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DEFORMED INFINITE SERIES METRIC IN CARTAN SPACES

[1]Brijesh Kumar Tripathi and [2]V. K. Chaubey

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Abstract Igarashi introduce the concept of (α, β)-metric in Cartan space ℓ^n analogously to one in Finsler space and obtained the basic important geometric properties and also investigate the special class of the space with (α, β)-metric in ℓ^n in terms of $'invariants'$. In the present paper , we determine the $'invariants'$ in two different cases of deformed infinite series metric, which characterize the special classes of Cartan spaces ℓ^n.Further, we investigate some classes of deformed infinite series spaces and obtain deformed infinite series class which will provide the example of Cartan spaces.

1 Introduction

In 2004, Lee and Park [8] introduced the concept of r^{th} series (α, β)-metric where r varies from $0, 1, 2, ..., \infty$ and gave an interesting example of special (α, β)-metric for the different values of r such as one-form metric, Randers metric, combination of Kropina and Randers metric, infinite series metric etc.

In 2001, Sorin Sabau and H. Shimada [17] investigated some classes of (α, β)-metric spaces and obtained Randers class, Kropina class and Matsumoto class. These classes provided a concrete examples of Finsler spaces with (α, β)-metrics.

In 1994, Igarashi [5, 6] introduce the concept of (α, β)-metric in Cartan space ℓ^n analogously to one in Finsler space and obtained the basic important geometric properties and also investigate the special class of the space with (α, β)-metric in ℓ^n in terms of $'invariants'$. The classes which he obtained includes the spaces corresponding to Randers and Kropina space. Further, he characterizes these spacial classes by means of $'invariants'$ in case of Finsler theory.

In the present paper, we determine the $'invariants'$ in two different cases of deformed infinite series metric, wherein first metric is defined as the product of infinite series and Riemannian metric. Whereas, the other one is the product of infinite series and one-form metric. Further, we characterize the special classes of Cartan spaces ℓ^n in case of these two metrics and also investigate the relation under which "invariants" are characterized as the special classes of ℓ^n.As a special case, we investigate some classes of deformed infinite series spaces and obtained deformed infinite series class which will provide the example of Cartan spaces.

2 Preliminaries

E. Cartan [3] introduced the concept of a Cartan Space, where the measure of its hypersurface element (x, y) is given a priori by homogeneous function $F(x, y)$ of degree one in y,i.e.,the "area" of a domain on hypersurface $S_{n-1} : x^i = x^i(v^1, v^2, v^3, ..., v^n), \quad i = 1, 2, 3, ..., n$ is given by

$$S = \underbrace{\int \cdots \int} F(x, y) dv^1 dv^2 dv^3 .....dv^{n-1} \tag{2.1}$$

where $y = (y_i)$ is the determinant of $(n-1, n-1)$ minor matrix omitting i^{th} row of $(n, n-1)$ matrix $(\frac{\partial x^i}{\partial v^\alpha})$, $\alpha = 1, 2, 3.....n-1$.The fundamental tensor given by

$$g^{ij} = G^{-\frac{1}{n-1}}, \quad G = det\|G^{ij}\|, \quad G^{ij} = \frac{\partial^2(\frac{1}{2}F^2)}{\partial y_i \partial y_j}, \quad (y_i) \neq 0. \tag{2.2}$$

As the special case for the fundamental tensor a_{ij} of Riemannian space, given as the $(n-1)$ dimensional area of a domain on hypersurface such that

$$S = \underbrace{\int \cdots \int} \sqrt{det\|a_{ij}(x)\frac{\partial x^i}{\partial v^\alpha}\frac{\partial x^j}{\partial v^\beta}\|} dv^1 dv^2 dv^3.....dv^{n-1}$$

Therefore, it clear that Riemannian space is a special case of Cartan space. On the other hand, Cartan space is considered as the dual notion of Finsler Space. Earlier, L. Berwald [1] has studied the relation between both spaces . Thereafter, H. Rund [15] and F.Brickel [2] carry forwarded the same study.R. Miron [13, 14] established new Carton geometry which shows totally different feature in the form of particularization the Hamilton space which defined as:

Definition 2.1. A Cartan space is a Hamilton space $\mathcal{H}^n = \{M, \underbrace{H(x,y)}\}$ in which the fundamental function $H(x,y)$ is positively 2-homogeneous in y_i on T^*M. We denote it by ℓ^n.

