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Generalization of (m, n)-closed ideals

Abdelhaq El Khalfi

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Abstract Let R be a commutative ring with nonzero identity. In this paper, we introduce and investigate a generalization of (m, n)-closed ideals. Let $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be a function where $\mathcal{I}(R)$ is the set of ideals of R. A proper ideal I of R is said to be a ϕ-(m, n)-closed ideal if $a^m \in I \setminus \phi(I)$ for $a \in R$ implies that $a^n \in I$. Moreover, we give some basic properties of this class of ideals and we study the ϕ-(m, n)-closed ideals of the localization of rings, the direct product of rings, the trivial ring extensions and amalgamation of rings.

1 Introduction

Throughout this paper, all rings are assumed to be commutative with nonzero identity and all modules are nonzero unital. If R is a ring, then $\sqrt{I}$ denotes the radical of an ideal I of R, in the sense of [17, page 17]. We denote the set of all ideals (resp. proper ideals) of a ring R by $\mathcal{I}(R)$ (resp. $\mathcal{I}^*(R)$).

Anderson and Smith [3], defined a weakly prime ideal as a proper ideal P of R with the property that for $a, b \in R$, $0 \neq ab \in P$ implies $a \in P$ or $b \in P$. Then the authors of [6] defined the notion of almost prime ideal, i.e., an ideal $P \in \mathcal{I}^*(R)$ with the property that if $a, b \in R$, $ab \in P \setminus P^2$, then either $a \in P$ or $b \in P$. Thus a weakly prime ideal is almost prime and any proper idempotent ideal is also almost prime. Moreover, an ideal P of R is almost prime if and only if P/P^2 is a weakly prime ideal of R/P^2. Anderson and Bataineh in [2], extended these concepts to ϕ-prime ideals. Let $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be a function. A proper ideal P of R is called ϕ-prime if for $x, y \in R$, $xy \in P \setminus \phi(P)$ implies $x \in P$ or $y \in P$. In fact, P is a ϕ-prime ideal of R if and only if $P/\phi(P)$ is a weakly prime ideal of $R/\phi(P)$. In 2017, J. Bagheri Harehdashti and H. Fazaeli Moghimi defined the ϕ-radical of an ideal I as the intersection of all ϕ-prime ideals of R containing I and investigated when the set of all ϕ-prime ideals of R has a Zariski topology analogous to that of the prime spectrum. Since $P \setminus \phi(P) = P \setminus (P \cap \phi(P))$, there is no loss of generality in assuming that $\phi(P) \subseteq P$. In [1], Anderson and Badawi introduced and studied the notion of (m, n)-closed ideal . Let m and n be positive integers. A proper ideal of R is said to be a (m, n)-closed ideal if $a^m \in I$ for $a \in R$ implies that $a^n \in I$. Also, recall from [5] that a proper ideal of R is called a weakly (m, n)-closed ideal if $0 \neq a^m \in I$ for $a \in R$ implies that $a^n \in I$.

Let A be a ring and E an A-module. Then $A \ltimes E$, the *trivial* (*ring*) *extension of* A *by* E, is the ring whose additive structure is that of the external direct sum $A \oplus E$ and whose multiplication is defined by $(a, e)(b, f) := (ab, af + be)$ for all $a, b \in A$ and all $e, f \in E$. (This construction is also known by other terminology and other notation, such as the *idealization* $A(+)E$.) The basic properties of trivial ring extensions are summarized in the books [14], [13]. Trivial ring extensions have been studied or generalized extensively, often because of their usefulness in constructing new classes of examples of rings satisfying various properties (cf. [4, 11, 12, 15, 16]).

