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Existence and uniqueness of positive solutions for a class of fractional integro-differential equations

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Abstract In this paper, we prove the existence and uniqueness of positive solutions of a fractional integro-differential equation involving Caputo-Hadamard fractional derivative with integral boundary conditions. The technique used to prove our results depends on the upper and lower solution, the Schauder fixed point theorem and the Banach contraction principle. An example is given which illustrate the effectiveness of the theoretical results.

1 Introduction

Fractional differential equations with and without delay arise from a variety of applications including in various fields of science and engineering such as applied sciences, practical problems concerning mechanics, the engineering technique fields, economy, control systems, physics, chemistry, biology, medicine, atomic energy, information theory, harmonic oscillator, nonlinear oscillations, conservative systems, stability and instability of geodesic on Riemannian manifolds, dynamics in Hamiltonian systems, etc. In particular, problems concerning qualitative analysis of linear and nonlinear fractional differential equations with and without delay have received the attention of many authors, see [1]–[19], [21]–[28] and the references therein.

Recently, Zhang in [28] investigated the existence and uniqueness of positive solutions for the nonlinear fractional differential equation

$$\begin{cases} D^{\alpha}x(t) = f\left(t, x\left(t\right)\right), \ 0 < t \leq 1, \\ x(0) = 0, \end{cases}$$

where D^{α} is the standard Riemann Liouville fractional derivative of order $0 < \alpha < 1$, and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a given continuous function. By using the method of the upper and lower solution and cone fixed-point theorem, the author obtained the existence and uniqueness of a positive solution.

The nonlinear fractional integro-differential equation with nonlinear conditions

$$\begin{cases} {}^{C}D^{\alpha}u(t) = f\left(t, u\left(t\right)\right) + \int_0^t k\left(t, s, u\left(s\right)\right) ds, \ t \in (0, T], \\ u(0) = u_0 - g\left(u\right), \end{cases}$$

was investigated in [5], where ${}^{C}D^{\alpha}$ is the standard Caputo fractional derivative of order $0 < \alpha < 1$, $u_0 \in \mathbb{R}$, g, f and k are given continuous functions. By employing the Krasnoselskii and Banach fixed point theorems, Ahmad and Sivasundaram obtained the existence and uniqueness results.

Wang, Wang and Zeng [25] discussed the existence of solutions of the following fractional differential equation with integral boundary conditions

$$\begin{cases} D^{\alpha}u(t) = f\left(t, u\left(t\right)\right), \ t \in (0, T], \\ u(0) = \lambda \int_0^1 u\left(s\right) ds + d, \end{cases}$$

where $0 < \alpha < 1, \lambda \geq 0$, $f : [0;T] \times\mathbb{R} \to \mathbb{R}$ is continuous. The authors applied the upper and lower solutions combined with a monotone iterative technique to obtain their main results.

In [1], Abdo, Wahash and Panchal discussed the existence and uniqueness of positive solutions of the following nonlinear fractional differential equation with integral boundary conditions

$$\begin{cases} {}^{C}D^{\alpha}u(t) = f\left(t, u\left(t\right)\right), \ t \in [0,1]\,, \\ u(0) = \lambda \int_0^1 u\left(s\right)ds + d, \lambda \geq 0, \ d > 0, \end{cases}$$

where $0 < \alpha < 1$ and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a given continuous function. By using the method of the upper and lower solutions and Schauder and Banach fixed point theorems, the existence and uniqueness of solutions has been established.

Inspired and motivated by the works mentioned above. In this paper, we used the upper and lower solution method, Schauder fixed point theorem and Banach contraction principle to obtain the existence and uniqueness of a positive solution for the following fractional differential equations with integral boundary conditions

$$\begin{cases} {}^{C}_{H}\mathfrak{D}^{\alpha}u(t) = f\left(t, u\left(t\right)\right) + {}^{H}\mathfrak{I}^{\beta}g(t,u(t)), \ t \in (1,T]\,, \\ u(1) = \lambda \int_1^T k(s)u\left(s\right)ds + d, \lambda \geq 0, \ d > 0, \end{cases} \tag{1.1}$$

where ${}^{C}_{H}\mathfrak{D}^{\alpha}$ is the Caputo Hadamard fractional derivative of order $0 < \alpha \leq 1$, and ${}^{H}\mathfrak{I}^{\beta}$ is the Hadamard fractional integral of order $\beta \in (0,1)$, $f : [1,T] \times [0,\infty) \to [0,\infty)$, $g : [1,T] \times [0,\infty) \to [0,\infty)$ and $k : [1,T] \to [0,\infty)$ are given continuous functions. g is non-decreasing on u.

