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Graded S-Artinian Modules and Graded S-Secondary Representations

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Abstract Let G be an abelian group and S a given multiplicatively closed subset of a commutative G-graded ring consisting of homogeneous elements. In this paper, we introduce G-graded S-Artinian rings and modules which are a generalization of S-Artinian rings and modules. We list some properties and characterizations of G-graded S-Artinian rings and modules. We also introduce and study the concept of G-graded S-secondary representable modules as a generalization of graded secondary representable modules. Moreover, we prove the existence of G-graded S-secondary representation in G-graded S-Artinian modules, which generalizes a well-known theory of secondary representation for the Artinian modules.

1 Introduction

The theory of rings and modules satisfying chain conditions was introduced in the celebrated works of E. Noether and E. Artin [4, 15], which played an important role in the development of the structure theory of commutative rings. Recall that a module over a ring is called an Artinian (resp., Noetherian) module if it satisfies descending (resp., ascending) chain condition on submodules, and a commutative ring is called an Artinian (resp., Noetherian) ring if it is an Artinian (resp., Noetherian) module over itself. Both the concepts are named for E. Artin (resp., E. Noether), which have been widely studied by many authors (see [5], [12], [13] and [20], for example), and hence it is one of the central subjects in the study of ring and module theory. Several authors have generalized these notions (see [1], [6], [9], [16] and [19], for example). As one of their crucial generalizations, many authors have studied graded Artinian and Noetherian modules (see [14], [17], [18] and [20]). Anderson and Dumitrescu [1] introduced S-Noetherian rings as a generalization of Noetherian rings and transferred many results. Let A be a commutative ring with identity and S a multiplicatively closed subset of A. Then A is called S-Noetherian if for any ideal I of A, there exist an $s \in S$ and a finitely generated ideal J of A such that $sI \subseteq J \subseteq I$. Later, as a generalization of the Artinian rings and modules, the S-Artinian rings and modules have been introduced and studied beautifully by Tekir et al. in [16] and [19]. An A-module M is called S-Artinian if for each descending chain $\{N_n\}_{n\in\mathbb{N}}$ of submodules of M, there exist an $s \in S$ and an index $j \geq 1$ such that $sN_j \subseteq N_i$ for every $i \geq j$. A commutative ring A is called S-Artinian if it is an S-Artinian module over itself, [16, 19]. They have extended many results on Artinian rings and modules to S-Artinian rings and modules. Moreover, several characterizations of S-Artinian modules are given in [16]. Recently, S-version of many special rings and modules has received some attention; see, for example, [1, 2, 6, 11, 16] and [19].

The theory of secondary representation has a history extending over more than fifty years. Macdonald [13] began the study of secondary representation for the Artinian modules and several authors have generalized this concept (see, [7], [8], [20] and [21], for example). This theory provides a useful tool for studying Artinian and injective modules. A nonzero A-module M is called secondary if for every $a \in A$, $aM = M$ or $a^nM = 0$ for some integer $n \geq 1$. An A-module M is called secondary representable (or representable) if it is a finite sum of secondary modules, [13]. Later, Sharp [20] introduced the concepts of graded secondary and graded secondary representable modules as a generalization of secondary and secondary representable modules to the graded case and used them as a tool for the study of asymptotic behavior of certain sets of attached prime ideals. Many authors have studied the graded secondary representations for the graded modules; see [7, 8, 20]. For example, graded secondary representations for graded injective and graded Artinian modules have been discussed in [7] and [20].

This paper discusses two objectives. Section 3 discusses the first objective of the paper which is devoted to extending the concepts of S-Artinian rings and modules to the graded case. We introduce the concepts of G-graded S-Artinian rings and modules as a generalization of S-Artinian rings and modules. Let G be an abelian group, A a commutative G-graded ring and S a multiplicatively closed subset of A consisting of homogeneous elements. We say that a G-graded A-module M is G-graded S-Artinian if for each descending chain $\{N_n\}_{n\in\mathbb{N}}$ of graded submodules of M, there exist an $s \in S$ and an index $j \geq 1$ such that $sN_j \subseteq N_i$ for every $i \geq j$. We show by Example 3.7 that this generalization is proper. We list a number of properties and characterizations of G-graded S-Artinian rings and modules. In particular, we characterize G-graded S-Artinian modules via graded S-MIN condition (Theorem 3.23). Also for study of the G-graded S-Artinian rings and modules deeply, we need S-versions of G-prime and G-maximal ideals. In [11], A. Hamed introduced the notion of S-prime ideals as a generalization of prime ideals and studied many properties of this class of ideals in S-Noetherian rings. We introduce and study the S-G-prime ideals as a generalization of the S-prime ideals to the graded case. Motivated by this, we also introduce and study the S-G-maximal ideals as a generalization of the G-maximal ideals. We transfer several properties of G-maximal ideals to S-G-maximal ideals (Proposition 3.35 and 3.37). Recall that an Artinian ring contains only finitely many maximal ideals and each prime ideal is maximal. We prove these results for the G-graded S-Artinian rings by using the concepts of S-G-prime and S-G-maximal ideals (Proposition 3.38 and Theorem 3.40).

Section 4 discusses the second objective of the paper which is devoted to the study of S-versions of the graded secondary and graded secondary representable modules. We introduce G-graded S-secondary and G-graded S-secondary representable modules as a generalization of the graded secondary and graded secondary representable modules. We give the basic properties of such modules and prove the existence of G-graded S-secondary representation in G-graded S-Artinian modules (Theorem 4.17) as a generalization of [20, Proposition 2.4] and [12, Theorem 1]. We investigate the extent to which this representation is unique, and introduce the notion of attached S-G-prime ideals which is a generalization of the attached prime ideals. A number of results on attached prime ideals have been generalized for the attached S-G-prime ideals (Theorem 4.11 and 4.14).

2 Preliminaries

Throughout this paper, G is an abelian group with identity e and all the rings are assumed to be commutative rings with identity unless otherwise stated.

A ring A is said to be G-graded if $A = \bigoplus_{g\in G} A_g$ for additive subgroups A_g and $A_gA_h \subseteq A_{gh}$ for all $g, h \in G$. The elements of the set $h(A) = \bigcup_{g\in G} A_g$ are said to be homogeneous. The grading is trivial if $A_g = 0$ for every non-identity $g \in G$. If A is a G-graded ring, then A_e is a subring of A containing 1_A. An ideal I of A is said to be graded if $I = \bigoplus_{g\in G}(I \cap A_g)$. Let $A = \bigoplus_{g\in G} A_g$ be a G-graded ring and M an A-module. Then M is said to be G-graded if $M = \bigoplus_{g\in G} M_g$ for additive subgroups M_g and $A_gM_h \subseteq M_{gh}$ for all $g, h \in G$. The elements of $h(M) = \bigcup_{g\in G} M_g$ are said to be homogeneous. If M is a G-graded A-module, then M_g is an A_e-module for every $g \in G$. Let N be a submodule of a G-graded A-module M. Then N is said to be graded if $N = \bigoplus_{g\in G}(N \cap M_g)$. Moreover, M/N becomes a G-graded A-module with $(M/N)_g = (M_g + N)/N$ for all $g \in G$. Let M and M' be G-graded A-modules. Then an A-module homomorphism $f : M \longrightarrow M'$ is said to be G-graded if $f(M_g) \subseteq M'_g$ for every $g \in G$.

Let A be a G-graded ring and M a G-graded A-module. A graded ideal P of A is said to be G-prime if $P \neq A$; and whenever $ab \in P$, we have $a \in P$ or $b \in P$, where $a, b \in h(A)$. The set of all G-prime ideals of A is denoted by $Spec^G(A)$. A proper graded ideal I of A is said to be

G-maximal if there is no proper graded ideal J of A such that $I \subset J$. A is called G-graded local if it has unique G-maximal ideal. A is called a G-graded field if each nonzero homogeneous element has a multiplicative inverse. M is said to be G-graded Artinian module if it satisfies the DCC on the graded submodules of M. A is said to be G-graded Artinian ring if it is G-graded Artinian module over itself. Let S be a multiplicatively closed subset of $h(A)$ containing 1_A. Then $S^{-1}M$ is a G-graded ring with $(S^{-1}M)_g = \{\frac{x}{s} : x \in M_\alpha, s \in S \cap A_\beta \textit{ with } g = \alpha\beta^{-1}\}$ for all $g \in G$. Similarly, $S^{-1}A$ is a G-graded ring called G-graded ring of fractions. The graded radical of a graded ideal I is $Gr(I) := \{a = \Sigma_{g \in G} a_g \in A : \textit{for every } g \in G, \textit{there exists an integer } n_g \geq 1 \textit{ such that } a_g^{n_g} \in I\}$. Clearly, $Gr(I)$ is a graded ideal of A. Also, we denote $\mathbb{N}$ by the set of all positive integers.

For more details of the graded rings and modules [10], [14], [17] and [18] are referred.

Proposition 2.1. *[17] Let A be a G-graded ring, M a G-graded A-module and N a graded submodule of M. Then $(N :_A M) := \{a \in A : aM \subseteq N\}$ is a graded ideal of A. In particular, $Ann(M) := (0 :_A M)$ is a graded ideal of A.*

Proposition 2.2. *[18] Let I and J be graded ideals in a G-graded ring A. Then $(I : J) := \{a \in A : aJ \subseteq I\}$ is a graded ideal of A. In particular, $(I : x) := \{a \in A : ax \in I\}$ is a graded ideal of A for every $x \in h(A)$.*

Definition 2.3. [20] Let M be a G-graded A-module. Then M is said to be G-graded secondary if $M \neq 0$ and for each $a \in h(A)$, either $aM = M$ or $a^n M = 0$ for some integer $n \geq 1$.

If M is a G-graded secondary A-module, then $P = Gr(Ann(M))$ is a G-prime ideal of A, and M is said to be *G-graded P-secondary* (see [20, Proposition 2.2]). We say that a G-graded A-module M has a *G-graded secondary representation* if it can be written as a sum $M = N_1 + N_2 + \cdots + N_r$ with each N_i G-graded P_i-secondary, where $P_i = Gr(Ann(N_i))$, [20].

Proposition 2.4. *[20, Proposition 2.4] Let M be a G-graded Artinian A-module. Then M has a G-graded secondary representation.*

Definition 2.5. [19, Definition 2.1, 2.2] Let M be an A-module and $S \subseteq A$ a multiplicatively closed subset. We say that M is an S-Artinian A-module if $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_n \supseteq \cdots$ is a descending chain of submodules of M, then there exist $s \in S$ and an index $j \geq 1$ such that $sN_j \subseteq N_i$ for all $i \geq j$. A ring A is said to be S-Artinian if it is an S-Artinian module over itself.

