}\ y\right) - \mathcal{T}_m^k f\left(\sqrt[p]{-m}\ (-y)\right), g(z)\Big\|_* \Big\} \\
&\le \max\Big\{ \alpha_m^k h_1\big((m+1)x^p,z\big) h_2\big((m+1)y^p,z\big)\ ,\ \alpha_m^k h_1\big((2m+1)x^p,z\big) h_2\big((2m+1)y^p,z\big) \\
&,\ \alpha_m^k h_1\big((m)x^p,z\big) h_2\big((m)y^p,z\big)\ ,\ \alpha_m^k h_1\big((-m)x^p,z\big) h_2\big((-m)y^p,z\big) \Big\} \\
&\le \alpha_m^k h_1(x^p,z) h_2(y^p,z) \max\Big\{ \lambda_1(m+1)\lambda_2(m+1)\ ,\ \lambda_1(2m+1)\lambda_2(2m+1)\ , \\
&\lambda_1(m)\lambda_2(m)\ ,\ \lambda_1(-m)\lambda_2(-m) \Big\} \\
&= \alpha_m^{k+1} h_1(x^p,z) h_2(y^p,z).
\end{aligned}$$

Thus, we have shown that (2.15) holds for $n \in \mathbb{N}_0$, and all $m \in \mathcal{U}$. Letting $n \to \infty$ in (2.15), we obtain

$$\mathcal{F}_m\left(\sqrt[p]{x^p+y^p}\right) + \mathcal{F}_m\left(\sqrt[p]{x^p-y^p}\right) = 2\mathcal{F}_m(x) + \mathcal{F}_m(y) + \mathcal{F}_m(-y),$$

for all $x, y \in \mathbb{R}_0$ such that $x \neq y$, $x \neq -y$ and $m \in \mathcal{U}$. This implies that $\mathcal{F}_m : \mathbb{R} \to X$ is a solution of the equation (1.5).
Therefore, we construct a sequence $\{\mathcal{F}_m\}_{m \in \mathcal{U}}$ of the solutions of equation (1.5) on $\mathbb{R}_0$ such that

$$\begin{aligned}\left\|\mathcal{F}_m(x) - f(x)\ ,\ g(z)\right\|_* &\leq \sup_{n\in\mathbb{N}} \Lambda_m^n \varepsilon_m(x,z)\\ &\leq \sup_{n\in\mathbb{N}} \left\{\lambda_1(m+1)\lambda_2(m)\alpha_m^n h_1(x^p,z)h_2(x^p,z)\right\},\end{aligned}$$

for all $x, z \in \mathbb{R}_0$ and all $m \in \mathcal{U}$. Letting $n \to \infty$ in the previous inequality and using (2.1), we deduce that f is a solution of the equation (1.5) on $\mathbb{R}_0$ which means that the equation (1.5) is hyperstable on $\mathbb{R}_0$. □

In a similar way, we can prove the following theorem.

Theorem 2.2. *Let p be an odd natural number, $(X\ ,\ \|\cdot,\cdot\|_*)$ be an ultrametric 2-Banach space, $g : \mathbb{R}_0 \to X$ be a mapping such that the set $g(\mathbb{R}_0) \subseteq X$ containing two linearly independent vectors and let $h : \mathbb{R}_0 \times \mathbb{R}_0 \to \mathbb{R}_+$ be a mapping such that*

$$\mathcal{U} := \left\{n \in \mathbb{N} : \alpha_n = \max\{\lambda(n+1)\ ,\ \lambda(2n+1)\ ,\ \lambda(n)\ ,\ \lambda(-n)\ \} < 1\right\} \neq \phi,$$

where

$$\lambda(n) = \inf\{t \in \mathbb{R}_+ : h(nx,z) \leq t\, h(x,z),\ \ x,z \in \mathbb{R}_0\},$$

for all $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \big(\lambda(n+1) + \lambda(n)\big) = 0.$$

Assume that $f : \mathbb{R} \to X$ satisfies the inequality

$$\left\|f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) - 2f(x) - f(y) - f(-y)\ ,\ g(z)\right\|_* \leq h(x^p,z) + h(y^p,z), \quad (2.16)$$

for all $x, y, z \in \mathbb{R}_0$ such that $x \neq y$ and $x \neq -y$. Then f is a solution of the equation (1.5) on $\mathbb{R}_0$.