Let A and B be two rings, let J be an ideal of B and let $f : A \longrightarrow B$ be a ring homomorphism. In this setting, we consider the following subring of $A \times B$, $A \bowtie^f J = \{(a, f(a) + j) | a \in A, j \in J\}$, called the amalgamation of A and B along J with respect to f. Moreover, other classical constructions (such as the $A + XB[X]$, $A + XB[[X]]$, and the $D + M$ constructions) can be studied as particular cases of the amalgamation (see [7, Examples 2.5 and 2.6]). A particular case of this construction is the amalgamated duplication of a ring along an ideal I (introduced and studied by D'Anna and Fontana in [7, 8, 9]). Let A be a ring, and let I be an ideal of A. $A \bowtie I := \{(a, a+i) : a \in A, i \in I\}$ is called the amalgamated duplication of A along the ideal I. See for instance [7, 8, 9, 10].

The purpose of this paper is to introduce and investigate a generalization of (m,n)-closed ideals. Let $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be a function where $\mathcal{I}(R)$ is the set of ideals of a ring R. A proper ideal I of R is said to be a ϕ-(m,n)-closed ideal if $a^m \in I \setminus \phi(I)$ for $a \in R$ implies that $a^n \in I$. Moreover, we give some basic properties of this class of ideals and we study the ϕ-(m,n)-closed ideals of the localization of rings, the direct product of rings, the trivial ring extensions and amalgamation of rings.

2 Main Results

We start this section by the following definition.

Definition 2.1. Let R be a ring, m, n nonzero positive integers and $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be a function. A proper ideal I of R is said to be a ϕ-(m,n)-closed ideal if $a^m \in I \setminus \phi(I)$ for $a \in R$ implies that $a^n \in I$.

Remark 2.2. Let R be a ring, m, n nonzero positive integers and $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be a function. Let I be a proper ideal of R. It is easy to see from the definition that if I is a ϕ-(m,n)-closed ideal of R, then I is ϕ-(m,n')-closed for every positive integer $n' \geq n$.

We next give some particular examples of ϕ-(m,n)-closed ideals

Example 2.3. Let R be a ring, I a proper ideal of R, $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ a function and let m, n be nonzero positive integers.

(1) If $\phi(I) = \emptyset$, then I is a ϕ-(m,n)-closed ideal of R if and only if I is a (m,n)-closed ideal.

(2) If $\phi(I) = 0$, then I is a ϕ-(m,n)-closed ideal of R if and only if I is a weakly (m,n)-closed ideal.

(i) Assume that R is a local ring with maximal ideal M such that $I \cap M^m \subseteq \phi(I)$. Then I is a ϕ-(m,n)-closed ideal of R. Moreover, if $I \neq M$ and $M^m \subseteq \phi(I)$, then I is not a $(m,1)$-closed ideal of R because we have $a^m \in \phi(I) \subseteq I$ for some $a \in M \setminus I$.

Definition 2.4. Let R be a ring, m and n positive integers, $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ a function and I a ϕ-(m,n)-closed ideal of R. Then $a \in R$ is a ϕ-(m,n)-unbreakable element of I if $a^m \in \phi(I)$ and $a^n \notin I$.

Remark 2.5. It is clear that a ϕ-(m,n)-closed ideal I has a ϕ-(m,n)-unbreakable element if and only if I is not (m,n)-closed.

Lemma 2.6. *Let R be a ring, m and n positive integers, $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ a function, and I a ϕ-(m,n)-closed ideal of R. If a is a ϕ-(m,n)-unbreakable element of I, then $(a+i)^m \in \phi(I)$ for every $i \in I$.*

Proof. Let $i \in I$ and a is a ϕ-(m,n)-unbreakable element of I. As $a^m \in \phi(I) \subseteq I$, we conclude that

$$(a+i)^m = a^m + \sum_{k=1}^{m} \binom{m}{k} a^{m-k} i^k \in I,$$

and similarly, $(a+i)^n \notin I$ since $a^n \notin I$. Thus $(a+i)^m \in \phi(I)$ because I is ϕ-(m,n)-closed ideal of R.