Definition 2.6. [11, Definition 1] Let A be a ring, $S \subseteq A$ a multiplicatively closed subset and P an ideal of A disjoint with S. Then P is said to be S-prime if there exists an $s \in S$ such that for every $a, b \in A$ with $ab \in P$, we have $sa \in P$ or $sb \in P$.

Let A be a ring and M an A-module. Following [13], M is called secondary if $M \neq 0$ and for every $a \in A$, $aM = M$ or $a^n M = 0$ for some integer $n \geq 1$. In this case $P = \sqrt{Ann(M)}$ is a prime ideal and M is called P-secondary. We say that M is secondary representable if there exist finitely many secondary submodules $N_1, N_2, \ldots, N_r$ such that $M = N_1 + N_2 + \cdots + N_r$. One may assume that the prime ideals $P_i = \sqrt{Ann(N_i)}$, $i = 1, 2, \ldots, r$, are all distinct. In this case the set $\{P_1, P_2, \ldots, P_r\}$ is independent from the choice of representation, and it is called the set of attached prime ideals of M and denoted by $Att(M)$. The Artinian modules are example of secondary representable module, [13].

Theorem 2.7. *[21, Theorem 1.10, 1.14] If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of secondary representable A-modules, then*

$$Att(M'') \subseteq Att(M) \subseteq Att(M') \cup Att(M'').$$

Corollary 2.8. *If $M_1, M_2, \ldots, M_k$ are secondary representable A-modules, then $\bigoplus_{i=1}^{k} M_i$ is secondary representable and*

$$Att(\bigoplus_{i=1}^{k} M_i) = \bigcup_{i=1}^{k} Att(M_i).$$

3 Graded S-Artinian Rings and Modules

In this section, we generalize the notion of S-Artinian rings and modules by introducing the concepts of G-graded S-Artinian rings and modules. To do so, we begin this section by introducing their definitions.

Definition 3.1. Let M be a G-graded A-module and $S \subseteq h(A)$ a multiplicatively closed subset. Then M is said to be a G-graded S-Artinian A-module if for every descending chain $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_n \supseteq \cdots$ of graded submodules of M, there exist $s \in S$ and an index $j \geq 1$ such that $sN_j \subseteq N_i$ for every $i \geq j$.

Definition 3.2. Let A be a G-graded ring and $S \subseteq h(A)$ a multiplicatively closed subset. Then A is said to be a G-graded S-Artinian ring if it is a G-graded S-Artinian module over itself.

Example 3.3. Every G-graded Artinian A-module is a G-graded S-Artinian A-module for every multiplicatively closed subset $S \subseteq h(A)$. The converse is also true if $S \subseteq U(h(A))$, where $U(h(A))$ denotes the set of all units of $h(A)$.

Example 3.4. Let M be a G-graded A-module and $S \subseteq h(A)$ a multiplicatively closed subset. If M is an S-Artinian module, then M is a G-graded S-Artinian module.

The converse of the Example 3.3 is not true in general. To see this, consider the following example.

Example 3.5. Let $G = \mathbb{Z}$, $A = \mathbb{Z} = A_0$ and $M = \mathbb{Z}_4^{(\mathbb{N})}$ (Direct sum of countable copies of $\mathbb{Z}_4$) be a naturally G-graded A-module. Take the multiplicatively closed subset $S = \{2^n : n \geq 0\}$. Put $s = 4$. Then M is a G-graded S-Artinian module since $sN = 0$ for every graded submodule N of M. However, it is easy to see that M is not a G-graded Artinian module.

Let A be a G-graded ring, S a multiplicatively closed subset of $h(A)$ and M a G-graded A-module. It is clear from the definition that M with

$$Gr(Ann(M)) \cap S \neq \phi$$

is trivially a G-graded S-Artinian A-module. Thus, from now, we assume that $Gr(Ann(M)) \cap S = \phi$ in this work. The previous example may seem a bit unfair since $Gr(Ann(M)) \cap S = \phi$ does not hold in that case. The following example presents a G-graded S-Artinian module that fulfills the above condition.

Example 3.6. Let $G = \mathbb{Z}_2$, $A = \mathbb{Z} = A_{\bar{0}}$, and $M = \mathbb{Z}_k[x] \oplus \mathbb{Z}(p^\infty)$ a G-graded A-module with $M_{\bar{0}} = \mathbb{Z}_k[x] \oplus 0$, $M_{\bar{1}} = 0 \oplus \mathbb{Z}(p^\infty)$ where $k \geq 1$ and $\mathbb{Z}(p^\infty) = \{\frac{a}{p^n} + \mathbb{Z} : a, n \in \mathbb{Z}, n \geq 0\}$, the p-Prüfer group for a fixed prime p. Notice that the graded submodules of M are the submodules of the form $N \oplus N'$ where N is a submodule of $\mathbb{Z}_k[x]$ and N' is a submodule of $\mathbb{Z}(p^\infty)$. Consider a multiplicatively closed subset $S = \{k^n : n \geq 0\}$ of A. Here we note that $S \cap Gr(Ann(M)) = \phi$ since $k^n M \neq 0$ for every integer $n \geq 0$. Let $\{N_i \oplus N_i'\}_{i \in \mathbb{N}}$ be a descending chain of graded submodules of M where N_i and N_i' are submodules of $\mathbb{Z}_k[x]$ and $\mathbb{Z}(p^\infty)$, respectively. Now, N_i' being a submodule of $\mathbb{Z}(p^\infty)$ implies that each N_i' is finite if it is proper (see [5, p. 74]). In particular, N_1' is finite if it is proper; whence there exists a positive integer j such that $N_j' = N_i'$ for all $i \geq j$. Put $s = k$. Then $s(N_j \oplus N_j') = 0 \oplus sN_j' \subseteq N_i \oplus N_i'$ for all $i \geq j$. This holds also if $N_1' = \mathbb{Z}(p^\infty)$. Thus, M is a G-graded S-Artinian A-module. However, it is easy to see that M is not a G-graded Artinian A-module since $\mathbb{Z}_k[x]$ is not an Artinian A-module. Moreover, this example remains valid if the p-Prüfer group $\mathbb{Z}(p^\infty)$ is replaced by any Artinian abelian group.

Let A be a G-graded ring and S a multiplicatively closed subset of A. If A is S-Artinian and $S \nsubseteq h(A)$, then obviously A is not a G-graded S-Artinian ring. However, if $S \subseteq h(A)$ and A is S-Artinian, then A is a G-graded S-Artinian ring. Thus the notion of G-graded S-Artinian rings generalizes the notion of S-Artinian rings. The following example shows that this generalization is proper in the sense that there is a G-graded S-Artinian ring which is not an S-Artinian ring.

Example 3.7. Let F be a field, $G = \mathbb{Z}$ and $A = F[x, x^{-1}]$ be the G-graded Laurent polynomial ring with $A_n = Fx^n$ for every $n \in \mathbb{Z}$. Then A is a G-graded field (see [10, p. 23]), and hence A is a G-graded Artinian ring (see also [20, p. 215]). So by Example 3.3, A is a G-graded S-Artinian ring for every multiplicatively closed subset $S \subseteq h(A)$. However, A is not an S-Artinian ring. Indeed, let $S = \{x^n : n \geq 0\}$ and $I = (x+1)F[x, x^{-1}]$. Then for a descending chain of ideals $I \supseteq I^2 \supseteq \cdots \supseteq I^n \supseteq \cdots$ in A, there do not exist $s \in S$ and integer $j \geq 1$ such that $sI^j \subseteq I^i$ for all $i \geq j$.

The above example also shows that the concepts of G-graded S-Artinian rings and modules are different from S-Artinian rings and modules. Now we study the basic properties of G-graded S-Artinian rings and modules.

Definition 3.8. [7, Definition 3.3] Let M be a G-graded A-module. A graded submodule N of M is said to be graded pure if $IN = N \cap IM$ for every graded ideal I of A.

Proposition 3.9. *Let M be a G-graded A-module. Then we have the following.*

(i) If $S_1 \subseteq S_2 \subseteq h(A)$ are multiplicatively closed subsets and M is a G-graded S_1-Artinian A-module, then M is a G-graded S_2-Artinian A-module.

(ii) If S^* is the saturation of multiplicatively closed subset $S \subseteq h(A)$, then M is a G-graded S-Artinian A-module if and only if M is a G-graded S^*-Artinian A-module.

(iii) If $S \subseteq h(A)$ is a multiplicatively closed subset and M is a G-graded S-Artinian A-module. Then $S^{-1}M$ is a G-graded Artinian A-module.

(iv) Let M be a G-graded A-module such that its each graded submodule is graded pure. Let $X = \{a \in h(A) : aM = M\}$ and $S \subseteq X$ a multiplicatively closed subset. Then M is a G-graded Artinian module if and only if M is a G-graded S-Artinian module.

Proof. (i) Clear.

(ii) The necessity is clear since $S \subseteq S^*$. Conversely, suppose M is a G-graded S^*-Artinian A-module. Let $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_n \supseteq \cdots$ be a descending chain of graded submodules of M. Since M is G-graded S^*-Artinian, there exist $s^* \in S^*$ and an integer $j \geq 1$ such that $s^*N_j \subseteq N_i$ for all $i \geq j$. Since $\frac{s^*}{1}$ is a unit in $S^{-1}A$, then $\frac{s^*}{1}\frac{a}{s} = 1$ for some $s \in S$ and $a \in h(A)$. This implies that there exists $t \in S$ such that $t(s^*a - s) = 0$, and so $ts^*a = ts$. Put $s' = ts$. Then $s' \in S$ and $s'N_j = tas^*N_j \subseteq s^*N_j \subseteq N_i$ for all $i \geq j$. Hence, M is a G-graded S-Artinian A-module.

(iii) Let $S^{-1}N_1 \supseteq S^{-1}N_2 \supseteq \cdots \supseteq S^{-1}N_n \supseteq \cdots$ be a descending chain of graded submodules of $S^{-1}M$, where each N_i is a graded submodule of M. Then $\pi^{-1}(S^{-1}N_1) \supseteq \pi^{-1}(S^{-1}N_2) \supseteq \cdots \supseteq \pi^{-1}(S^{-1}N_n) \supseteq \cdots$ is a descending chain of graded submodules of M, where $\pi : M \longrightarrow S^{-1}M$ is a natural graded homomorphism given by $\pi(m) = \frac{m}{1}$. Since M is G-graded S-Artinian, there exist $s \in S$ and an integer $j \geq 1$ such that $s\pi^{-1}(S^{-1}N_j) \subseteq \pi^{-1}(S^{-1}N_i)$ for all $i \geq j$. This implies that $\pi(s\pi^{-1}(S^{-1}N_j)) \subseteq \pi(\pi^{-1}(S^{-1}N_i))$, and so $\frac{s}{1}(S^{-1}N_j) \subseteq S^{-1}N_i$ for all $i \geq j$. But then $S^{-1}N_j \subseteq S^{-1}N_i$ for all $i \geq j$ since $\frac{s}{1}$ is a unit in $S^{-1}A$, and therefore $S^{-1}N_j = S^{-1}N_i$ for all $i \geq j$. Hence, $S^{-1}M$ is a G-graded Artinian A-module.