The organization of the rest of the paper divided of four sections. In Section 2, some notations, definitions of fractional calculus and fixed point theorems are presented. In Section 3, Some useful results about the existence and uniqueness of positive solution for problem (1.1) are obtained. In Section 4, An example is presented which illustrates the effectiveness of the theoretical results.

2 Preliminaries

Some definitions, notations and results of the fractional calculus are introduced throughout this section which will be utilized in this paper. For more details we refer the reader to see [13, 19].

Let $J = [1,T]$. Denote by $C\left(J\right)$ the Banach space of all continuous functions defined on J endowed with the norm

$$\|u\| = \sup\left\{|u\left(t\right)| : t \in J\right\}.$$

And A a nonempty closed subset of $C\left(J\right)$ defined as

$$A = \left\{u(t) \in C\left(J\right) : u(t) \geq 0, t \in J\right\}.$$

$C^{n}\left(J\right)$ denotes the class of all real valued functions defined on J which have a continuous n th order derivative.

Definition 2.1. [13] The Hadamard fractional integral of order $\alpha > 0$ for a continuous function $u : [1,\infty) \to \mathbb{R}$ is defined as

$${}^{H}\mathfrak{I}^{\alpha}u\left(t\right) = \frac{1}{\Gamma\left(\alpha\right)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} u\left(s\right)\frac{ds}{s}, \ \alpha > 0.$$

where Γ denotes the Gamma function.

Definition 2.2. [11] The Caputo-Hadamard fractional derivative of order $\alpha > 0$ for a function $u : [1,\infty) \to \mathbb{R}$ is defined as

$${}^{C}_{H}\mathfrak{D}^{\alpha}u\left(t\right) = \frac{1}{\Gamma\left(n-\alpha\right)}\int_1^t \left(\log\frac{t}{s}\right)^{n-a-1} \delta^{n}u\left(s\right)\frac{ds}{s}, \ n-1 < \alpha < n,$$

where $\delta^{n} = \left(t\frac{d}{dt}\right)^{n}, n \in \mathbb{N}$.

Lemma 2.3. *[11] Let $n-1<\alpha\leq n$, $n\in\mathbb{N}$ and $u\in C^{n}(J)$. Then the Caputo-Hadamard fractional differential equation*

$$ {}_{H}^{C}\mathfrak{D}^{\alpha}u(t)=0, $$

has a solution

$$ u(t)=\sum_{k=0}^{n-1}c_{k}(\log t)^{k}, $$

and the following formula holds:

$$ {}^{H}\mathfrak{J}^{\alpha}\left({}_{H}^{C}\mathfrak{D}^{\alpha}u(t)\right)=u(t)-\sum_{k=0}^{n-1}c_{k}(\log t)^{k}, $$

where $c_{k}\in\mathbb{R}$, $k=0,1,...,n-1$.

Theorem 2.4. *(Schauder's fixed point theorem [20]) Let Ω be a nonempty closed convex subset of a Banach space S and $\Phi:\Omega\rightarrow\Omega$ be a continuous compact operator. Then, Φ has a fixed point in Ω.*

Theorem 2.5. *(Banach contraction principle [20]) Let Ω be a non-empty closed subset of a Banach space $(S,\|.\|)$, then any contraction mapping Φ of Ω into itself has a unique fixed point.*

Definition 2.6. A function $u\in C^{1}(J)$ is said to be a solution of problem (1.1) if u satisfies the equation ${}_{H}^{C}\mathfrak{D}^{\alpha}u(t)=f(t,u(t))+^{H}\mathfrak{J}^{\beta}g(t,u(t))$, $t\in J$ with integral boundary conditions $u(1)=\lambda\int_{1}^{T}k(s)u(s)\,ds+d$.

Definition 2.7. A function $u\in C^{1}(J)$ is called a positive solution of problem (1.1) if $u(t)\geq 0$ for all $t\in J$ and u satisfies the problem (1.1).

Definition 2.8. Let $a,b\in\mathbb{R}^{+}$, and $b>a$: For any $u\in[a,b]$, we define the upper-control function $U(t,u)=\sup_{a\leq\rho\leq u}f(t,\rho)$, and lower-control function $L(t,u)$ =$\inf_{u\leq\rho\leq b}f(t,\rho)$.