□

Theorem 2.7. *Let R be a ring, m and n positive integers, $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ a function, and I a ϕ-(m,n)-closed ideal of R. If I is not (m,n)-closed, then $I \subseteq \sqrt{\phi(I)}$.*

Proof. As I is a ϕ-(m,n)-closed ideal of R that is not (m,n)-closed, we get that I has a ϕ-(m,n)-unbreakable element a. Thus $a^m \in \phi(I)$, and $(a+i)^m \in \phi(I)$ for every $i \in I$ by Lemma 2.6. Which implies that $a \in \sqrt{\phi(I)}$ and $a+i \in \sqrt{\phi(I)}$. Hence $i = (a+i) - a \in \sqrt{\phi(I)}$ and thus $I \subseteq \sqrt{\phi(I)}$. □

Let S be a multiplicatively closed subset of a ring R. Given a function $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\varnothing\}$, as in [2] we define $\phi_S : \mathcal{I}(S^{-1}R) \to \mathcal{I}(S^{-1}R) \cup \{\varnothing\}$ by $\phi_S(J) = S^{-1}\phi((J \cap R))$ and $\phi_S(J) = \varnothing$ if $\phi(J \cap R) = \varnothing$. Also, let J be an ideal of R, define $\phi_J : \mathcal{I}(R/J) \to \mathcal{I}(R/J) \cup \{\varnothing\}$ by $\phi_J(I/J) = (\phi(I) + J)/J$ for $I \supseteq J$ and $\phi(I/J) = \varnothing$ if $\phi(I) = \varnothing$. Then we have the following result.

Proposition 2.8. *Let R be a ring, m and n positive integers and $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\varnothing\}$ be a function. Let I be a ϕ-(m, n)-closed ideal of R.*

(1) *If J is an ideal R with $J \subseteq I$, then I/J is a ϕ_J-(m, n)-closed ideal of R/J.*

(2) *Suppose that S is a multiplicatively closed subset of R with $I \cap S = \varnothing$. and $S^{-1}\phi(I) \subseteq \phi_S(S^{-1}I)$. Then $S^{-1}I$ is a ϕ_S-(m, n)-closed ideal of $S^{-1}R$.*

Proof. (1) Let $a \in R$ such that $\bar{a}^m \in I/J \setminus \phi_J(I/J) = I/J \setminus (\phi(I) + J)/J$. Thus $a^m \in I \setminus (\phi(I) + J)$. Hence $a^m \in I \setminus \phi(I)$, and so $a^n \in I$. Therefore $\bar{a}^n \in I/J$ and thus I/J is ϕ_J-(m, n)-closed.
(2) Let $(\frac{a}{s})^m \in S^{-1}I \setminus \phi_S(S^{-1}I)$. Thus $ta^m \in I$ for some $t \in S$. But $sa^m \notin \phi_S(S^{-1}I) \cap R$ for every $s \in S$. Now let $sa^m \in \phi(I)$, then $(\frac{a}{s})^m \in S^{-1}\phi(I) \subseteq \phi_S(S^{-1}I)$ which gives a contradiction. Hence $(ta)^m \in I \setminus \phi(I)$ and so I is a ϕ-(m, n)-closed ideal gives $t^n a^n \in I$. Which implies that $(\frac{a}{s})^n \in S^{-1}I$ and so $S^{-1}I$ is a ϕ_S-(m, n)-closed ideal of $S^{-1}R$.. □

We next study when certain ideals of $A \ltimes E$ are ϕ-(m, n)-closed ideals.

Proposition 2.9. *Let A be a ring and E an A-module. Let m and n positive integers, $\phi : \mathcal{I}(A) \to \mathcal{I}(A) \cup \{\emptyset\}$ and $\psi : \mathcal{I}(A \ltimes E) \to \mathcal{I}(A \ltimes E) \cup \{\emptyset\}$ be two functions such that $\psi(I \ltimes F) = \phi(I) \ltimes F$ and $\psi(I \ltimes F) = \varnothing$ if $\phi(I) = \varnothing$ where F is a submodule of E. Then*

(i) If $I \ltimes F$ is a ψ-(m, n)-closed ideal of $A \ltimes E$, then I is a ϕ-(m, n)-closed ideal of A.

(ii) $I \ltimes E$ is a ψ-(m, n)-closed ideal of $A \ltimes E$ if and only if I is a ϕ-(m, n)-closed ideal of A.