(iv) The necessity is clear (see Example 3.3). Conversely, suppose M is a G-graded S-Artinian module. Let $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_n \supseteq \cdots$ be a descending chain of graded submodules of M. Since M is G-graded S-Artinian, there exist $s \in S$ and an integer $j \geq 1$ such that $sN_j \subseteq N_i$ for all $i \geq j$. Also since each N_j is graded pure and $sM = M$, we have $sN_j = N_j \cap sM = N_j \cap M = N_j$. Therefore we have $N_j \subseteq N_i$ for all $i \geq j$, and we conclude that $N_j = N_i$ for all $i \geq j$. Hence, M is a G-graded Artinian A-module.

□

The following corollary is an immediate consequence of the Proposition 3.9.

Corollary 3.10. *Let A be a G-graded ring.*

(i) If $S_1 \subseteq S_2 \subseteq h(A)$ are multiplicatively closed subsets and A is a G-graded S_1-Artinian ring, then A is a G-graded S_2-Artinian ring.

(ii) If S^* is the saturation of multiplicatively closed subset $S \subseteq h(A)$, then A is a G-graded S-Artinian ring if and only if A is a G-graded S^*-Artinian ring.

(iii) If $S \subseteq h(A)$ is a multiplicatively closed subset and A is a G-graded S-Artinian ring. Then $S^{-1}A$ is a G-graded Artinian ring.

Proposition 3.11. *Let $f : M \longrightarrow M'$ be a G-graded A-homomorphism and $S \subseteq h(A)$ a multiplicatively closed subset.*

(i) If f is injective and M' is G-graded S-Artinian, then M is a G-graded S-Artinian A-module.

(ii) If f is surjective and M is G-graded S-Artinian, then M' is a G-graded S-Artinian A-module.

Proof. (i) Let $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_n \supseteq \cdots$ be a descending chain of graded submodules of M. Notice that each $f(N_i)$ is a graded submodule of M' since f is G-graded. Therefore $f(N_1) \supseteq f(N_2) \supseteq \cdots \supseteq f(N_n) \supseteq \cdots$ is a descending chain of graded submodules of M'. Since M' is G-graded S-Artinian, there exist $s \in S$ and an integer $j \geq 1$ such that $sf(N_j) \subseteq f(N_i)$ for all $i \geq j$. This immediately shows that $sN_j = f^{-1}(f(sN_j)) \subseteq f^{-1}(f(N_i)) = N_i$ for all $i \geq j$ since f is injective. Hence, M is a G-graded S-Artinian module.

(ii) Let $N'_1 \supseteq N'_2 \supseteq \cdots \supseteq N'_n \supseteq \cdots$ be a descending chain of graded submodules of M'. Evidently, each $f^{-1}(N'_i)$ is a graded submodule of M since f is G-graded. Therefore $f^{-1}(N'_1) \supseteq f^{-1}(N'_2) \supseteq \cdots \supseteq f^{-1}(N'_n) \supseteq \cdots$ is a descending chain of graded submodules of M. Since M is G-graded S-Artinian, there exist $s \in S$ and an integer $j \geq 1$ such that $sf^{-1}(N'_j) \subseteq f^{-1}(N'_i)$ for all $i \geq j$. But then we have $sN'_j = sf(f^{-1}(N'_j)) = f(sf^{-1}(N'_j)) \subseteq f(f^{-1}(N'_i)) = N'_i$ for all $i \geq j$ since f is surjective. Hence, M' is a G-graded S-Artinian module.

□

As an immediate consequence, we have the following result.

Corollary 3.12. *Let M be a G-graded S-Artinian A-module and N be its graded submodule. Then N and M/N are G-graded S-Artinian A-modules.*

Theorem 3.13. *Let M be a G-graded A-module, $S \subseteq h(A)$ a multiplicatively closed subset and N a graded submodule of M. Then M is a G-graded S-Artinian A-module if and only if N and M/N are G-graded S-Artinian A-modules.*

Proof. The necessary part follows from Corollary 3.12. Conversely, suppose N and M/N are G-graded S-Artinian A-modules. Let $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_n \supseteq \cdots$ be a descending chain of graded submodules of M. Since sum and quotient of graded submodules are graded, therefore we have two descending chains $N_1 \cap N \supseteq N_2 \cap N \supseteq \cdots \supseteq N_n \cap N \supseteq \cdots$ and $(N_1 + N)/N \supseteq (N_2 + N)/N \supseteq \cdots \supseteq (N_n + N)/N \supseteq \cdots$ of graded submodules of N and M/N, respectively. Since N and M/N are G-graded S-Artinian, there exist $s', s'' \in S$ and positive integers j', j'' such that $s'(N_{j'} \cap N) \subseteq N_i \cap N$ and $s''N_{j''} + N \subseteq N_i + N$ for all $i \geq j'$ and $i \geq j''$, respectively. Put $s = s's''$ and j = maximum of j' and j'', then $s(N_j \cap N) \subseteq N_i \cap N$ and $sN_j \subseteq N_i + N$ for all $i \geq j$. Let $x \in N_j$. For each $i \geq j$, write $sx = x_i + y_i$ for some $x_i \in N_i$ and $y_i \in N$. This implies that $y_i = sx - x_i \in N_j$ for all $i \geq j$, in fact, $y_i \in N_j \cap N$ for all $i \geq j$, and so $sy_i \in s(N_j \cap N) \subseteq N_i \cap N$ for all $i \geq j$. Consequently, $s^2x = sx_i + sy_i \in N_i$ for all $i \geq j$; whence $s^2N_j \subseteq N_i$ for all $i \geq j$. Therefore, M is a G-graded S-Artinian module. □

The following results are immediate consequences of the Theorem 3.13.

Corollary 3.14. *Let A be a G-graded ring, $S \subseteq h(A)$ a multiplicatively closed subset and M_1, M_2 be G-graded A-modules. Then $M_1 \oplus M_2$ is a G-graded S-Artinian A-module if and only if M_1 and M_2 are G-graded S-Artinian A-modules.*

Corollary 3.15. *Let A be a G-graded S-Artinian ring and M a finitely generated G-graded A-module. Then M is a G-graded S-Artinian A-module.*

Proof. Follows from Corollary 3.14 and Theorem 3.13.

□

Let M be a G-graded A-module and H a subgroup of G. Following [14],

$$A_H := \bigoplus_{h \in H} A_h$$

is an H-graded ring. In fact A_H is a G-graded ring. Also, let $g \in G$ and gH be coset of H in G, then

$$M_{gH} := \bigoplus_{h \in H} M_{gh}$$

is a G-graded A_H-submodule of M. In particular, M_H is a G-graded A_H-module. Now we characterize G-graded S-Artinian modules. For this, we need the following known lemma.

Lemma 3.16. *[14, Lemma 5.4.1] Let M be a G-graded A-module, H a subgroup of G and $g \in G$. Consider A_H as a G-graded ring, and let N be a graded A_H-submodule of M_{gH}. If AN is the graded A-submodule of M generated by N, then*

$$AN \cap M_{gH} = N.$$

Theorem 3.17. *Let A be a G-graded ring, H a subgroup of G and $S \subseteq h(A_H)$ a multiplicatively closed subset. If M is a G-graded S-Artinian A-module, then M_{gH} is a G-graded S-Artinian A_H-module for every $g \in G$. Conversely, if $[G : H] < \infty$ and M_{gH} is G-graded S-Artinian A_H-module for every $g \in G$, then M is a G-graded S-Artinian A-module.*

Proof. Let $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_n \supseteq \cdots$ be a descending chain of graded A_H-submodules of M_{gH}. Then $AN_1 \supseteq AN_2 \supseteq \cdots \supseteq AN_n \supseteq \cdots$ is a descending chain of graded A-submodules of M. As M is G-graded S-Artinian A-module, there exist $s \in S$ and an index $j \geq 1$ such that $sAN_j \subseteq AN_i$ for every $i \geq j$. By Lemma 3.16,

$$sN_j = sAN_j \cap M_{gH} \subseteq AN_i \cap M_{gH} = N_i$$

for every $i \geq j$. Thus M_{gH} is G-graded S-Artinian A_H-module for every $g \in G$. For the converse, write

$$M = \bigoplus_{g \in G} M_{gH}.$$

This direct sum is finite as $[G : H] < \infty$. Now since each M_{gH} is G-graded S-Artinian A_H-module, so by Corollary 3.14, M is a G-graded S-Artinian A_H-module, and so M is a G-graded S-Artinian A-module, as desired. □

Corollary 3.18. *Let G be a finite abelian group, A a G-graded ring, $S \subseteq A_e$ a multiplicatively closed subset and M a G-graded A-module. Then the following are equivalent:*

(i) M is a G-graded S-Artinian A-module.

(ii) M_g is an S-Artinian A_e-module for every $g \in G$.

(iii) M is an S-Artinian A_e-module.

(iv) M is an S-Artinian A-module.

Proof. (1) $\Longrightarrow$ (2): Follows from Theorem 3.17 for $H = \{e\}$.
(2) $\Longrightarrow$ (3): Follows from Corollary 3.14 since G is finite.
(3) $\Longrightarrow$ (4): It is obvious.
(4) $\Longrightarrow$ (1): Follows from Example 3.4.

□

Corollary 3.19. *Let A be a G-graded ring, H a subgroup of G and $S \subseteq h(A_H)$ a multiplicatively closed subset. If A is a G-graded S-Artinian ring, then A_H is an H-graded S-Artinian ring.*

Proof. It follows from Theorem 3.17 for $M = A$.

□

Remark 3.20. Let M be a G-graded A-module, $S \subseteq h(A)$ a multiplicatively closed subset. Let N be a submodule of M and N^* denotes the largest graded submodule of M contained in N. It is straightforward to see that if N is an S-Artinian submodule of M, then N^* is a G-graded S-Artinian submodule of M. It is also straightforward to see that if M/N is an S-Artinian module, then M/N^* is a G-graded S-Artinian module. From this, we can conclude that if M is S-Artinian A-module, then M is isomorphic to some quotient of a G-graded S-Artinian A-module.