The fundamental tensor field of ℓ^n and its reciprocal $g_{ij}(x,y)$ is given by

$$g^{ij}(x,y) = \frac{1}{2}\dot{\partial}^i\dot{\partial}^j H, \tag{2.3}$$

$$g_{ij}(x,y)g^{jk}(x,y) = \delta_i^k \tag{2.4}$$

The homogeneity of $H(x,y)$ is expressed by

$$y_j\dot{\partial}^j H = 2H, \ which\ also\ implies\ H = g^{ij}y_i y_j \tag{2.5}$$

where $g^{ij}(x,y)$ and its reciprocal $g_{ij}(x,y)$ are both symmetric and homogeneous of degree 0 in y_i.

In 1996, D. Hrimiuc and H. Shimada [4] expressed the definitions of Lagrange,Hamilton and Cartan Spacces with examples:

Definition 2.2. A regular Lagrangian (Hamiltonian)on a domain $D \subset TM$ $(D^* \subset T^*M)$ is a real smooth function $L : D \to R(H : D^* \to R)$such that the matrix with entries

$$g_{ij}(x,y) = \dot{\partial}_i\dot{\partial}_j L(x,y) \quad (g^{*ij}(x,y) = \dot{\partial}^i\dot{\partial}^j H(x,y)) \tag{2.6}$$

is everywhere nondegenerate on $D(D^*)$.

A Lagrange (Hamilton) manifold is a pair (M, L) $((M, H))$ where M is smooth manifold and $L(H)$ is a regular Lagrangian (Hamiltonian)on $D(D^*)$.

Examples:

1: Every Finsler Space $F^n = (M, F(x,y))$ is Lagrange Manifold with $L = \frac{F^2}{2}$.

2: Every Cartan Space $\ell^n = (M, \bar{F}(x,p))$ is a Hamilton Manifold with $H = \frac{1}{2}\bar{F}^2$.(Here $\bar{F}$ is positively $1-$homogeneous in p_i and the tensor $\bar{g}^{ij} = \frac{1}{2}\partial^i\partial j\bar{F}^2$ is nondegenerate.)

3:(M, L) and (M, H) with

$$\begin{aligned} L(x,y) &= a_{ij}(x)y^i y^j + b_i(x)y^i + c(x) \\ H(x,p) &= \bar{a}^{ij}(x)p_i p_j + \bar{b}^i(x)p_i + \bar{c}(x) \end{aligned}$$

are Lagrange and Hamilton Manifold respectively.

On the other hand, the Finsler spaces with (α, β)-metric were considered by G. Randers [15], V. K. Kropina [7] and M. Matsumoto [10, 11, 12], especially the last paper shows the great success for investigation of these spaces.

In [13], R. Miron expected the existence of Randers type metric:

$$H(x,y) = \{\alpha(x,y) + \beta(x,y)\}^2 \tag{2.7}$$

and of Kropina's one :

$$H(x,y) = \{\frac{[\alpha(x,y)]^2}{\beta(x,y)}\}^2, \quad \beta(x,y) \neq 0. \tag{2.8}$$

in Cartan spaces. Here we have put as

$$\alpha^2(x,y) = a^{ij}(x)y_iy_j, \quad \beta(x,y) = b^i(x)y_i, \tag{2.9}$$

$a^{ij}(x)$ being a Riemannian metric on the base manifold M and $b^i(x)$ a vector field on M such that $\beta > 0$ on a region of $T^*M \stackrel{def}{=} T^*M - \{0\}$.