Proof. (1) Let $a^m \in I \setminus \phi(I)$ for $a \in A$. Hence $(a, 0)^m = (a^m, 0) \in I \ltimes F \setminus \phi(I) \ltimes F = \psi(I \ltimes F)$. Thus $(a, 0)^n \in I \ltimes F$ since $I \ltimes F$ is ψ-(m, n)-closed. Which implies that $a^n \in I$ and so I is a ϕ-(m, n)-closed ideal of A.
(2) By (1), it suffices to prove the "if" assertion. Let $(a, e)^m \in I \ltimes E \setminus \phi(I) \ltimes E$. Thus, $a^m \in I \setminus \phi(I)$ which implies that $a^n \in I$ because I is a ϕ-(m, n)-closed ideal of A. Therefore $(a, e)^n = (a^n, na^{n-1}e) \in I \ltimes E$ and this completes the proof of (2). □

Theorem 2.10. *Let A be a ring, E an A-module, m and n positive integers, and $\phi : \mathcal{I}(A) \to \mathcal{I}(A) \cup \{\emptyset\}$ a function. Let N be a submodule of E and $\psi : \mathcal{I}(A \ltimes E) \to \mathcal{I}(A \ltimes E) \cup \{\emptyset\}$ be a function defined by :*

$$\psi(H) = \begin{cases} (\phi(I_H) \ltimes N) \cap H & \text{if } \phi(I_H) \neq \emptyset \\ \emptyset & \text{if } \phi(I_H) = \emptyset \end{cases}$$

where $I_H = \{a \in A \mid (a, e) \in H\}$. We consider a submodule F of E. Then

(i) If $I \ltimes F$ is a ψ-(m, n)-closed ideal of $A \ltimes E$, then I is a ϕ-(m, n)-closed ideal of A and $m(a^{m-1}F) \subseteq (N \cap F)$ for every ϕ-(m, n)-unbreakable element a of I.

(ii) $I \ltimes E$ is a ψ-(m, n)-closed ideal of $A \ltimes E$ that is not (m, n)-closed if and only if I is a ϕ-(m, n)-closed ideal of A that is not (m, n)-closed and $m(a^{m-1}E) \subseteq N$ for every ϕ-(m, n)-unbreakable element a of I.

Proof. (1) Let $J = I \ltimes F$. Assume that $a^m \in I \setminus \phi(I)$ for $a \in A$. Thus $(a, 0)^m = (a^m, 0) \in J \setminus \psi(J)$. Hence $(a, 0)^n = (a^n, 0) \in J$ and so $a^n \in I$. Thus I is a ϕ-(m, n)-closed ideal of A. Now, let a be a ϕ-(m, n)-unbreakable element of I and $e \in F$. Then $(a, e)^m = (a^m, ma^{m-1}e) \in J$. Since $a^n \notin I$, we have $(a, e)^m = (a^m, ma^{m-1}e) \in \psi(J) = \phi(I) \ltimes (N \cap F)$. Therefore $ma^{m-1}F \subseteq N \cap F$.
(2) Suppose that $J = I \ltimes E$ is a ϕ-(m, n)-closed ideal of $A \ltimes E$ that is not (m, n)-closed.

Hence, I has an ϕ-(m,n)-unbreakable element. Thus I is not a (m,n)-closed ideal of A. The rest follows by (1). Conversely, as I is a ϕ-(m,n)-closed ideal of A that is not (m,n)-closed, I has a ϕ-(m,n)-unbreakable element a. Then $(a,0)$ is a ψ-(m,n)-unbreakable element of J. Thus J is not an (m,n)-closed ideal of A. Assume that $(b,f)^m = (b^m, mb^{m-1}f) \in J \setminus \psi(J)$. So, $b^m \in I$. If b is a ϕ-(m,n)-unbreakable element of I, then the hypothesis gives that $(b,f)^m \in \phi(I) \ltimes N = \psi(J)$, a contradiction. Hence $b^n \in I$ and thus $(b^n, nb^{n-1}f) = (b,f)^n \in J$. Therefore J is a ψ-(m,n)-closed ideal of $A \ltimes E$ that is not (m,n)-closed.