Let A be a G graded ring and $a \in h(A)$. Then $S_a := \{a^n : n \geq 0\}$ is a multiplicatively closed subset of $h(A)$. Also, $U(A)$ denotes the set of all units of A. In the following theorem, we obtain a characterization of G-graded Artinian modules in terms of G-graded S-Artinian modules.

Theorem 3.21. *Let A be a G-graded ring which is not G-graded local and M be a G-graded A-module. Then the following are equivalent:*

(i) M is a G-graded Artinian A-module.

(ii) M is a G-graded S_a-Artinian A-module for every $a \in h(A) \setminus U(A)$.

Proof. $(i) \implies (ii)$: Follows from Example 3.3.

$(ii) \implies (i)$: Let $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_n \supseteq \cdots$ be a descending chain of graded submodules of M. Since M is G-graded S_a-Artinian for every $a \in h(A) \setminus U(A)$, there exist nonnegative integers l_a, k_a such that $a^{k_a} N_{l_a} \subseteq N_{i_a}$ for all $i_a \geq l_a$. In fact $a^{k_a} N_{l_a} \subseteq N_t$ for all $t \geq 1$. Let T be the ideal generated by the set $\{a^{k_a} : a \in h(A) \setminus U(A)\}$. Since T is generated by the homogeneous elements of A, so T is a graded ideal of A. If $T \neq A$, then there exists a G-maximal ideal L of A such that $T \subseteq L$. But then $a^{k_a} \in L$ for all $a \in h(A) \setminus U(A)$, and so $a \in L$ since L is a G-prime ideal. Consequently, $h(A) \setminus U(A) \subseteq L$ which shows that L is the unique G-maximal ideal of A, a contradiction. Hence $T = A$. This immediately shows that there exist $a_1, a_2, \ldots, a_r \in A$ and nonnegative integers $k_{a_1}, k_{a_2}, \ldots, k_{a_r}$ with $a_i^{k_{a_i}} N_{l_{a_i}} \subseteq N_t$ for all $t \geq 1$, $i = 1, 2, \ldots, r$ such that $1 = b_1 a_1^{k_{a_1}} + b_2 a_2^{k_{a_2}} + \cdots + b_r a_r^{k_{a_r}}$ for some $b_1, b_2, \ldots, b_r \in A$. Therefore, $A = \sum_{i=1}^{r} A a^{k_{a_i}}$. Now, let l be the maximum of $l_{a_1}, l_{a_2}, \ldots, l_{a_r}$, then $N_l \subseteq N_{l_{a_i}}$ for $i = 1, 2, \ldots, r$ which implies that $a_i^{k_{a_i}} N_l \subseteq a_i^{k_{a_i}} N_{l_{a_i}} \subseteq N_t$ for all $t \geq 1$, $i = 1, 2, \ldots, r$. Consequently, $N_l = (\sum_{i=1}^{r} A a^{k_{a_i}}) N_l = \sum_{i=1}^{r} A a^{k_{a_i}} N_l \subseteq N_t$ for all $t \geq 1$. But already we have $N_t \subseteq N_l$ for all $t \geq l$. Therefore, $N_l = N_t$ for all $t \geq l$, as desired. □

Our next result gives a characterization of G-graded S-Artinian modules. In fact, it is an S-version of a well-known characterization of G-graded Artinian modules via graded MIN condition. Recall that a G-graded module M satisfies *graded MIN condition* if each non-empty family of graded submodules of M has a minimal element with respect to inclusion of sets. Before stating the result, we define S-version of graded MIN condition.

Definition 3.22. Let M be a G-graded A-module and $S \subseteq h(A)$ a multiplicatively closed subset. Let X be a non-empty family of graded submodules of M. Then $N \in X$ is said to be S-minimal element of X if there exists $s \in S$ such that whenever $L \subseteq N$ for some $L \in X$, then $sN \subseteq L$. We say that M satisfies graded S-MIN condition if each non-empty family of graded submodules of M has an S-minimal element.

Theorem 3.23. *Let M be a G-graded A-module and $S \subseteq h(A)$ a multiplicatively closed subset. Then M is G-graded S-Artinian if and only if M satisfies graded S-MIN condition.*

Proof. Suppose M satisfies graded S-MIN condition. Let $X = \{N_i\}_{i \in \mathbb{N}}$ be a descending chain of graded submodules of M. Then X has a graded S-minimal element say N_j. Clearly, $N_i \subseteq N_j$ for all $i \geq j$. This implies that there exists $s \in S$ such that $sN_j \subseteq N_i$ for all $i \geq j$ since N_j is an S-minimal element of X. Hence, M is a G-graded S-Artinian module.
Conversely, suppose M is a G-graded S-Artinian A-module. Let X be an arbitrary non-empty set of graded submodules of M. Let X' be the set of those graded submodule N of M for which there exist $s \in S, N' \in X$ such that $sN' \subseteq N$. Evidently, $X \subseteq X'$. Now we prove that X' has a minimal element. For this, let $\{N_i\}_{i \in \mathbb{N}}$ be a descending chain in X'. Then there exist

$s' \in S$ and an integer $j \geq 1$ such that $s'N_j \subseteq N = \bigcap_{i\geq 1} N_i$ since M is G-graded S-Artinian. Also since $N_j \in X'$, there exist $N_j' \in X$ and $s'' \in S$ such that $s''N_j' \subseteq N_j$. This implies that $s's''N_j' \subseteq s'N_j \subseteq N$, and so $N \in X'$. Therefore by Zorn's lemma, X' has a minimal element say L. Consequently, there exist $L' \in X$ and $s \in S$ such that $sL' \subseteq L$ since $L \in X'$. Now we claim that L' is an S-minimal element of X. To see this, suppose $T \in X$ such that $T \subseteq L'$. This implies that $sT \subseteq sL' \subseteq L$, and so $sT \subseteq L \cap T$ which yields that $L \cap T \in X'$. But then $L \cap T = L$ since L is a minimal element of X'. Consequently, $L \subseteq T$, and so $sL' \subseteq L \subseteq T$ which conclude that L' is an S-minimal element of X. Hence, M satisfies graded S-MIN condition.

□

For the case when S consists of homogeneous units, we can recover the following well-known result.

Corollary 3.24. *Let M be a G-graded A-module. Then M is a G-graded Artinian module if and only if M satisfies graded MIN condition.*

Much of the theory of G-graded Artinian rings and modules are centered around the G-prime and G-maximal ideals. For example, a G-graded Artinian ring contains only finitely many G-maximal ideals, and each G-prime ideal is G-maximal. The attached G-prime ideals also play an important role in the study of G-graded Artinian modules. Thus, if we wish to obtain results analogous to G-graded Artinian rings and modules for G-graded S-Artinian rings and modules which depend upon G-prime and G-maximal ideals, then we need S-versions of G-prime and G-maximal ideals. S-version of prime ideals (namely, S-prime ideal) is already introduced and studied by Hamed et al. in [11]. To study the G-graded S-Artinian rings and modules in more detail, we introduce S-G-prime ideals as a generalization of S-prime ideals to the graded case.

Let A be a G-graded ring and S a multiplicatively closed subset of $h(A)$. A graded ideal I of A is said to be *S-proper* if $I \cap S = \phi$. It is straightforward to see that every S-proper graded ideal of A is a proper ideal but the converse need not be true.

Definition 3.25. Let A be a G-graded ring, $S \subseteq h(A)$ a multiplicatively closed subset and P an S-proper graded ideal of A. Then P is said to be an S-G-prime ideal if there exists an $s \in S$ such that for every $a, b \in h(A)$ with $ab \in P$, we have $sa \in P$ or $sb \in P$.

Remark 3.26. Let P be an S-G-prime ideal of a G-graded ring A. Let $a \in h(A)$ with $s'a \notin P$ for all $s' \in S$, and let $b \in A$ such that $ab \in P$. Then there exists an $s \in S$ such that $sb \in P$. Indeed, write $b = b_1 + b_2 + \cdots + b_n$ where each b_i is homogeneous. Evidently, $ab_1 + ab_2 + \cdots + ab_n = ab \in P$ which implies that $ab_i \in P$ for every i since P is a graded ideal of A. But then there exists an $s \in S$ such that $sb_i \in P$ for all i since P is S-G-prime. Consequently, $sb \in P$, as required.

Example 3.27. Let F be a field, $A = F[x, y]$ the polynomial ring over F and $G = \mathbb{Z}$. Then A is a G-graded ring with deg(x) = deg(y) = 1. Consider the multiplicatively closed subset $S = \{y^n : n \geq 0\}$ and a graded ideal $P = xyA$ of A. Put $s = y$. Then P is an S-G-prime ideal of A. Indeed, let $f, g \in h(A)$ such that $fg \in P \subseteq xA$. Then x divides f or x divides g; whence $sf \in P$ or $sg \in P$, as required.

Proposition 3.28. *Let A be a G-graded ring, $S \subseteq h(A)$ a multiplicatively closed subset. Let P be an S-prime ideal of A in non-graded case and P^* be the largest graded ideal contained in P. Then P^* is an S-G-prime ideal of A.*

Proof. Let $a, b \in h(A)$ such that $ab \in P^*$. Then $ab \in P$, and so there exists an $s \in S$ such that either $sa \in P$ or $sb \in P$. Since sa and sb are homogeneous elements of P and P^* is the largest graded ideal contained in P, we have $sa \in P^*$ or $sb \in P^*$, as desired. □

The following two propositions are the straightforward adaptations of the results of [11], and therefore we only indicate the relevant results of [11] and leave the details to the readers.

Proposition 3.29. *Let A be a G-graded ring, $S \subseteq h(A)$ a multiplicatively closed subset consisting of nonzero divisors and P an S-proper graded ideal of A. Then the following are equivalent:*

(i) P *is an* S*-*G*-prime ideal of* A*.*

(ii) $(P : s)$ *is a* G*-prime ideal of* A *for some* $s \in S$*.*

(iii) $S^{-1}P$ *is a* G*-prime ideal of* $S^{-1}A$ *and* $S^{-1}P \cap A = (P : s)$ *for some* $s \in S$*.*

Proof. Use the similar arguments as in the ungraded case (see [11, Proposition 1, Remark 1]). □

Proposition 3.30. *Let* A *be a* G*-graded ring,* $S \subseteq h(A)$ *a multiplicatively closed subset and* P *an* S*-proper graded ideal of* A*. Then* P *is* S*-*G*-prime if and only if there exists* $s \in S$ *such that for all graded ideal* I, J *of* A*, if* $IJ \subseteq P$*, then* $sI \subseteq P$ *or* $sJ \subseteq P$*.*

Proof. Use the similar arguments as in the ungraded case (see [11, Theorem 1]). □

Now we include some basic facts about S-G-prime ideals.