3 Cartan spaces with (α, β)-metric.

Cartan spaces with (α,β)-metric [5, 6, 13] can be defiend ;

Definition 3.1. A Cartan space $\ell^n = \{M, H(x,y)\}$ is known as Cartan space with (α,β)-metric if its fundamental metric $H(x,y)$ is a function of $\alpha(x,y)$ and $\beta(x,y)$ only i.e.

$$H(x,y) = \breve{H}\{\alpha(x,y), \beta(x,y)\} \tag{3.1}$$

It is clear that $\breve{H}$ satisfy the conditions imposed to the function $H(x,y)$ as a fundamental function for ℓ^n. Then

$$t\alpha(x,y) = \alpha(x,ty), \quad t\beta(x,y) = \beta(x,ty), \quad H(x,ty) = t^2H(x,y) \quad t > 0. \tag{3.2}$$

It follows:

Proposition 3.2. *The function $\breve{H}\{\alpha(x,y),\beta(x,y)\}$ is positively homogeneous of degree 2 in both α and β.*

By this reason, there maybe no confusion if we adopt the notation $H(\alpha,\beta)$ itself instead of $\breve{H}(\alpha,\beta)$.It can also be written;

$$H_\alpha = \frac{\partial H}{\partial \alpha}, \ H_\beta = \frac{\partial H}{\partial \beta}, \ H_{\alpha\beta} = \frac{\partial^2 H}{\partial\alpha\partial\beta}, \ etc \tag{3.3}$$

Proposition 3.3. *The following identities hold :*

$$\alpha H_{\alpha\alpha} + \beta H_{\beta\beta} = H, \ \alpha H_{\alpha\beta} + \beta H_{\beta\beta} = H_\beta, \tag{3.4}$$

$$\alpha^2 H_{\alpha\alpha} + 2\alpha\beta H_{\alpha\beta} + \beta^2 H_{\beta\beta} = \alpha H_\alpha + \beta H_\beta = 2H.$$

Differentiating α and β with respect to y_i we have

$$\dot\partial^i\alpha = \alpha^{-1}a^{ij}y_j = \alpha^{-1}Y^i, \quad \dot\partial^i\beta = b^i(x), \tag{3.5}$$

where

$$Y^i(x,y) = a^{ij}(x)y_i, \quad \{Y = (Y^i)\} \neq 0 \tag{3.6}$$

and the vector field Y^i satisfies the relation

$$Y^iy_i = \alpha^2, \tag{3.7}$$

Further let

$$B_i(x) = a_{ij}(x)b^j(x), \quad B^2(x) = a^{ij}B_iB_j = a_{ij}b^ib^j, \tag{3.8}$$

$$y^i(x,y) = g^{ij}(x,y)y_j = \frac{1}{2}\frac{\partial H}{\partial y_i} \tag{3.9}$$

Also, we have the relation similar to (3.7):

$$y^i y_i = H(x,y). \tag{3.10}$$

Differentiating (3.6) and (3.9) by y_i succeedingly, we have

$$\dot\partial^j Y^i = a^{ij}(x), \quad \dot\partial^k \dot\partial^j Y^i = 0, \tag{3.11}$$

$$\dot\partial^j y^i = g^{ij}, \quad \dot\partial^k \dot\partial^j y^i = \dot\partial^k g^{ij} = -2C^{ijk}, \tag{3.12}$$

And using the same manner for (3.5), we get

$$\dot\partial^j \dot\partial^i \alpha = \alpha^{-1} a^{ij}(x) - \alpha^{-3} Y^i Y^j, \ \ \dot\partial^j \dot\partial^i \beta = 0, \quad \dot\partial^j \dot\partial^i (\frac{1}{2}\alpha^2) = a^{ij}. \tag{3.13}$$

On account of (3.5) and $y^i = \frac{1}{2}(H_\alpha \dot\partial^i \alpha + H_\beta \dot\partial^i \beta)$, we have

Lemma 3.4. *The Liouville vector field y^i is expressed in the form*

$$y^i = \rho_1 b^i + \rho Y^i, \tag{3.14}$$

where

$$\rho_1 = \frac{1}{2} H_\beta, \quad \rho = \frac{1}{2\alpha} H_\alpha \tag{3.15}$$

Taking into account that the relation $\dot\partial^i \rho = \frac{\partial \rho}{\partial \alpha}\dot\partial^i \alpha + \frac{\partial \rho}{\partial \beta}\dot\partial^i \beta$ holds, we have

Lemma 3.5. *The quantities ρ_1 and ρ satify the relations*

$$\dot\partial^i \rho_1 = \rho_0 b^i + \rho_{-1} Y^i, \quad \dot\partial^i \rho = \rho_{-1} b^i + \rho_{-2} Y^i \tag{3.16}$$

respectively, where

$$\rho_0 = \frac{1}{2} H_{\beta\beta}, \quad \rho_{-1} = \frac{1}{2\alpha} H_{\alpha\beta}, \quad \rho_{-2} = \frac{1}{2\alpha^2}(H_{\alpha\alpha} - \alpha^{-1} H_\alpha) \tag{3.17}$$