□

Remark 2.11. Assume that A is a reduced ring. Thus, for any function $\psi : \mathcal{I}(A \ltimes E) \to \mathcal{I}(A \ltimes E) \cup \{\emptyset\}$ and a submodule F of E, the ideal $0 \ltimes F$ is always a ψ-(m,n)-closed ideal of $A \ltimes E$ for $n \geq 2$. Indeed. Let $(a,e)^m \in 0 \ltimes F \setminus \psi(0 \ltimes F)$ for $(a,e) \in A \ltimes E$. Then $a^m = 0$ and so $a = 0$. Now, the fact that $(0,e)^n = (0,0) \in 0 \ltimes F$ implies that $0 \ltimes F$ is ψ-(m,n)-closed.

Now, we study the ϕ-(m,n)-closed ideals of the direct product of rings.

Proposition 2.12. *Let R_1 and R_2 be rings, I_1 a proper ideal of R_1 and let $\phi_i : \mathcal{I}(R_i) \to \mathcal{I}(R_i) \cup \{\emptyset\}$ be two functions. Let $\psi = \phi_1 \times \phi_2$. Then $I_1 \times R_2$ is a ψ-(m,n)-closed ideal of $R_1 \times R_2$ if and only if I_1 is a ϕ_1-(m,n)-closed ideal of R_1 which must be (m,n)-closed if $\phi_2(R_2) \neq R_2$.*

Proof. Assume that $I_1 \times R_2$ is a ψ-(m,n)-closed ideal of $R_1 \times R_2$. Let $a^m \in I_1 \setminus \phi_1(I_1)$ for $a \in R_1$. Hence $(a,0)^m \in I_1 \times R_2 \setminus \phi_1(I_1) \times \phi_2(R_2)$ which gives that $(a,0)^n \in I_1 \times R_2$. Therefore, $a^n \in I_1$ and thus I_1 is ϕ_1-(m,n)-cloded. Now, suppose that $\phi_2(R_2) \neq R_2$. If I_1 is not (m,n)-closed, then I_1 has a ϕ_1-(m,n)-unbreakable element a. Hence, $(a,1)^m \in I_1 \times R_2 \setminus \psi(I_1 \times R_2) = \phi_1(I_1) \times \phi_2(R_2)$ and $(a,1)^n \notin I_1 \times R_2$, a contradiction. Thus I_1 is a (m,n)-closed ideal of R_1. Conversely, assume that I_1 is ϕ_1-(m,n)-closed and $\phi_2(R_2) = R_2$. Let $(a,b)^m \in I_1 \times R_2 \setminus \psi(I_1 \times R_2) = \phi_1(I_1) \times R_2$ for $(a,b) \in R_1 \times R_2$. Thus $a^m \in I_1 \setminus \phi_1(I_1)$ and so $a^n \in I_1$. Which implies that $(a,b)^n \in I_1 \times R_2$. If $\phi_2(R_2) \neq R_2$, then I_1 is (m,n)-closed and so the result follows from [1, Theorem 2.12]. □

Theorem 2.13. *Let $R = R_1 \times R_2$, where R_1 and R_2 are two rings and m and n positive integers. Let $\phi_i : \mathcal{I}(R_i) \to \mathcal{I}(R_i) \cup \{\emptyset\}$ be two functions and $\psi = \phi_1 \times \phi_2$. Then $I_1 \times I_2$ is a ψ-(m,n)-closed ideal of R that is not (m,n)-closed for proper ideals I_1 of R_1 and I_2 of R_2 if and only if either*

(i) I_1 is a ϕ_1-(m,n)-closed ideal of R_1 that is not (m,n)-closed, $b^m \in \phi_2(I_2)$ whenever $b^m \in I_2$ for $b \in R_2$, and if $a^m \in I_1 \setminus \phi_1(I_1)$ for some $a \in R_1$, then I_2 is an (m,n)-closed ideal of R_2, or

(ii) I_2 is a ϕ_2-(m,n)-closed ideal of R_2 that is not (m,n)-closed, $a^m \in \phi_1(I_1)$ whenever $a^m \in I_1$ for $a \in R_1$, and if $b^m \in I_2 \setminus \phi_2(I_2)$ for some $b \in R_2$, then I_1 is an (m,n)-closed ideal of R_1.