Remarks and Examples 3.31. (i) Let A be a G-graded ring and $S \subseteq h(A)$ a multiplicatively closed subset. Then an S-proper G-prime ideal of A is an S-G-prime ideal. The converse is not true in general. For instance, in Example 3.27, the graded ideal $P = xyA$ is an S-G-prime ideal of $A = F[x, y]$ but not a G-prime ideal.

(ii) An S-prime ideal of a G-graded ring A is not an S-G-prime ideal in general. Indeed, let $A = \mathbb{Z}[x]$ and $G = \mathbb{Z}_2$. Then A is a G-graded ring with $A_{\bar{0}} = \mathbb{Z} + \mathbb{Z}x^2 + \mathbb{Z}x^4 + \cdots$, $A_{\bar{1}} = \mathbb{Z}x + \mathbb{Z}x^3 + \mathbb{Z}x^5 + \cdots$. Consider the multiplicatively closed subset $S = \{2^n : n \geq 0\}$ and an ideal $P = 4(x + 1)\mathbb{Z}[x]$. Put $s = 4$. Then P is an S-prime ideal of A (see [11, Example 1(3)]). However, P is not an S-G-prime ideal since it is not a graded ideal.

(iii) An S-G-prime ideal of a G-graded ring A is not an S-prime ideal in general. Indeed, let $A = \mathbb{Z}[i]$ (The Gaussian integers) and $G = \mathbb{Z}_2$. Then A is a G-graded ring with $A_{\bar{0}} = \mathbb{Z}$, $A_{\bar{1}} = i\mathbb{Z}$. Consider the G-prime ideal $P = 2\mathbb{Z}[i]$ and multiplicatively closed subset $S = \{3^n : n \geq 0\}$. Clearly, P is an S-proper G-prime ideal of A. Then by (i), P is an S-G-prime ideal of A. On the other hand, P is not an S-prime ideal since $(1 + i)(1 - i) = 2 \in P$ but $3^n(1 + i) \notin P$ and $3^n(1 - i) \notin P$ for all $n \geq 1$.

(iv) Let P be an S-proper graded ideal of a G-graded ring A. Then P is an S-G-prime ideal of A if and only if $P[x]$ is an S-G-prime ideal of $A[x]$. We can prove it by using similar arguments as in the ungraded case (see [11, Example 4]). Here we note that the G-gradation of A can be extended to $A[x]$ by taking the components $(A[x])_g = A_g[x]$ for every $g \in G$.

(v) Following [18, Example 4], if P is a G-prime ideal of a G-graded ring A, then $Gr(P) = P$. However, it is not true for an S-G-prime ideal in general. Indeed, let $A = \mathbb{Z}[x]$, $G = \mathbb{Z}_2$ and $S = \{2^n : n \geq 0\}$. Then A is a G-graded ring, as in (ii). Also, $P = 4x\mathbb{Z}[x]$ is an S-prime ideal of A (see [11, Example 1(3)]). Then P is an S-G-prime ideal of A since it is graded. Here $Gr(P) = 2x\mathbb{Z}[x] \neq P$.

(vi) If P is an S-G-prime ideal of a G-graded ring A, then $Gr(P)$ is an S-G-prime ideal. Indeed, let $a, b \in h(A)$ such that $ab \in Gr(P)$. This implies that $(ab)^n \in P$ for some integer $n \geq 1$. Since P is S-G-prime, there exists an $s \in S$ such that either $sa^n \in P$ or $sb^n \in P$. Consequently, $sa \in Gr(P)$ or $sb \in Gr(P)$, as required.
The converse of this statement is not true in general. Indeed, let $A = \mathbb{Z}[i]$, $G = \mathbb{Z}_2$ and $S = \{3^n : n \geq 0\}$. Then A is a G-graded ring, as in (iii). Notice that $P = 4\mathbb{Z}[i]$ is an S-proper graded ideal of A. Clearly, $Gr(P) = 2\mathbb{Z}[i]$ which is an S-proper G-prime ideal and hence by (i), $Gr(P)$ is an S-G-prime ideal of A. On the other hand, P is not an S-G-prime ideal since $4 \in P$ but $2s \notin P$ for all $s \in S$.

(vii) If P is an S-G-prime ideal of a G-graded ring A, then there exists $s \in S$ such that $sGr(P) \subseteq P$. Indeed, let $a \in Gr(P)$ be a homogeneous element. Then $a^n \in P$ for some integer $n \geq 1$. Since P is S-G-prime, there exists an $s \in S$ such that $sa \in P$. Consequently, $sGr(P) \subseteq P$.

As noted before, we also need an S-version of the G-maximal ideals. Inspired by the S-G-prime ideals, we introduce and study the S-G-maximal ideals as a generalization of the G-maximal ideals.

Definition 3.32. Let A be a G-graded ring, $S \subseteq h(A)$ a multiplicatively closed subset and L an S-proper graded ideal of A. We say that L is an S-G-maximal ideal if there exists an $s \in S$ such that whenever $L \subseteq I$ for some graded ideal I of A, then either $sI \subseteq L$ or $sA \subseteq I$.

Let A be a G-graded ring and S a multiplicatively closed subset of $h(A)$. It can be easily seen by Zorn's lemma that every S-proper graded ideal of A is contained in an S-G-maximal ideal. Moreover, if L is an S-G-maximal ideal and I is an S-proper graded ideal of A such that $L \subseteq I$, then $sI \subseteq L$ for some $s \in S$.

Example 3.33. Let A be a G-graded ring and $S \subseteq h(A)$ a multiplicatively closed subset. If L is an S-proper G-maximal ideal of A, then L is an S-G-maximal ideal.

Let M be an A-module. Following [3], $A(+)M = A \oplus M$ with coordinate-wise addition and multiplication $(a_1, m_1)(a_2, m_2) = (a_1a_2, a_1m_2 + a_2m_1)$ is a commutative ring called the *idealization of* M.

The converse of Example 3.33 is not true in general. For this, consider the following example.

Example 3.34. Let $A = \mathbb{Z}(+)\mathbb{Z}_4$ be the idealization of the $\mathbb{Z}$-module $\mathbb{Z}_4$. Then A is a $G = \mathbb{Z}_2$-graded ring with $A_{\bar{0}} = \mathbb{Z} \oplus 0$, $A_{\bar{1}} = 0 \oplus \mathbb{Z}_4$. Consider the multiplicatively closed subset $S = \{(2,0)^n : n \geq 0\}$ and a graded ideal $L = 6\mathbb{Z}(+)\mathbb{Z}_4$ of A. Clearly, $L \cap S = \phi$ and $S \subseteq h(A)$. We show that L is an S-G-maximal ideal. For this, let I be a graded ideal of A such that $L \subseteq I$. Notice that I is of the form $m\mathbb{Z}(+)\mathbb{Z}_4$ for some integer m (see [3, p. 5]). Also, L being a subset of I implies that $6\mathbb{Z} \subseteq m\mathbb{Z}$; hence I is one of the $2\mathbb{Z}(+)\mathbb{Z}_4$, $3\mathbb{Z}(+)\mathbb{Z}_4$, $6\mathbb{Z}(+)\mathbb{Z}_4$ or $\mathbb{Z}(+)\mathbb{Z}_4$. Put $s = (2,0)$. Then it is easy to see that either $sI \subseteq L$ or $sA \subseteq I$ for all the cases of I. Thus, L is an S-G-maximal ideal of A. On the other hand, L is not a G-maximal ideal since L is contained in a proper graded ideal $3\mathbb{Z}(+)\mathbb{Z}_4$.

The following proposition collects some immediate properties of the S-G-maximal ideals of a G-graded ring.

Proposition 3.35. *Let A be a G-graded ring, $S \subseteq h(A)$ a multiplicatively closed subset and L an S-G-maximal ideal of A. Then*

(i) L is an S-G-prime ideal of A.

(ii) $S^{-1}L$ is a G-maximal ideal of $S^{-1}A$.

(iii) $Gr(L)$ is an S-G-maximal ideal of A such that $sGr(L) \subseteq L$ for some $s \in S$.

(iv) $(L : s)$ is a G-maximal ideal among all S-proper graded ideals of A for some $s \in S$.

Proof. (i) Let $a, b \in h(A)$ such that $ab \in L$. Clearly, $L \subseteq L + aA$. Since L is S-G-maximal, there exists $s \in S$ such that either $sA \subseteq L + aA$ or $s(L + aA) \subseteq L$. If $s(L + aA) \subseteq L$, then $sa \in L$. If $sA \subseteq L + aA$, then $sbA \subseteq bL + abA \subseteq L$ since $ab \in L$; hence $sb \in L$. Thus, L is an S-G-prime ideal of A.

(ii) Clearly $S^{-1}L$ is a proper graded ideal of $S^{-1}A$ since L is an S-proper graded ideal of A. Now, suppose $S^{-1}L \subseteq S^{-1}I$ for some graded ideal I of A and $S^{-1}I \neq S^{-1}A$. Then $\pi^{-1}(S^{-1}I)$ is an S-proper graded ideal of A and $\pi^{-1}(S^{-1}L) \subseteq \pi^{-1}(S^{-1}I)$, where $\pi : A \to S^{-1}A$ is a natural graded homomorphism given by $\pi(a) = \frac{a}{1}$. This implies that $L \subseteq \pi^{-1}(S^{-1}I)$. But then there exists $s \in S$ such that $s\pi^{-1}(S^{-1}I) \subseteq L$ since L is S-G-maximal. Consequently, $\frac{s}{1}S^{-1}I = \pi(s\pi^{-1}(S^{-1}I)) \subseteq \pi(L) \subseteq S^{-1}L$. This yields that $S^{-1}I \subseteq S^{-1}L$, and so $S^{-1}I = S^{-1}L$. Hence, $S^{-1}L$ is a G-maximal ideal of $S^{-1}A$.

(iii) Suppose $Gr(L) \subseteq I$ for some graded ideal I of A. This implies that $L \subseteq I$, and so there exists $s \in S$ such that either $sI \subseteq L \subseteq Gr(L)$ or $sA \subseteq I$; hence, $Gr(L)$ is an S-G-maximal ideal of A. Also, $sGr(L) \subseteq L$ for some $s \in S$ since $L \subseteq Gr(L)$ and $Gr(L)$ is an S-proper graded ideal.