Contracting $\dot\partial^i \rho_1$, $\dot\partial^i \rho$ in (3.16) by y_i,we have

$$y_i \dot\partial^i \rho_1 = \alpha^2 \rho_{-1} + \beta \rho_0 = \rho_1, \quad y_i \dot\partial^i \rho = \alpha^2 \rho_{-2} + \beta \rho_{-1} = 0. \tag{3.18}$$

Analogously to deduction of Lemma (3.2), we have also

Lemma 3.6. *The quantities ρ_0 and ρ_{-1} satisfy the relations*

$$\dot\partial^i \rho_0 = r_{-1} b^i + r_{-2} Y^i, \quad \dot\partial^i \rho_{-1} = r_{-2} b^i + r_{-3} Y^i, \tag{3.19}$$

respectively, where

$$r_{-1} = \frac{1}{2} H_{\beta\beta\beta}, \quad r_{-2} = \frac{1}{2\alpha} H_{\alpha\beta\beta}, \quad r_{-3} = \frac{1}{2\alpha^2}(H_{\alpha\alpha\beta} - \alpha^{-1} H_{\alpha\beta}). \tag{3.20}$$

Corresponding to (3.18), it follows

$$y_i \dot\partial^i \rho_0 = \alpha^2 r_{-2} + \beta r_{-1} = 0, \quad y_i \dot\partial^i \rho_{-1} = \alpha^2 r_{-3} + \beta r_{-2} = -\rho_{-1}. \tag{3.21}$$

Furthermore for ρ_{-2}, we have

Lemma 3.7. *The quantities ρ_{-2} satisfies the relations*

$$\dot\partial^i \rho_{-2} = r_{-3} b^i + r_{-4} Y^i \tag{3.22}$$

$$r_{-4} = \frac{1}{2\alpha^3}(H_{\alpha\alpha\alpha} - 3\alpha^{-1} H_{\alpha\alpha} + 3\alpha^{-2} H_\alpha). \tag{3.23}$$

Also following homogeneity holds good:

$$y_i \dot{\partial}^i \rho_{-2} = \alpha^2 r_{-4} + \beta r_{-3} = -2\rho_{-2}. \tag{3.24}$$

It is easy to conclude for the scalars (or invariants) $\rho_1, \rho, \rho_0, \rho_{-1}, ..., r_{-1}, r{-}2, ...$ in the above lemmas that subscript of ρ,s and r,s represent degree of their own homogeneity in (α, β) or y_i,where the ρ without subscript means of degree 0.

We have these properties from expressions in (3.18),(3.19) and (3.21) and the following relations

$$y_i \dot{\partial}^i r_{-1} = -r_{-1}, \quad y_i \dot{\partial}^i r_{-2} = -2r_{-2}, \tag{3.25}$$

$$y_i \dot{\partial}^i r_{-3} = -3r_{-3}, \quad y_i \dot{\partial}^i r_{-4} = -4r_{-4},$$

because of the homogeneity of

$$\alpha H_{\alpha\alpha\alpha\alpha} + \beta H_{\alpha\alpha\alpha\beta} = -H\alpha\alpha\alpha, \quad \alpha H_{\alpha\alpha\alpha\beta} + \beta H_{\alpha\alpha\beta\beta} = -H\alpha\alpha\beta, \tag{3.26}$$

$$\alpha H_{\alpha\alpha\beta\beta} + \beta H_{\alpha\beta\beta\beta} = -H\alpha\beta\beta, \quad \alpha H_{\alpha\beta\beta\beta} + \beta H_{\beta\beta\beta\beta} = -H\beta\beta\beta.$$

We shall use the previous results to study the fundamental geometric objects of the space ℓ^n with (α, β)-metric.All these scalar functions $\rho_1, \rho, \rho_0, \rho_{-1}, ...$ as well as $r_{-1}, r_{-2}, ...$ will be called the invariants of the Cartan space ℓ^n with the fundamental function $H(\alpha, \beta)$.