Proof. Set $J = I_1 \times I_2$ and suppose that J is ψ-(m,n)-closed ideal of R that is not (m,n)-closed. Since J is not an (m,n)-closed ideal of R, either I_1 is a ϕ_1-(m,n)-closed ideal of R_1 that is not (m,n)-closed or I_2 is a ϕ_2-(m,n)-closed ideal of R_2 that is not (m,n)-closed. Assume that I_1 is a ϕ_1-(m,n)-closed ideal of R_1 that is not (m,n)-closed. Hence I_1 has a ϕ_1-(m,n)-unbreakable element r. Assume that $b^m \in I_2$ for $b \in R_2$. Since r is a ϕ_1-(m,n)-unbreakable element of I_1 and $(r,b)^m \in J$, we get that $(r,b)^m \in \psi(J) = \phi_1(I_1) \times \phi_2(I_2)$. Hence $b^m \in \phi_2(I_2)$. Now suppose that $a^m \in I_1 \setminus \phi_1(I_1)$ for some $a \in R_1$. Let $b \in R_2$ such that $b^m \in I_2$. Then $(a,b)^m \in J \setminus \phi_1(I_1) \times \phi_2(I_2)$. Then $(a,b)^n \in J$ and so $b^n \in I_2$. Thus I_2 is an (m,n)-closed ideal of R_2. Similarly, if I_2 is a ϕ_2-(m,n)-closed ideal of R_2 that is not (m,n)-closed, then $a^m \in \phi_1(I_1)$ whenever $a^m \in I_1$ for $a \in R_1$, and if $b^m \in I_2 \setminus \phi_2(I_2)$ for some $b \in R_2$, then I_1 is an (m,n)-closed ideal of R_1. Conversely, without loss of generality assume that I_1 is a ϕ_1-(m,n)-closed proper ideal of R_1 that is not (m,n)-closed, $b^m \in \phi_2(I_2)$ whenever $b^m \in I_2$ for $b \in R_2$, and if $a^m \in I_1 \setminus \phi_1(I_1)$ for some $a \in R_1$, then I_2 is a (m,n)-closed ideal of R_2. Let r be a ϕ_1-(m,n)-unbreakable element of I_1. Then $(r,0)$ is a ψ-(m,n)-unbreakable element of J. Thus J is not an (m,n)-closed ideal of R. Now assume that $(a,b)^m \in J \setminus \psi(J) = \phi_1(I_1) \times \phi_2(I_2)$ for $a \in R_1$ and $b \in R_2$. Then $b^m \in \phi_2(I_2)$ and $a^m \in I_1 \setminus \phi_1(I_1)$. Since I_1 is a ϕ_1-(m,n)-closed ideal of R_1 and I_2 is an (m,n) -closed ideal of R_2, we colclude that $(a,b)^n \in J$. This completes the proof. □

Next, we study the ϕ-(m,n)-closed ideals in the amalgamation of rings.

Theorem 2.14. *Let A and B be two rings, $f : A \to B$ be a ring homomorphism and J an ideal of B. Let $\phi : \mathcal{I}(A) \to \mathcal{I}(A) \cup \{\emptyset\}$ and $\psi : \mathcal{I}(A \bowtie^f J) \to \mathcal{I}(A \bowtie^f J) \cup \{\emptyset\}$ be two functions such that*

$$\psi(P \bowtie^f J) = \begin{cases} \phi(P) \bowtie^f K & \text{if } \phi(P) \neq \emptyset \\ \emptyset & \text{if } \phi(P) = \emptyset \end{cases}$$

where P is an ideal of A and K a subideal of J. Then, $P \bowtie^f J$ is a ψ-(m,n)-closed ideal if and only if P is a ϕ-(m,n)-closed ideal and for every ϕ-(m,n)-unbreakable element p of P we have $(f(p)+i)^m - f(p)^m \in K$ for all $i \in J$.