Obviously, $U(t,u)$ and $L(t,u)$ are monotonous non-decreasing on $[a,b]$ and

$$ L(t,u)\leq f(t,u)\leq U(t,u). $$

3 Main Results

In this section, we shall give existence and uniqueness results of problem (1.1) and prove it. Before starting and proving the main result, we introduce the following lemma:

Lemma 3.1. *Let $u\in C(J)$, u' exists, then u is a solution of problem (1.1) if and only if u is a solution of the integral equation*

$$ \begin{aligned} u(t)=&\frac{1}{\Gamma(\alpha)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}f(s,u(s))\frac{ds}{s}+\frac{1}{\Gamma(\alpha+\beta)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\alpha+\beta-1}g(s,u(s))\frac{ds}{s}\\ &+\lambda\int_{1}^{T}k(s)u(s)\,ds+d,\ t\in J. \end{aligned} \tag{3.1} $$

Proof. Suppose $u(t)$ satisfies the problem (1.1), then applying the Hadamard fractional integral ${}^{H}\mathfrak{J}^{\alpha}$ to both sides of (4.1), we have

$$ {}^{H}\mathfrak{J}^{\alpha}\,{}_{H}^{C}\mathfrak{D}^{\alpha}u(t)=^{H}\mathfrak{J}^{\alpha}\left(f(t,u(t))+^{H}\mathfrak{J}^{\beta}g(t,u(t))\right), $$

In view of Lemma (2.3), we get

$$ u(t)=\frac{1}{\Gamma(\alpha)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}f(s,u(s))\frac{ds}{s}+\frac{1}{\Gamma(\alpha+\beta)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\alpha+\beta-1}g(s,u(s))\frac{ds}{s}+c_{0}. \tag{3.2} $$

Using the integral condition, we find

$$c_0 = \lambda \int_1^T k(s)\, u(s)\, ds + d.$$

Substituting the value of c_0 into (3.2), we obtain the integral equation (3.1). The converse is followed by a direct computation that finishes the proof. □

To transform (3.1) to be applicable to Schauder's fixed point, we define the operator $\Phi : A \to A$ by

$$\begin{aligned}(\Phi u)(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, u(s)) \frac{ds}{s} + \frac{1}{\Gamma(\alpha+\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha+\beta-1} g(s, u(s)) \frac{ds}{s} \\ &\quad + \lambda \int_1^T k(s)\, u(s)\, ds + d,\ t \in J. \end{aligned} \tag{3.3}$$

where figured fixed point must satisfy the identity operator equation $\Phi u = u$.

We introduce the following assumptions

(H1) Let $u^*, u_* \in A$ such that $a \le u_*(t) \le u^*(t) \le b$ and

$$\begin{cases} {}_H^C\mathfrak{D}^\alpha u^*(t) - {}^H\mathfrak{I}^\beta g(t, u^*(t)) \ge U(t, u^*(t)), \\ {}_H^C\mathfrak{D}^\alpha u_*(t) - {}^H\mathfrak{I}^\beta g(t, u_*(t)) \le L(t, u_*(t)), \end{cases}$$

for any $t \in J$.

(H2) For $t \in J$, and $u, v \in [0, \infty)$, there exist two positives number l_f and l_g such that

$$\begin{aligned} |f(t,u) - f(t,v)| &\le l_f |u - v|, \\ |g(s,u) - g(s,v)| &\le l_g |u - v|. \end{aligned}$$

The function $u^*$and u_* are respectively called the pair of upper and lower solutions for problem (1.1).

The first result is based on the Schauder fixed point theorem.

Theorem 3.2. *Assume that (H1) is satisfied, then problem(1.1) has at least one positive solution.*

Proof. Let $\Omega = \{u \in A : u_*(t) \le u(t) \le u^*(t), t \in J\}$ endowed with the norm $\|u\| = \max_{t\in J} |u(t)|$, then we have $\|u\| \le b$. Hence, Ω is convex bounded and closed subset of the Banach space $C(J)$. Notice that continuity of the operator Φ follows from that of the functions f and g. Now, if $u \in \Omega$, there exist two positives constants c_f, c_g such that