4 The fundamental tensor of the space ℓ^n with (α, β)-metric.

We need to derive the fundamental tensor from the fundamental function $H(x, y)$ of the Cartan space ℓ^n.

Theorem 4.1. *The fundamental tensor g^{ij} of Cartan space ℓ^n with (α, β)-metric is given by*

$$g^{ij} = \rho a^{ij} + \rho_0 b^i b^j + \rho_{-1}(b^i Y^j + b^j Y^i) + \rho_{-2} Y^i Y^j, \tag{4.1}$$

where $\rho, \rho_0, \rho_{-1}\rho_{-2}$ are the invariants given by (3.15) and (3.17).

Proof. Making use of (3.12) and (3.14),we have

$$g^{ij} = \dot{\partial}^j y^i = \dot{\partial}^j(\rho_1 b^i + \rho Y^i) = (\dot{\partial}^j \rho_1) b^i + (\dot{\partial}^j \rho) Y^i + \rho a^{ij}.$$

Taking into account Lemma (3.2), we have (3.1). Q.E.D.
In order to check the fitness of this tensor g^{ij} for the fundamental tensor of (α, β)-metric,we verify the homogeneity of g^{ij}.Contracting g^{ij} by y_i and y_j,we have

$$\begin{aligned} g^{ij} y_i y_j = \tfrac{1}{2}\{&\alpha^{-1} H_\alpha a^{ij} y_i y_j + (\alpha^{-2} H_{\alpha\alpha} - \alpha^{-3} H_\alpha) Y^i Y^j y_i y_j \\ &+ \alpha^{-1} H_{\alpha\beta}(Y^i y_i b^j y_j + b^i y_i Y^j y_j) + H_{\beta\beta} b^i y_i b^j y_j\} \\ = \tfrac{1}{2}\{&\alpha H_\alpha + (\alpha^{-2} H_{\alpha\alpha} - \alpha^{-3} H_\alpha)\alpha^4 + \alpha^{-1} H_{\alpha\beta}.2\alpha^2\beta + H_{\beta\beta}\} \\ &= \tfrac{1}{2}.2H = H. \end{aligned}$$

which shows that our conclusion is right.
Let us rewrite this expression in the form

$$g^{ij} = A^{ij} + C^i C^j, \tag{4.2}$$

where

$$A^{ij} = \rho a^{ij}, \quad C^i = q_0 b^i + q^{-1} Y^i, \tag{4.3}$$

$$\rho_0 = q_0^2, \quad \rho_{-1} = q_0 q_{-1}, \quad \rho_{-2} = q_{-1}^2 \tag{4.4}$$

and

$$\rho_0 \rho_{-2} = \rho_{-1}^2 \tag{4.5}$$

The reciprocal tensor g_{ij} of g^{ij} are given by

$$g_{ij} = A_{ij} - \frac{1}{1+C^2} C_i C_j, \tag{4.6}$$

where

$$det\|A_{ij}\| = (1+C^2) det\|A^{ij}\| \quad (if 1+C^2 \neq 0), \tag{4.7}$$

and A_{ij},C_i are given by

$$A_{ij}A^{jk} = \delta^i_k, \quad C^i C_j = \delta^i_j C^2, \quad C^i = A^{ij}C_j, \quad C_i = A_{ij}C^j. \tag{4.8}$$

Consequently, we have $g^{ij}(x,y)g_{jk}(x,y) = \delta^i_k$,and
(3.2') $\quad rank\|g^{ij}(x,y)\| = n$
because of
(3.2") $\quad det\|g^{ij}(x,y)\| = (1+C^2)det\|A^{ij}\| = (1+C^2)det\|a^{ij}(x)\| \neq 0$
The following relations are useful afterwards :

$$A_{ij} = \frac{1}{\rho} a_{ij}, \quad det\|A^{ij}\| = \rho^n det\|a^{ij}\|, \ \ (A = det\|a^{ij}\| \neq 0), \tag{4.9}$$

$$C^2 = \frac{1}{\rho}(\rho_0 B^2 + \rho_{-1}\beta), \quad det\|g^{ij}\| = \rho^{n-1}\tau, \tag{4.10}$$

where we use the notations

$$B^2 = a^{ij}b_i b_j, \quad \tau = \rho + \rho_0 B^2 + \rho_{-1}\beta \tag{4.11}$$