Proof. We will suffice with the basic idea of the proof. Replacing x by $\sqrt[p]{m+1}\ x$ and y by $\sqrt[p]{m}\ x$ in the inequality (2.16) where $x \in \mathbb{R}_0$, $m \in \mathcal{U}$, we get

$$\begin{aligned}&\left\|2f\left(\sqrt[p]{m+1}\ x\right) - f\left(\sqrt[p]{2m+1}\ x\right) + f\left(\sqrt[p]{m}\ x\right) + f\left(\sqrt[p]{-m}\ x\right) - f(x)\ ,\ g(z)\right\|_*\\ &\leq h((m+1)x^p, z) + h((m)x^p, z)\\ &\leq \Big(\lambda(m+1) + \lambda(m)\Big) h(x^p, z),\end{aligned}$$

for all $m \in \mathcal{U}$ and all $x, z \in \mathbb{R}_0$. We define operators $\mathcal{T}_m : X^{\mathbb{R}_0} \to X^{\mathbb{R}_0}$ and $\Lambda_m : \mathbb{R}_+^{\mathbb{R}_0\times\mathbb{R}_0} \to \mathbb{R}_+^{\mathbb{R}_0\times\mathbb{R}_0}$ by

$$\mathcal{T}_m\xi(x) := 2\xi\left(\sqrt[p]{m+1}\ x\right) - \xi\left(\sqrt[p]{2m+1}\ x\right) + \xi\left(\sqrt[p]{m}\ x\right) + \xi\left(\sqrt[p]{-m}\ x\right),$$

for all $\xi \in X^{\mathbb{R}_0}$ and all $x \in \mathbb{R}_0$ and

$$\Lambda_m\delta(x,z) := \max\left\{\delta\left(\sqrt[p]{m+1}\ x, z\right), \delta\left(\sqrt[p]{2m+1}\ x, z\right), \delta\left(\sqrt[p]{m}\ x, z\right), \delta\left(\sqrt[p]{-m}\ x, z\right)\ \right\}.$$

Moreover, we write

$$\varepsilon_m(x,z) = h\big((m+1)x^p, z\big) + h\big((m)x^p, z\big) \leq \big(\lambda(m+1) + \lambda(m)\big)h(x^p,z), \quad x,z \in \mathbb{R}_0.$$

As in Theorem 2.1, we observe that if (2.16) takes the following form

$$\left\|f(x) - \mathcal{T}_m, g(z)\right\|_* \leq \varepsilon_m(x,z), x, z \in \mathbb{R}_0, m \in \mathcal{U},$$

then we complete the proof by similar steps of the proof of Theorem 2.1. □

3 Applications

In this section, we get, as particular cases of of our main results, the hyperstability results in the sens of Hyers-Ulam-Rassiass. Also, we get the same results for the inhomogeneous general p-radical functional equation

$$f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)=2f(x)+f(y)+f(-y)+G(x,y). \tag{3.1}$$

Corollary 3.1. *Let p be an odd natural number, $\left(X\ ,\ \|\cdot,\cdot\|_*\right)$ be an ultrametric 2-Banach space, $g:\mathbb{R}_0\to X$ be a mapping such that the set $g(\mathbb{R}_0)\subseteq X$ containing two linearly independent vectors and let $c,r,s,t\in\mathbb{R}$ such that $s+t<0$, $r\geq 0$ and $c\geq 0$. Assume that a mapping $f:\mathbb{R}\to X$ satisfies the inequality*

$$\left\|f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-2f(x)-f(y)-f(-y)\ ,\ g(z)\right\|_*\leq c|Q_1(x^p)|^s|Q_2(y^p)|^t|z|^r, \tag{3.2}$$

for all $x,y,z\in\mathbb{R}_0$ where $Q_1,Q_2:\mathbb{R}_0\to\mathbb{R}_+$ are two quadratic functions. Then f is a solution of the equation (1.5) on $\mathbb{R}_0$.

Proof. The proof follows from Theorem 2.1 by taking $h_1,h_2:\mathbb{R}_0\times\mathbb{R}_0\to\mathbb{R}_+$ as follows:

$$h_1(x^p,z)=c_1|Q_1(x^p)|^s|z|^{r_1}$$

and

$$h_2(x^p,z)=c_2|Q_2(x^p)|^t|z|^{r_2}$$

for all $x,y,z\in\mathbb{R}_0$ where $c_1,c_2 r_1,r_2\in\mathbb{R}_+$ such that $c_1c_2=c\geq 0$ and $r_1+r_2=r$. For each $m\in\mathbb{N}$, we define $\lambda_1(m)$ as in Theorem 2.1 by