(iv) Since L is S-G-maximal, there exists $s \in S$ such that if $L \subseteq I$ for some graded ideal I of A then either $sI \subseteq L$ or $sA \subseteq I$. By Proposition 2.2, $(L : s)$ is a graded ideal of A. If $(L : s) \cap S \neq \phi$, then there exists $s' \in S$ such that $ss' \in L$, absurd since L is S-proper. Thus $(L : s)$ is an S-proper graded ideal, and so a proper graded ideal of A. We show that $(L : s)$

is a G-maximal ideal among all S-proper graded ideals. For this, let $(L : s) \subseteq J$ for some S-proper graded ideal J of A. We need to show $(L : s) = J$. Clearly, $L \subseteq (L : s) \subseteq J$; whence either $sJ \subseteq L$ or $sA \subseteq J$ since L is S-G-maximal. But J being an S-proper graded ideal implies that $sA \nsubseteq J$, and so we conclude that $sJ \subseteq L$ which implies that $J \subseteq (L : s)$. Hence $(L : s) = J$, as desired.

□

The converse of Proposition 3.35(i) is not true in general. For this consider the following example.

Example 3.36. Let $A = \mathbb{Z}[x]$, $G = \mathbb{Z}$ and $S = \{2^n : n \geq 0\}$ a multiplicatively closed subset. Then A is a G-graded ring with deg(x) = 1. Take a graded ideal $P = 4x\mathbb{Z}[x]$ of A. By [11, Example 1(3)], P is an S-prime ideal of A; hence P is an S-G-prime ideal since it is graded. On the other hand, P is not S-G-maximal since $P \subseteq I = 4x\mathbb{Z}[x] + 3\mathbb{Z}[x]$ but $2^n A \nsubseteq I$ and $2^n I \nsubseteq P$ for any integer $n \geq 0$.

The next proposition gives a characterization of the S-G-maximal ideals in terms of G-maximal ideals .

Proposition 3.37. *Let A be a G-graded ring, $S \subseteq h(A)$ a multiplicatively closed subset and L an S-proper graded ideal of A. Then*

(i) L is an S-G-maximal ideal of A if and only if $S^{-1}L$ is a G-maximal ideal of $S^{-1}A$ and $S^{-1}L \cap A = (L : s)$ for some $s \in S$.

(ii) L is an S-G-maximal ideal of A if $(L : s)$ is a G-maximal ideal of A for some $s \in S$.

Proof. (i) Suppose L is an S-G-maximal ideal of A. Then by Proposition 3.35(ii), $S^{-1}L$ is a G-maximal ideal of $S^{-1}A$. Also, since L is S-G-maximal, there exists $s \in S$ such that if $L \subseteq I$ for some graded ideal I of A then either $sI \subseteq L$ or $sA \subseteq I$. We show that $S^{-1}L \cap A = (L : s)$. For this, let $a \in (L : s)$. Then $sa \in L$, and so $\frac{a}{1} \in S^{-1}L$. Thus $a \in S^{-1}L \cap A$, and so $(L : s) \subseteq S^{-1}L \cap A$. For the reverse containment, let $a \in S^{-1}L \cap A$. Since L is S-G-maximal and $L \subseteq L + aA$, we have either $sA \subseteq L + aA$ or $s(L + aA) \subseteq L$. If $sA \subseteq L + aA$, then $S^{-1}A = S^{-1}(sA) \subseteq S^{-1}(L + aA) = S^{-1}L + S^{-1}(aA)$, and so $S^{-1}A = S^{-1}L$ since $a \in S^{-1}L$, absurd because $S^{-1}L$ is G-maximal. Therefore, $s(L + aA) \subseteq L$ which implies that $sa \in L$; hence $a \in (L : s)$. Thus $S^{-1}L \cap A \subseteq (L : s)$, as required.
Conversely, suppose $S^{-1}L$ is a G-maximal ideal of $S^{-1}A$ and $S^{-1}L \cap A = (L : s)$ for some $s \in S$. Notice that L is S-proper since $S^{-1}L$ is a proper graded ideal. Now let $L \subseteq I$ for some S-proper graded ideal I of A. Then $S^{-1}L \subseteq S^{-1}I$. Since $S^{-1}L$ is a G-maximal ideal of $S^{-1}A$ and $I \cap S = \phi$, we obtain that $S^{-1}L = S^{-1}I$. This implies that $I \subseteq S^{-1}I \cap A = S^{-1}L \cap A = (L : s)$; hence $sI \subseteq L$. Thus L is an S-G-maximal ideal of A.

(ii) Suppose $(L : s)$ is a G-maximal ideal of A for some $s \in S$. Clearly, $(L : s)$ is S-proper since L is S-proper. Let $L \subseteq I$ for some graded ideal I of A. Then $(L : s) \subseteq (I : s)$ which implies that either $(I : s) = A$ or $(L : s) = (I : s)$ since $(L : s)$ is G-maximal. If $(I : s) = A$, then $sA \subseteq I$. If $(L : s) = (I : s)$, then $sI \subseteq L$. Hence, L is an S-G-maximal ideal of A.

□

Now we are in a position to explore the analogous properties of G-graded Artinian rings and modules depending upon G-prime and G-maximal ideals for the G-graded S-Artinian rings and modules. It is well known that in a G-graded Artinian ring, each G-prime ideal is G-maximal. The following proposition generalizes this result for the G-graded S-Artinian rings.

Proposition 3.38. *Let A be a G-graded S-Artinian ring, where $S \subseteq h(A)$ is a multiplicatively closed subset. Then each S-G-prime ideal of A is an S-G-maximal ideal.*

Proof. Let P be an S-G-prime ideal of A. Suppose $P \subseteq I$ for some S-proper graded ideal I of A. Since P is S-G-prime, there exists an $s \in S$ such that for all $a, b \in h(A)$ with $ab \in P$, either $sa \in P$ or $sb \in P$. We claim that $sI \subseteq P$. On contrary, suppose $sI \nsubseteq P$. Let $x \in sI \setminus P$ be a homogeneous element. Consider the descending chain $P + Ax \supseteq P + Ax^2 \supseteq P + Ax^3 \supseteq \cdots \supseteq P + Ax^n \supseteq \cdots$ of graded ideals of A. Since A is G-graded S-Artinian, there exist $s' \in S$ and an integer $j \geq 1$ such that $s'(P + Ax^j) \subseteq P + Ax^{j+1}$. Let $y \in P$. Then $s'(y + x^j) = y' + x^{j+1}z$ for some $y' \in P$ and $z \in A$. This implies that $x^j(s' - xz) = y' - s'y \in P$. Write $x = sw$ for some homogeneous $w \in I$. Then $sw^j \notin P$ since $x = sw \notin P$ and P is an S-G-prime ideal. Consequently, $sx^j = s^{j+1}w^j \notin P$. Since P is S-G-prime and x^j is a homogeneous element with $sx^j \notin P$, then by Remark 3.26, we have $s(s' - zx) \in P \subseteq I$ which implies that $s^2(s' - zx) \in sI$. But then $s's^2 \in sI \subseteq I$ since $s^2zx \in sI$. This is a contradiction since I is S-proper. Hence $sI \subseteq P$, and therefore P is an S-G-maximal ideal of A. □

In the sequel of generalizing the results of G-graded Artinian rings and modules to G-graded S-Artinian rings and modules, our next result generalizes the fact that G-graded Artinian rings have only finitely many G-maximal ideals.

Lemma 3.39. *Let A be a G-graded S-Artinian ring, where $S \subseteq h(A)$ is a multiplicatively closed subset. Then A has only finitely many S-proper G-prime ideals.*

Proof. Use the similar arguments as in the ungraded case (see [19, Theorem 2.4]). □

Theorem 3.40. *Let A be a G-graded S-Artinian ring. Then there exist finitely many S-G-maximal ideals $L_1, L_2, \ldots, L_r$ of A such that if L is any S-G-maximal ideal of A, then there exists an $s \in S$ such that $sL \subseteq L_j$ and $sL_j \subseteq L$ for some j.*

Proof. Consider the set $X = \{S^{-1}L : L$ is an S-G-maximal ideal of $A\}$. Then by Proposition 3.37(i), $X \subseteq Spec^G(S^{-1}A)$. By Lemma 3.39, $S^{-1}A$ has only finitely many G-prime ideals which implies that X is a finite set. Consequently, there exist only finitely many S-G-maximal ideals $L_1, L_2, \ldots, L_r$ in A such that $S^{-1}L_i \neq S^{-1}L_j$ if $i \neq j$.
Let L be an S-G-maximal ideal of A. If $L = L_i$ for some $i = 1, 2, \ldots, r$, then we are done. Suppose $L \neq L_i$ for any i. Then obviously $S^{-1}L = S^{-1}L_j$ for some j. By Proposition 3.37(1), $S^{-1}L \cap A = (L : s)$ and $S^{-1}L_j \cap A = (L_j : s')$ for some $s, s' \in S$. But then $(L : s) = (L_j : s')$ since $S^{-1}L = S^{-1}L_j$. This yields that $s'L \subseteq L_j$ and $sL_j \subseteq L$. Consequently, $ss'L \subseteq L_j$ and $ss'L_j \subseteq L$, as desired. □

Corollary 3.41. *Let A be a G-graded Artinian ring. Then A has only a finite number of G-maximal ideals.*

Proof. Follows form Theorem 3.40 for $S = \{1\}$. □

Corollary 3.42. *Let A be an Artinian ring. Then A has only a finite number of maximal ideals.*

Proof. Follows form Corollary 3.41 for $G = \{e\}$. □

4 Graded S-Secondary Modules and Graded S-Secondary Representations

It is known that the class of G-graded Artinian modules is one of the examples of the existence of graded secondary representation (Proposition 2.4). As we have transferred several results on G-graded Artinian modules to G-graded S-Artinian modules so far; in that sequence, if we wish to obtain analogous representation in G-graded S-Artinian modules, we must replace G-graded secondary modules by their S-version, namely, G-graded S-secondary modules. We begin this section by introducing its definition.

Definition 4.1. Let M be a G-graded A-module and $S \subseteq h(A)$ a multiplicatively closed subset. Then M is said to be G-graded S-secondary if $M \neq 0$ and there exists $s \in S$ such that for each $a \in h(A)$, either $sM \subseteq aM$ or $(sa)^n M = 0$ for some integer $n \geq 1$. By a G-graded S-secondary submodule, we mean a graded submodule which is also a G-graded S-secondary module.

If $S \subseteq h(A)$ is a multiplicatively closed subset such that $0 \in S$, then every G-graded A-module M is G-graded S-secondary. More generally, if $S \cap Gr(Ann(M)) \neq \phi$, then M is trivially a G-graded S-secondary A-module. Thus, from now, we assume that $Gr(Ann(M)) \cap S = \phi$ in this work.