Therefore we can prove without difficulty:

Proposition 4.2. *The covariant form of the fundamental tensor is given by*

$$g_{ij} = \sigma a_{ij} - \sigma_0 B_i B_j + \sigma_{-1}(B_i y_j + B_j y_i) + \sigma_{-2} y_i y_j, \tag{4.12}$$

where we use the notations

$$\sigma = \frac{1}{\rho}, \quad \sigma_0 = \frac{\rho_0}{\rho\tau}, \quad \sigma_{-2} = \frac{\rho_{-2}}{\rho\tau}. \tag{4.13}$$

We can get another result from the Lemma (3.4) such that

Theorem 4.3. *The Cartan tensor C^{ijk} of a Cartan space ℓ^n with (α,β)-metric is given by*

$$C^{ijk} = \frac{-1}{2}[r_{-1}b^i b^j b^k + \varPi_{ijk}\{\rho_{-1}a^{ij}b^k + \rho_{-2}a^{ij}Y^k + r_{-2}b^i b^j Y^k + r_{-3}b^i Y^j Y^k\} + r_{-4}Y^i Y^j Y^k] \tag{4.14}$$

where the notation $\varPi_{ijk}$ means the cyclic symmetrization of the quantity in the brackets with respect to indices i, j, k.

We can deduce the other important geometric object fields for ℓ^n with (α,β)-metric. For instance, N_{ij},H^i_{jk},C^{jk}_i etc. without difficulty.

5 Cartan spaces with infinite series of (α,β)-metric

In 2004, Lee and Park [8] introduced a r-th series (α,β)-metric

$$L(\alpha,\beta) = \beta\sum_{k=0}^{r}(\frac{\alpha}{\beta})^k, \tag{5.1}$$

where they assume $\alpha < \beta$. If $r = 1$ then $L = \alpha + \beta$ is a Randers metric. If $r = 2$ then $L = \alpha + \beta + \frac{\alpha^2}{\beta}$ is a combination of Randers metric and Kropina metric. If $r = \infty$ then above metric is expressed as

$$L(\alpha, \beta) = \frac{\beta^2}{\beta - \alpha} \tag{5.2}$$

and the metric (5.2) named as infinite series (α, β)-metric. This metric is very remarkable because it is the difference of Randers and Matsumoto metric.

In this section, we consider two cases of Cartan Finsler spaces with special (α, β)-metrics of deformed infinite series metric which are defined as

I $H(\alpha, \beta) = \frac{\alpha\beta^2}{\beta-\alpha}$ i.e. the product of infinite series and Riemannian metric.

II $H(\alpha, \beta) = \frac{\beta^3}{\beta-\alpha}$ i.e. the product of infinite series and one-form metric.

5.1 Cartan space ℓ^n for $H(\alpha, \beta) = \frac{\alpha\beta^2}{\beta-\alpha}$

In the first case, partial derivatives of the fundamental function $H(\alpha, \beta)$ lead us the followings:

$$H_\alpha = \frac{\beta^3}{(\beta-\alpha)^2}, \quad H_\beta = \frac{\alpha\beta^2 - 2\alpha^2\beta}{(\beta-\alpha)^2}$$
$$H_{\alpha\alpha} = \frac{2\beta^3}{(\beta-\alpha)^3} \quad H_{\alpha\beta} = \frac{\beta^3 - 3\alpha\beta^2}{(\beta-\alpha)^3} \quad H_{\beta\beta} = \frac{2\alpha^3}{(\beta-\alpha)^3}$$
$$H_{\alpha\alpha\alpha} = \frac{6\beta^3}{(\beta-\alpha)^4} \quad H_{\alpha\alpha\beta} = \frac{-6\alpha\beta^2}{(\beta-\alpha)^4} \quad H_{\beta\beta\beta} = \frac{-6\alpha^3}{(\beta-\alpha)^4} \quad H_{\alpha\beta\beta} = \frac{6\alpha^2\beta}{(\beta-\alpha)^4}$$