$$\begin{aligned}\lambda_1(m)&=\inf\left\{t\in\mathbb{R}_+: h_1\left(mx^p,z\right)\leq th_1(x^p,z)\right\}\\ &=\inf\left\{t\in\mathbb{R}_+: c_1\Big|Q_1\left(mx^p\right)\Big|^s|z|^{r_1}\leq tc_1\Big|Q_1(x^p)\Big|^s|z|^{r_1}\right\}\\ &=\inf\left\{t\in\mathbb{R}_+: m^{2s}\Big|Q_1(x^p)\Big|^s|z|^{r_1}\leq t\ \Big|Q_1(x^p)\Big|^s|z|^{r_1}\right\}\\ &=m^{2s},\end{aligned}$$

for all $x,z\in\mathbb{R}_0$. Also, for each $m\in\mathbb{N}$, we have $\lambda_2(m)=m^{2t}$. It is clear that there exists $m_0\in\mathbb{N}$ such that, for each $m\geq m_0$, we get

$$\begin{aligned}\alpha_m&=\max\left\{\lambda_1(m+1)\lambda_2(m+1)\ ,\ \lambda_1(2m+1)\lambda_2(2m+1)\ ,\ \lambda_1(m)\lambda_2(m)\ ,\ \lambda_1(m)\lambda_2(m)\ \right\},\\ &=\max\left\{(m+1)^{2(s+t)}\ ,\ \ (2m+1)^{2(s+t)}\ ,\ m^{2(s+t)}\ ,\ (-m)^{2(s+t)}\ \right\}<1\end{aligned}$$

According to Theorem 2.1, there exists a unique mapping $\mathcal{F}_m:\mathbb{R}_0\to X$ satisfying the equation (1.5) such that

$$\begin{aligned}\|\mathcal{F}_m-f(x)\ ,\ g(z)\|_*&\leq c\sup_{n\in\mathbb{N}}\left\{\lambda_1(m+1)\lambda_2(m)\alpha_m^n|Q_1(x^p)|^s|Q_2(x^p)|^t|z|^r\right\}\\ &=c(m+1)^{2s}\ m^{2t}|Q_1(x^p)|^s|Q_2(x^p)|^t|z|^r\sup_{n\in\mathbb{N}}\left\{\alpha_m^n\right\},\end{aligned}$$

for all $x,z\in\mathbb{R}_0$. On the other hand, since $s+t<0$, one of s,t must be negative. Assume that $s<0$. Then

$$\lim_{m\to\infty}\lambda_1(m+1)\lambda_2(m)=\lim_{m\to\infty}m^{2(s+t)}=0 \tag{3.3}$$

We get the desired result. □

Corollary 3.2. *Let p be an odd natural number, $\left(X\ ,\ \|\cdot,\cdot\|_*\right)$ be an ultrametric 2-Banach space, $g:\mathbb{R}_0\to X$ be a mapping such that the set $g(\mathbb{R}_0)\subseteq X$ containing two linearly independent* *vectors and let $c, r, s \in \mathbb{R}$ such that $c \geq 0$, $r \geq 0$ and $s < 0$. Assume that a function $f : \mathbb{R} \to X$ satisfies the inequality*

$$\left\|f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-2f(x)-f(y)-f(-y), g(z)\right\|_* \leq c\Big(|Q(x^p)|^s+|Q(y^p)|^s\Big)|z|^r \tag{3.4}$$

for all $x, y, z \in \mathbb{R}_0$ where $Q : \mathbb{R}_0 \to \mathbb{R}_+$ is a quadratic function. Then f is a solution of the equation (1.5) on $\mathbb{R}_0$.

Proof. The proof is similar to the proof of Corollary 3.1 with taking $h : \mathbb{R}_0 \times \mathbb{R}_0 \to \mathbb{R}_+$ defined by $h(x^p, z) = c\left|Q(x^p)\right|^s |z|^r$ for all $x, z \in \mathbb{R}_0$ where $c \geq 0$, $r \geq 0$ and $s < 0$. □

In the following corollaries, we get the hyperstability results for the inhomogeneous general p-radical functional equation related to quadratic mappings.

Corollary 3.3. *Let p be an odd natural number, $c, r, s, t \in \mathbb{R}$ such that $c, r \geq 0$ and $s + t < 0$, $\big(X\ ,\ \|\cdot,\cdot\|_*\big)$ be an ultrametric 2-Banach space, $g : \mathbb{R}_0 \to X$ be a mapping such that the set $g(\mathbb{R}_0) \subseteq X$ containing two linearly independent vectors, $G : \mathbb{R} \times \mathbb{R} \to X$ be a mapping such that $G(0,0) = 0$ and let $f : \mathbb{R} \to X$ be a mapping such that $f(0) = 0$. Assume that f and G satisfy the inequality*

$$\begin{aligned}&\left\|f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-2f(x)-6f(y)-f(-y)-G(x,y)\ ,\ g(z)\right\|_* \\ &\leq c\,|Q_1(x^p)|^s|Q_2(y^p)|^t|z|^r,\end{aligned} \tag{3.5}$$

for all $x, y, z \in \mathbb{R}_0$, where $Q_1, Q_2 : \mathbb{R}_0 \to \mathbb{R}_+$ are two quadratic functions. If the functional equation

$$f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-2f(x)-f(y)-f(-y)-G(x,y)=0 \tag{3.6}$$

has a solution $f_0 : \mathbb{R} \to X$ on $\mathbb{R}_0$, then f is a solution of the equation (3.6) on $\mathbb{R}_0$.