Proposition 4.2. *Let M be a G-graded S-secondary A-module. Then $P = Gr(Ann(M))$ is an S-G-prime ideal of A, and we say that M is G-graded S-P-secondary.*

Proof. Let $a, b \in h(A)$ such that $ab \in P = Gr(Ann(M))$ and $sa \notin P$ for all $s \in S$. Then $(ab)^m M = 0$ for some integer $m \geq 1$ and $(sa)^n M \neq 0$ for all $s \in S$ and integers $n \geq 1$. But then there exists $s' \in S$ such that $s'M \subseteq aM$ since M is G-graded S-secondary. This implies that $(s'b)^m M \subseteq (ab)^m M = 0$, and so $s'b \in P = Gr(Ann(M))$. Also, P is S-proper since $Gr(Ann(M)) \cap S = \phi$. Hence $P = Gr(Ann(M))$ is an S-G-prime ideal of A. □

Now we include some basic facts about G-graded S-secondary modules.

Remarks and Examples 4.3. (i) Every G-graded secondary A-module is a G-graded S-secondary A-module for every multiplicatively closed subset $S \subseteq h(A)$. The converse is also true if $S \subseteq U(h(A))$, where $U(h(A))$ denotes the set of all units of $h(A)$.

(ii) A G-graded S-secondary module is not a G-graded secondary module in general. Indeed, let $G = \mathbb{Z}$, $A = \mathbb{Z} = A_0$ a G-graded ring and $M = \mathbb{Z}_6[x]$ a G-graded A-module with deg(x) = 1. Consider a multiplicatively closed subset $S = \{6^n : n \geq 0\}$. Put $s = 6$. Then $0 = sM \subseteq aM$ for all $a \in h(A) = \mathbb{Z}$; hence M is a G-graded S-secondary module. On the other hand, M is not a G-graded secondary module since $3M \neq M$ and $3^n M \neq 0$ for all integers $n \geq 1$.

(iii) The converse of Proposition 4.2 is not true in general. Indeed, let $G = \mathbb{Z}_2$, $A = \mathbb{Z} = A_{\bar{0}}$ and $M = \mathbb{Z}[i]$ a G-graded A-module with $M_{\bar{0}} = \mathbb{Z}$, $M_{\bar{1}} = i\mathbb{Z}$. Take the multiplicatively closed subset $S = \{2^n : n \geq 0\}$. Then $Gr(Ann(M)) = 0$, a G-prime ideal, and so an S-G-prime ideal of A. However, M is not G-graded S-secondary since $sM \nsubseteq 3M$ and $(3s)^n M \neq 0$ for all integers $n \geq 1$ and $s \in S$.

(iv) Let $M = \mathbb{Z}_6[x]$ be a G-graded A-module, as in (ii). Take the multiplicatively closed subset $S = \{5^n : n \geq 0\}$. Then $Gr(Ann(M)) = 6\mathbb{Z}$ is not an S-G-prime ideal of A since $6 \in 6\mathbb{Z}$ but $3s \notin 6\mathbb{Z}$ and $2s \notin 6\mathbb{Z}$ for any $s \in S$. Hence by Proposition 4.2, M is not G-graded S-secondary.

(v) Let M be a G-graded A-module. Let $X = \{a \in h(A) : aM = M\}$ and $S \subseteq X$ a multiplicatively closed subset. Then it is straightforward to see that M is G-graded S-secondary if and only if it is G-graded secondary.

Now we study the basic properties of the G-graded S-secondary modules, especially under localization and homomorphism.

Proposition 4.4. *Let M be a G-graded A-module and $S \subseteq h(A)$ a multiplicatively closed subset.*

(i) If N_1 and N_2 are G-graded S-P-secondary submodules of M. Then $N_1 + N_2$ is a G-graded S-P-secondary submodule of M.

(ii) If M is a G-graded S-secondary module, then $S^{-1}M$ is a G-graded secondary $S^{-1}A$-module.

Proof. (i) Let $N = N_1 + N_2$. Then $Gr(Ann(N)) = Gr(Ann(N_1)) \cap Gr(Ann(N_2)) = P$ since $P = Gr(Ann(N_1)) = Gr(Ann(N_2))$. Now, let $a \in h(A)$. Suppose $(sa)^n N \neq 0$ for all $s \in S$ and $n \geq 1$. Then $sa \notin P = Gr(Ann(N))$ for every $s \in S$. Consequently, $(sa)^n N_1 \neq 0$ and $(sa)^n N_2 \neq 0$ for all $s \in S$ and $n \geq 1$. This implies that there exist $s_1, s_2 \in S$ such that $s_1 N_1 \subseteq aN_1$ and $s_2 N_2 \subseteq aN_2$ since N_1 and N_2 are G-graded S-secondary. Put $s' = s_1 s_2$. Then $s'N = s'N_1 + s'N_2 \subseteq aN_1 + aN_2 = aN$. Hence, N is a G-graded S-P-secondary submodule of M.

(ii) Let $\frac{a}{t} \in h(S^{-1}A)$, where $a \in h(A)$ and $t \in S$. Since M is G-graded S-secondary, there exist $s \in S$ and an integer $n \geq 1$ such that $(sa)^n M = 0$ or $sM \subseteq aM$. Suppose $(sa)^n M = 0$. Let $\frac{x}{t'} \in S^{-1}M$, where $x \in M$ and $t' \in S$. Then $s^n(a^n x) = (sa)^n x = 0$; whence $\frac{a^n x}{1} = 0$, and so $(\frac{a}{t})^n \frac{x}{t'} = 0$. This implies that $(\frac{a}{t})^n S^{-1}M = 0$.
Now, suppose $sM \subseteq aM$. Let $\frac{x}{t'} \in S^{-1}M$, where $x \in M$ and $t' \in S$. Write $sx = ay$ for some $y \in M$. This implies that $\frac{sx}{1} = \frac{ay}{1}$ in $S^{-1}M$, and so $\frac{x}{t'} = (\frac{a}{t})(\frac{ty}{st'}) \in \frac{a}{t} S^{-1}M$. Consequently, $\frac{a}{t} S^{-1}M = S^{-1}M$. Hence, $S^{-1}M$ is a G-graded secondary $S^{-1}A$-module. □

Proposition 4.5. *Let $f : M \longrightarrow M'$ be a G-graded A-homomorphism and $S \subseteq h(A)$ a multiplicatively closed subset.*

(i) If f is injective and M' is G-graded S-P-secondary such that $f(M)$ is a graded pure submodule of M', then M is G-graded S-P-secondary.

(ii) If f is surjective and M is G-graded S-P-secondary, then M' is G-graded S-P-secondary.

Proof. (i) Let $a \in h(A)$. If $a \in P = Gr(Ann(M'))$, then $f(a^n M) = a^n f(M) \subseteq a^n M' = 0$ for some integer $n \geq 1$. Consequently, $a^n M = 0$ since f is injective. If $sa \notin P$ for all $s \in S$, then there exists $s' \in S$ such that $s'M' \subseteq aM'$ since M' is G-graded S-P-secondary. This implies that $s'f(M) = f(M) \cap s'M' \subseteq f(M) \cap aM' = af(M)$ since $f(M)$ is a graded pure submodule of M'. Therefore $s'M \subseteq aM$ since f is injective. Hence, M is a G-graded S-P-secondary module.

(ii) Let $a \in h(A)$. If $a \in P = Gr(Ann(M))$, then $a^n M = 0$ for some $n \geq 1$. This implies that $a^n M' = a^n f(M) = f(a^n M) = 0$ since f is surjective. If $sa \notin P$ for all $s \in S$, then there exists $s' \in S$ such that $s'M \subseteq aM$ since M is G-graded S-P-secondary. This implies that $s'M' = s'f(M) = f(s'M) \subseteq f(aM) = af(M) = aM'$ since f is surjective. Hence, M' is a G-graded S-P-secondary module.

□

In general, a graded submodule of a G-graded S-P-secondary module need not be G-graded S-P-secondary. The following corollary asserts that under some conditions graded submodules of a G-graded S-P-secondary module are G-graded S-P-secondary.

Corollary 4.6. *Let M be a G-graded S-P-secondary A-module and N a proper graded submodule of M.*

(i) If N is a graded pure submodule of M, then N is G-graded S-P-secondary.

(ii) M/N is G-graded S-P-secondary.

The following corollary is a generalization of [8, Proposition 2.7].

Corollary 4.7. *Let M be a G-graded A-module and N a nonzero graded pure submodule of M. Then M is a G-graded S-P-secondary module if and only if both N and M/N are G-graded S-P-secondary.*

Proof. Assume that M is G-graded S-P-secondary. Then by Corollary 4.6, N and M/N are G-graded S-P-secondary. Conversely, assume that N and M/N are G-graded S-P-secondary. Clearly, $P = Gr(Ann(N)) = Gr(Ann(M/N))$. Now let $a \in h(A)$. If $a \in P$, then $a^{n_1}M \subseteq N$, $a^{n_2}N = 0$ for some integers $n_1, n_2 \geq 1$. Let n be the maximum of n_1 and n_2. Then $0 = a^n N = N \cap a^n M = a^n M$ since N is graded pure and $a^n M \subseteq N$. If $sa \notin P$ for all $s \in S$, then there exist $s_1, s_2 \in S$ such that $s_1 N \subseteq aN$ and $s_2(M/N) \subseteq a(M/N)$ since N and M/N are G-graded S-P-secondary. Put $s' = s_1 s_2$. Then $N \cap s'M = s'N \subseteq aN = N \cap aM$ and $s'(M/N) \subseteq a(M/N)$; hence $s'M \subseteq aM$, as required. □

Now we introduce an S-version of G-graded secondary representation, namely, G-graded S-secondary representation.

Definition 4.8. Let A be a G-graded ring, $S \subseteq h(A)$ a multiplicatively closed subset and M a G-graded A-module. We say that M has a G-graded S-secondary representation (or M is G-graded S-secondary representable) if it can be written as a sum

$$M = N_1 + N_2 + \cdots + N_r$$

with each N_i G-graded S-P_i-secondary, where $P_i = Gr(Ann(N_i))$ for $i = 1, 2, \ldots, r$.
The representation is minimal if the S-G-prime ideals P_i are all distinct and none of the N_i is redundant.

It is straightforward to see that G-graded secondary representable modules are G-graded S-secondary representable. Thus the notion G-graded S-secondary representation generalizes the notion of graded secondary representation. The next result follows by Proposition 4.4(i).

Theorem 4.9. *(Existence of minimal G-graded S-secondary representations) Let M be a G-graded S-secondary representable A-module. Then it has a minimal G-graded S-secondary representation.*

Remark 4.10. If $M = \sum_{i=1}^{r} N_i$ is a minimal G-graded S-secondary representation for a G-graded A-module M, then it follows from Proposition 4.4(ii) that

$$S^{-1}M = \sum_{i=1}^{r} S^{-1}N_i$$

is a G-graded secondary representation for $S^{-1}M$. Thus if M is G-graded S-secondary representable, then $S^{-1}M$ is a G-graded secondary representable module.