Proof. Suppose that $P \bowtie^f J$ is a ψ-(m,n)-closed ideal of $A \bowtie^f J$. Let $a \in A$ such that $a^m \in P \setminus \phi(P)$. Then, $(a, f(a))^m \in P \bowtie^f J \setminus \psi(P \bowtie^f J)$. Hence, $(a, f(a))^n \in P \bowtie^f J$. Therefore, $a^n \in P$ and thus P is a ϕ-(m,n)-closed ideal of A. Now, let p be a ϕ-(m,n)-unbreakable element of P such that $p^m \in \phi(P)$ and $p^n \notin P$ and assume that $(f(p)+i)^m - f(p)^m \notin K$ for some i of J. Then, $(p, f(p)+i)^m = (p^m, f(p)^m + (f(p)+i)^m - f(p)^m) \in P \bowtie^f J \setminus \phi(P) \bowtie^f K$ since $(f(p)+i)^m - f(p)^m \notin K$. Thus, $(p, f(p)+i)^n \in P \bowtie^f J$ and so $p^n \in P$, which is a contradiction. Thus, for every ϕ-(m,n)-unbreakable element p of P we have $(f(p)+i)^m - f(p)^m \in K$ for all $i \in J$.
Conversely, without loss of generality we may assume that $\phi(P) \neq \varnothing$. Let $(a, f(a)+i)^m \in P \bowtie^f J \setminus \phi(P) \bowtie^f K$ for $(a, f(a)+i) \in A \bowtie^f J$. Then, $a^m \in P$. Two cases are possible :

Case 1 : $a^m \in \phi(P)$. Suppose that $a^n \notin P$, then $(a^m, (f(a)+i)^m) \in \phi(P) \bowtie^f K$ since $(f(a)+i)^m - f(a)^m \in K$, which is a contradiction. Hence, $a^n \in P$ and thus $(a, f(a)+i)^n \in P \bowtie^f J$.

Case 2 : $a^m \notin \phi(P)$. Then, $a^n \in P$ since P is ϕ-(m,n)-closed. Hence, $(a, f(a)+i)^n \in P \bowtie^f J$.

In both cases we have $(a, f(a)+i)^n \in P \bowtie^f J$ and so $P \bowtie^f J$ is a ψ-(m,n)-closed ideal of $A \bowtie^f J$.

□

The next corollaries are immediate applications of Theorem 2.14.

Corollary 2.15. *Let A and B be two rings, $f : A \to B$ be a ring homomorphism and J an ideal of B. Then, $P \bowtie^f J$ is a weakly (m,n)-closed ideal if and only if P is a weakly (m,n)-closed ideal and for every element p of P such that $p^m = 0$ and $p^n \notin P$ we have $(f(p)+i)^m - f(p)^m = 0$ for all $i \in J$.*

Corollary 2.16. *Let A be a ring and I an ideal of A. Let $\phi : \mathcal{I}(A) \to \mathcal{I}(A) \cup \{\emptyset\}$ and $\psi : \mathcal{I}(A \bowtie I) \to \mathcal{I}(A \bowtie I) \cup \{\emptyset\}$ be two functions such that*

$$\psi(P \bowtie I) = \begin{cases} \phi(P) \bowtie K & \text{if } \phi(P) \neq \emptyset \\ \emptyset & \text{if } \phi(P) = \emptyset \end{cases}$$

where P is an ideal of A and K a subideal of I. Then, $P \bowtie I$ is a ψ-(m,n)-closed ideal if and only if P is a ϕ-(m,n)-closed ideal and for every ϕ-(m,n)-unbreakable element p of P we have $(p+i)^m - p^m \in K$ for all $i \in I$.

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Author information

Abdelhaq El Khalfi, Laboratory of Topology, Algebra, Geometry and Discrete Mathematics, Faculty of Sciences Ain Chock, Hassan II University of Casablanc, Morocco.
E-mail: abdelhaqelkhalfi@gmail.com

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