$$\max \{f(t, u(t)) : t \in J,\ u(t) \le b\} < c_f,$$

$$\max \{g(t, u(t)) : t \in J, u(t) \le b\} < c_g,$$

Then

$$\begin{aligned} &|(\Phi u)(t)| \\ &\le \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |f(s, u(s))| \frac{ds}{s} + \frac{1}{\Gamma(\alpha+\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha+\beta-1} |g(s, u(s))| \frac{ds}{s} \\ &+ \lambda \int_1^T k(s) |u(s)|\, ds + d \\ &\le \frac{c_f (\log T)^\alpha}{\Gamma(\alpha+1)} + \frac{c_g (\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \lambda b k^* (T-1) + d, \end{aligned}$$

where $k^* = \sup_{t\in J} k(t)$, thus

$$\|\Phi u\| \le \frac{c_f (\log T)^\alpha}{\Gamma(\alpha+1)} + \frac{c_g (\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \lambda b k^* (T-1) + d.$$

Hence, $\Phi(\Omega)$ is uniformly bounded. Next, we prove the equicontinuity of $\Phi(\Omega)$. For each $u \in \Omega$. Then for $t_1, t_2 \in [1, T]$ with $t_1 < t_2$, we have

$$\begin{aligned}
|\Phi u(t_2) - \Phi u(t_1)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left[(\log \frac{t_1}{s})^{\alpha-1} - (\log \frac{t_2}{s})^{\alpha-1} \right] |f(s, u(s))| \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (\log \frac{t_2}{s})^{\alpha-1} |f(s, u(s))| \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha+\beta)} \int_1^{t_1} \left[\left(\log \frac{t_2}{s} \right)^{\alpha+\beta-1} - \left(\log \frac{t_1}{s} \right)^{\alpha+\beta-1} \right] |g(s, u(s))| \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s} \right)^{\alpha+\beta-1} |g(s, u(s))| \frac{ds}{s} \\
&\leq \frac{c_f}{\Gamma(\alpha+1)} \left[2(\log \frac{t_2}{t_1})^{\alpha} + (\log t_1)^{\alpha} - (\log t_2)^{\alpha} \right] \\
&+ \frac{c_g}{\Gamma(\alpha+\beta+1)} \left[(\log t_2)^{\alpha+\beta} - (\log t_1)^{\alpha+\beta} \right] \\
&\leq \frac{2c_f}{\Gamma(\alpha+1)} (\log \frac{t_2}{t_1})^{\alpha} + \frac{c_g}{\Gamma(\alpha+2)} \left((\log t_2)^{\alpha+\beta} - (\log t_1)^{\alpha+\beta} \right).
\end{aligned}$$

As $t_1 \to t_2$. We see that the right hand side of the above inequality tends to zero and the convergence is independent of u in Ω, which means $\Phi(\Omega)$ is equicontinuous. The Arzela-Ascoli Theorem implies that $\Phi : \Omega \to A$ is compact. The only thing to apply Schauder fixed point is to prove that $\Phi(\Omega) \subset \Omega$. For any $u \in \Omega$, then $u_*(t) \leq u(t) \leq u^*(t)$ and by (H1), we have

$$\begin{aligned}
&(\Phi u)(t) \\
&= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} |f(s, u(s))| \frac{ds}{s} + \frac{1}{\Gamma(\alpha+\beta)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha+\beta-1} |g(s, u(s))| \frac{ds}{s} \\
&+ \lambda \int_1^T k(s) u(s) ds + d \\
&\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} U(s, u(s)) \frac{ds}{s} + \frac{1}{\Gamma(\alpha+\beta)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha+\beta-1} |g(s, u(s))| \frac{ds}{s} \\
&+ \lambda \int_1^T k(s) u(s) ds + d \\
&\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} U(s, u^*(s)) \frac{ds}{s} + \frac{1}{\Gamma(\alpha+\beta)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha+\beta-1} |g(s, u^*(s))| \frac{ds}{s} \\
&+ \lambda \int_1^T k(s) u^*(s) ds + d \\
&\leq u^*(t),
\end{aligned}$$

and

$$
\begin{aligned}
&(\Phi u)(t)\\
&=\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}|f(s,u(s))|\frac{ds}{s}+\frac{1}{\Gamma(\alpha+\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha+\beta-1}|g(s,u(s))|\frac{ds}{s}\\
&+\lambda\int_1^T k(s)\,u(s)\,ds+d\\
&\geq\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}L(s,u(s))\frac{ds}{s}+\frac{1}{\Gamma(\alpha+\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha+\beta-1}|g(s,u(s))|\frac{ds}{s}\\
&+\lambda\int_1^T k(s)\,u(s)\,ds+d\\
&\geq\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}L(s,u_*(s))\frac{ds}{s}+\frac{1}{\Gamma(\alpha+\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha+\beta-1}|g(s,u_*(s))|\frac{ds}{s}\\
&+\lambda\int_1^T k(s)\,u_*(s)\,ds+d\\
&\geq u_*(t).
\end{aligned}
$$