Let M be a G-graded A-module. Recall that a G-prime ideal P of A is called an attached G-prime ideal of M if M has a G-graded P-secondary quotient. In order to investigate the extent to which the G-graded S-secondary representation is unique, we need S-version of the attached G-prime ideals, namely, the attached S-G-prime ideals. An S-G-prime ideal P of A is called an *attached S-G-prime ideal of M* if M has a G-graded S-P-secondary quotient. The set of all attached S-G-prime ideals of M is denoted by $S\text{-}Att^G_A(M)$.

Theorem 4.11. *Let M be a G-graded S-secondary representable A-module. If $M = N_1 + N_2 + \cdots + N_r$ is a minimal G-graded S-secondary representation of M and $P_i = Gr(Ann(N_i))$ for $i = 1, 2, \ldots, r$, then*

$$S\text{-}Att^G_A(M) = \{P_1, P_2, \ldots, P_r\}.$$

Proof. Let $N^i = \sum_{j=1, j\neq i}^{r} N_j$ for $i = 1, 2, \ldots, r$. Then for each i,

$$M/N^i \cong N_i/(N_i \cap N^i)$$

is a nonzero quotient of N_i; hence by Corollary 4.6, M/N^i is G-graded S-P_i-secondary. Thus $\{P_1, P_2, \ldots, P_r\} \subseteq S\text{-}Att^G_A(M)$. For the reverse containment, let $P \in S\text{-}Att^G_A(M)$ and let U be a G-graded S-P-secondary quotient of M. Evidently, $S\text{-}Att^G_A(U)=\{P\}$. Write $U = M/N$, where N is graded submodule of M. Let $\bar{N}_i = (N_i + N)/N$ for $i = 1, 2, \ldots, r$. Then for each i,

$$\bar{N}_i \cong N_i/(N_i \cap N)$$

is a nonzero quotient of N_i; hence again by Corollary 4.6, $\bar{N}_i$ is G-graded S-P_i-secondary. Consequently, $U = \bar{N}_1 + \bar{N}_2 + \cdots + \bar{N}_r$ is a G-graded S-secondary representation of U. From this we obtain a minimal G-graded S-secondary representation $U = \bar{N}_{i_1} + \bar{N}_{i_2} + \cdots + \bar{N}_{i_k}$, and then

$$\{P_{i_1}, P_{i_2}, \ldots, P_{i_k}\} \subseteq S\text{-}Att^G_A(U) = \{P\}.$$

This yields that $P = P_j$ for some j $(1 \leq j \leq r)$, as desired. □

Corollary 4.12. *(Uniqueness Theorem) Let M be a G-graded S-secondary representable A-module. If $\sum_{i=1}^{r} N_i$ and $\sum_{j=1}^{k} N'_j$ are two minimal G-graded S-secondary representations of M with N_i is G-graded S-P_i-secondary and N'_j is G-graded S-P'_j-secondary, then $k = r$ and*

$$\{P_1, P_2, \ldots, P_r\} = \{P'_1, P'_2, \ldots, P'_r\}.$$

Proof. By Theorem 4.11, $\{P'_1, P'_2, \ldots, P'_k\} = S\text{-}Att^G_A(M) = \{P_1, P_2, \ldots, P_r\}$ which implies that $k = r$, as required. □

Remark 4.13. Let M be a G-graded S-secondary representable A-module and N a proper graded submodule of M. Then M/N is G-graded S-secondary representable and $S\text{-}Att^G_A(M/N) \subseteq S\text{-}Att^G_A(M)$. Indeed, let $M = \sum_{i=1}^{r} N_i$ be a minimal G-graded S-secondary representation of M, then $M/N = \sum_{i=1}^{r}(N_i + N)/N$. Notice that

$$(N_i + N)/N \cong N_i/(N \cap N_i)$$

is a quotient of N_i; hence by Corollary 4.6, each $(N_i + N)/N$ is G-graded S-secondary, as required. Next, let $P \in S\text{-}Att_A^G(M/N)$. Then M/N has a G-graded S-P-secondary quotient, and hence M has a G-graded S-P-secondary quotient. Consequently, $P \in S\text{-}Att_A^G(M)$, as desired.

Now we study an important property of the attached S-G-prime ideals in the next theorem which is a generalization of Theorem 2.7.

Theorem 4.14. *Let M be a G-graded S-secondary representable A-module. If N is a G-graded S-secondary representable submodule of M. Then*

$$S\text{-}Att_A^G(M/N) \subseteq S\text{-}Att_A^G(M) \subseteq S\text{-}Att_A^G(N) \cup S\text{-}Att_A^G(M/N).$$

Proof. The first containment follows from Remark 4.13. For the next containment, let $P \in S\text{-}Att_A^G(M)$. Let L be a graded submodule of M such that M/L is G-graded S-P-secondary. If $L + N \neq M$, then

$$\frac{M/N}{(L+N)/N} \cong M/(L+N) \cong \frac{M/L}{(L+N)/L}$$

and hence by Corollary 4.6, $M/(L + N)$ is G-graded S-P-secondary since M/L is G-graded S-P-secondary. This implies that M/N has a G-graded S-P-secondary quotient. Therefore, $P \in S\text{-}Att_A^G(M/N)$. If $L + N = M$, then

$$M/L = (N + L)/L \cong N/(L \cap N).$$

Consequently, $N/(L \cap N)$ is G-graded S-P-secondary since M/L is G-graded S-P-secondary. Hence $P \in S\text{-}Att_A^G(N)$, as desired. □

As an immediate consequence, we have the following result which is a generalization of Corollary 2.8.

Corollary 4.15. *If $M_1, M_2, \ldots, M_k$ are G-graded S-secondary representable A-modules, then $\bigoplus_{i=1}^{k} M_i$ is G-graded S-secondary representable and we have*

$$S\text{-}Att_A^G(\bigoplus_{i=1}^{k} M_i) = \bigcup_{i=1}^{k} S\text{-}Att_A^G(M_i).$$

Now we give an S-version of graded secondary representation for graded Artinian modules. For this, we need S-version of the G-graded sum-irreducible modules. Recall that a non-zero G-graded module is said to be *G-graded sum-irreducible* if it is not the sum of two proper graded submodules [20, Definition 2.1(iii)]. A nonzero G-graded A-module M is called *G-graded S-sum-irreducible* if whenever $sM \subseteq M_1 + M_2$ for some $s \in S$ and graded submodules M_1, M_2 of M, then there exists $s' \in S$ such that either $s'sM \subseteq M_1$ or $s'sM \subseteq M_2$.

We end this section by proving the existence of G-graded S-secondary representation in G-graded S-Artinian modules. For this, we need the following lemma which is a generalization of [20, Proposition 2.3].

Lemma 4.16. *Let M be a G-graded S-Artinian, G-graded S-sum-irreducible A-module, where $S \subseteq h(A)$ is a multiplicatively closed subset. Then M is a G-graded S-secondary module.*

Proof. On the contrary, suppose M is not a G-graded S-secondary module. Then there exists $a \in h(A)$ such that $sM \nsubseteq aM$ and $(sa)^n M \neq 0$ for every $s \in S$ and $n \geq 1$. This implies that $aM \neq M$ and $a^n M \neq 0$ for all $n \geq 1$. Consider a descending chain $aM \supseteq a^2 M \supseteq \cdots \supseteq a^n M \supseteq \cdots$ of graded submodules of M. Since M is G-graded S-Artinian, there exist $s' \in S$ and an integer $j \geq 1$ such that $s'a^j M \subseteq a^i M$ for all $i \geq j$. Let $x \in M$. Then there exists $y \in M$ such that $s'a^j x = a^{j+1} y$. This implies that $s'x - ay \in (0 :_M a^j)$, and so $s'M \subseteq aM + (0 :_M a^j)$. Notice that aM and $(0 :_M a^j) := \{m \in M : a^j m = 0\}$ are graded submodules of M. Now, since M is G-graded S-sum-irreducible, there exists $s'' \in S$ such that either $s''s'M \subseteq aM$ or $s''s'M \subseteq (0 :_M a^j)$. This is a contradiction since $sM \nsubseteq aM$ and $sM \nsubseteq (0 :_M a^n)$ for every $s \in S$ and $n \geq 1$. Hence, M is a G-graded S-secondary module. □

Theorem 4.17. *Let M be a G-graded S-Artinian A-module, where $S \subseteq h(A)$ is a multiplicatively closed subset. Then M contains a graded submodule which has a G-graded S-secondary representation.*

Proof. If M is G-graded S-secondary representable, then we are done. Suppose M is not G-graded S-secondary representable. Let X be the set of nonzero graded submodules of M which is not G-graded S-secondary representable. Then X is non-empty as $M \in X$. By Theorem 3.23, X has a graded S-minimal element N say. Then N itself is not a G-graded S-secondary submodule and hence by Lemma 4.16, N is not G-graded S-sum-irreducible. This implies that that there exist an $s \in S$ and graded submodules N_1, N_2 of N with $s'sN \nsubseteq N_1$, $s'sN \nsubseteq N_2$ for all $s' \in S$ such that $sN \subseteq N_1 + N_2$. Consequently, N_1, $N_2 \notin X$ since $s'N \nsubseteq N_1$, $s'N \nsubseteq N_2$ for all $s' \in S$ and N is a graded S-minimal element of X. This implies that N_1 and N_2 are G-graded S-secondary representable. Notice that N_1 and N_2 are nonzero. Hence, $N_1 + N_2$ has a G-graded S-secondary representation, as desired. □

Corollary 4.18. *Let M be a G-graded Artinian A-module. Then M has a G-graded secondary representation.*

Proof. Follows from Theorem 4.17 for $S = \{1\}$. □

Let A be a G-graded ring, S a multiplicatively closed subset of $h(A)$, and M a G-graded S-secondary A-module. If the grading is trivial, then M is called an *S-secondary module*. Clearly, the notion of S-secondary module is a generalization of secondary modules. By Proposition 4.2, if an A-module M is S-secondary, then $\sqrt{Ann(M)}$ is an S-prime ideal of A, and in this case we say that M is an *S-P-secondary module*.

Corollary 4.19. *Let A be a ring, S a multiplicatively closed subset of A and M an S-Artinian A-module. Then M contains a submodule which can be written as the sum of finite number of S-secondary modules.*

Proof. Follows from Theorem 4.17 for $G = \{e\}$. □

Notice that the case $S = \{1\}$ of the result above is nothing but a well-known fact that Artinian modules have secondary representations. Thus Theorem 4.17 generalizes the graded secondary representation for graded Artinian modules and at the same time is the extension of secondary representation for Artinian modules given by Macdonald in [13].

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