Proof. Let $\psi : \mathbb{R} \to X$ be a mapping defined by $\psi(x) := f(x) - f_0(x)$ for all $x \in \mathbb{R}$. Then we get that

$$\begin{aligned}&\left\|\psi\left(\sqrt[p]{x^p+y^p}\right)+\psi\left(\sqrt[p]{x^p-y^p}\right)-2\psi(x)-\psi(y)-\psi(-y), g(z)\right\|_* \\ &= \Big\|f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-2f(x)-f(y)-f(-y) \\ &-G(x,y)-f_0\left(\sqrt[p]{x^p+y^p}\right)-f_0\left(\sqrt[p]{x^p-y^p}\right) \\ &+2f_0(x)+f_0(y)+f_0(-y)+G(x,y)\ ,\ g(z)\Big\|_* \\ &\leq \max\Big\{\left\|f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-2f(x)-f(y)-f(-y)-G(x,y)\ ,\ g(z)\right\|_* \\ &,\ \left\|f_0\left(\sqrt[p]{x^p+y^p}\right)+f_0\left(\sqrt[p]{x^p-y^p}\right)-2f_0(x)-f_0(y)-f_0(-y)-G(x,y)\ ,\ g(z)\right\|_*\Big\} \\ &\leq \left\|f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-2f(x)-f(y)-f(-y)-G(x,y)\ ,\ g(z)\right\|_* \\ &\leq c\left|Q_1(x^p)\right|^s\left|Q_2(y^p)\right|^t|z|^r,\end{aligned}$$

for all $x, y, z \in \mathbb{R}_0$. By using Corollary 3.1, we deduce that ψ is a solution of equation (1.5). Moreover, for all $x, y \in \mathbb{R}_0$, we have

$$\begin{aligned}&f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-2f(x)-f(y)-f(-y)-G(x,y) \\ &=\psi\left(\sqrt[p]{x^p+y^p}\right)+\psi\left(\sqrt[p]{x^p-y^p}\right)-2\psi(x)-\psi(y)-\psi(-y) \\ &+f_0\left(\sqrt[p]{x^p+y^p}\right)+f_0\left(\sqrt[p]{x^p-y^p}\right)-2f_0(x)-f_0(y)-f_0(-y)-G(x,y)=0,\end{aligned}$$

which means that f is a solution of (3.6) on $\mathbb{R}_0$. □

With an analogous proof of Corollary 3.3, we can prove the following corollary.

Corollary 3.4. *Let p be an odd natural number, $c, r, s \in \mathbb{R}$ such that $c, r \geq 0$ and $s < 0$, $\big(X\ ,\ \|\cdot,\cdot\|_*\big)$ be an ultrametric 2-Banach space, $g : \mathbb{R}_0 \to X$ be a mapping such that the set $g(\mathbb{R}_0) \subseteq Y$ containing two linearly independent vectors and let $G : \mathbb{R} \times \mathbb{R} \to X$ be a mapping such that $G(0,0) = 0$ and $f : \mathbb{R} \to X$ be a mapping such that $f(0) = 0$. Assume that f and G satisfy the inequality*

$$\begin{aligned} &\left\| f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) - 2f(x) - f(y) - f(-y) - G(x,y)\ ,\ g(z)\right\|_* \\ &\leq c\ \left(|Q(x^p)|^s + |Q(y^p)|^s\right)|z|^r, \end{aligned} \tag{3.7}$$

for all $x, y, z \in \mathbb{R}_0$, where $Q : \mathbb{R}_0 \to \mathbb{R}_+$ is a quadratic function. If the functional equation

$$f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) - 2f(x) - f(y) - f(-y) - G(x,y) = 0, \tag{3.8}$$

has a solution $f_0 : \mathbb{R} \to X$ on $\mathbb{R}_0$, then f is a solution of the equation (3.8) on $\mathbb{R}_0$.

4 Conclusion

This paper indeed presents a relationship between three various disciplines: the theory of Banach spaces, the theory of stability of functional equations, and the fixed point theory. We established some hyperstability results concerning a general radical functional equation in ultrametric 2-Banach spaces by using the fixed point approach with some particular cases and applications.

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