Hence, $u_*(t)\leq(\Phi u)(t)\leq u^*(t)$, $t\in J$, that is , $\Phi(\Omega)\subset\Omega$. According to Schauder fixed point theorem, the operator Φ has at least one fixed point $u\in\Omega$.Therefore, problem (1.1) has at least one positive solution. □

The second result is based on the Banach contraction principle.

Theorem 3.3. *Assume that (H2) is satisfied and*

$$
\left(\frac{l_f(\log T)^{\alpha}}{\Gamma(\alpha+1)}+\frac{l_g(\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\lambda k^*(T-1)\right)<1. \tag{3.4}
$$

Then problem (1.1) has a unique positive solution.

Proof. From Theorem (3.2), it follows that problem (1.1) has at least one positive solution. Hence, we need only to prove that the operator defined in (3.3) is a contraction in Ω. In fact, for each $u,v\in\Omega$, we have

$$
\begin{aligned}
|(\Phi u)(t)-(\Phi v)(t)|\leq&\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}|f(s,u(s))-f(s,v(s))|\frac{ds}{s}\\
&+\frac{1}{\Gamma(\alpha+\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha+\beta-1}|g(s,u(s))-g(s,v(s))|\frac{ds}{s}\\
&+\lambda k^*\int_1^T|u(s)-v(s)|\,ds\\
\leq&\left(\frac{l_f(\log T)^{\alpha}}{\Gamma(\alpha+1)}+\frac{l_g(\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\lambda k^*(T-1)\right)\|u-v\|.
\end{aligned}
$$

From (3.4), Φ is a contraction. As a result of Theorem 2.5, Φ has a unique fixed point that is the corresponding unique positive solution of the problem (1.1). This finishes the proof. □

4 Example

As an application of our result, we consider the following fractional integro-differential equation with integral boundary condition

$$
\begin{cases}
{}^{C}_{H}\mathfrak{D}^{\frac{1}{2}}u(t)=\frac{\sin(t)}{\exp(t^2-1)+7}\left(\frac{|u|}{|u|+1}\right)+\frac{1}{4}I^{\frac{1}{4}}\frac{u(t)}{\exp(t-1)},\ t\in(1,e]\,,\\
u(1)=\frac{1}{6}\int_1^e\frac{1}{t}u(s)\,ds+\frac{1}{10}.
\end{cases} \tag{4.1}
$$

Where $\alpha = \frac{1}{2}$, $\beta = \frac{1}{4}$, $\lambda = \frac{1}{6}$, $d = \frac{1}{10}$, $f(t,u) = \frac{\sin(t)}{\exp(t^2-1)+7}\left(\frac{|u|}{|u|+1}\right)$, $g(t,u) = \frac{u}{4\exp(t-1)}$ and $k(t) = \frac{1}{t}$.

Since f, g and k are continuous positive functions, g is non-decreasing on u. For $u, v \in [0,\infty)$, we have

$$\begin{aligned} |f(t,u) - f(t,v)| &= \left|\frac{\sin(t)}{\exp(t^2-1)+7}\left(\frac{|u|}{|u|+1} - \frac{|v|}{|v|+1}\right)\right| \\ &\leq \frac{1}{\exp(t^2-1)+7}\left(\frac{|u-v|}{(1+|u|)(1+|v|)}\right) \\ &\leq \frac{1}{8}|u-v|, \end{aligned}$$

and

$$\begin{aligned} |g(t,u) - g(t,v)| &= \frac{1}{4\exp(t-1)}|u-v| \\ &\leq \frac{1}{4}|u-v|. \end{aligned}$$

Thus, the assumption (H2) is satisfied with $l_f = \frac{1}{8}$, and $l_g = \frac{1}{4}$. We will check that condition (3.4) is satisfied. Indeed

$$\begin{aligned} &\frac{l_f(\log T)^{\alpha}}{\Gamma(\alpha+1)} + \frac{l_g(\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \lambda k^*(T-1) \\ &\simeq 0.7 < 1. \end{aligned}$$

Then by Theorem (3.3), the problem (4.1) has a unique positive solution.

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