Using equation (3.15) and (3.17) we have following invariants

$$\rho_1 = \frac{\alpha\beta^2 - 2\alpha^2\beta}{2(\beta-\alpha)^2}, \quad \rho = \frac{\beta^3}{2(\beta-\alpha)^2}, \quad \rho_0 = \frac{\alpha^3}{(\beta-\alpha)^3}, \tag{5.3}$$

$$\rho_{-1} = \frac{\beta^3 - 3\alpha\beta^2}{2\alpha(\beta-\alpha)^3} \quad \rho_{-2} = \frac{\beta^3(3\alpha-\beta)}{2\alpha^3(\beta-\alpha)^3}$$

Proposition 5.1. *The invariants ρ never vanishes in a Cartan space ℓ^n equipped with deformed infinite series metric function $H(\alpha, \beta) = \frac{\alpha\beta^2}{\beta-\alpha}$ -metric on $\widetilde{T^*M}$. Converesely, we have $H_\alpha \neq 0$ on $\widetilde{T^*M}$.*

Again using equation (3.20) and (3.23) we have following invariants

$$r_{-1} = \frac{-3\alpha^3}{(\beta-\alpha)^4}, \qquad r_{-2} = \frac{3\alpha\beta}{(\beta-\alpha)^4}, \tag{5.4}$$

$$r_{-3} = \frac{4\alpha\beta^3 - \beta^4 - 9\alpha^2\beta^2}{2\alpha^3(\beta-\alpha)^4} \quad r_{-4} = \frac{15\alpha^2\beta^3 + 3\beta^5 - 12\alpha\beta^4}{2\alpha^5(\beta-\alpha)^4}$$

Proposition 5.2. *The invariants of Cartan tensor C^{ijk} in Cartan space ℓ^n which equipped with deformed infinite series metric function $H(\alpha, \beta) = \frac{\alpha\beta^2}{\beta-\alpha}$ is given by (5.3)and (5.4).*

The invariants of equations (5.3) and (5.4) satisfies the following relations

$$\alpha^2\rho_{-1} + \beta\rho_0 = \rho_1, \quad \alpha^2\rho_{-2} + \beta\rho_{-1} = 0, \tag{5.5}$$

$$\alpha^2 r_{-2} + \beta r_{-1} = 0, \quad \alpha^2 r_{-3} + \beta r_{-2} = -\rho_{-1},$$

$$\alpha^2 r_{-4} + \beta r_{-3} = -2\rho_{-2},$$

Theorem 5.3. *The Cartan space ℓ^n equipped with deformed infinite series metric function $H(\alpha, \beta) = \frac{\alpha\beta^2}{\beta-\alpha}$ has the invariants in equations (5.3) and (5.4) are satifies the relations in (5.5).*

From equation (5.3) and (5.4), the fundamental tensor $g^{ij}(x,y)$ is of the form

$$g^{ij}(x,y)=\frac{\beta^3}{2(\beta-\alpha)^2}a^{ij}+\frac{\alpha^3}{(\beta-\alpha)^3}b^ib^j+\frac{\beta^3-3\alpha\beta^2}{2(\beta-\alpha)^3}(b^iY^j+b^jY^i) \tag{5.6}$$

$$+\frac{\beta^3(3\alpha-\beta)}{2\alpha^3(\beta-\alpha)^3}Y^iY^j$$

Corollary 5.4. *The fundamental tensor $g^{ij}(x,y)$ of the space ℓ^n endoewd with the metric function $H(\alpha,\beta)=\frac{\alpha\beta^2}{\beta-\alpha}$ is given by the equation (5.6).*

Converesely, we obtain

Theorem 5.5. *The Cartan space with (α,β)-metric which have the invariants such that (5.3)and (5.4) is the spaces ℓ^n with the fundamental function $H(\alpha,\beta)=\frac{\alpha\beta^2}{\beta-\alpha}$, i.e. deformed infinite series metric.*

5.2 Cartan space ℓ^n for $H(\alpha,\beta)=\frac{\beta^3}{\beta-\alpha}$

In the second case, partial derivatives of the fundamental function $H(\alpha,\beta)$ lead us the followings:

$$H_\alpha=\tfrac{\beta^3}{(\beta-\alpha)^2},\quad H_\beta=\tfrac{2\beta^3-3\alpha\beta^2}{(\beta-\alpha)^2}$$
$$H_{\alpha\alpha}=\tfrac{2\beta^3}{(\beta-\alpha)^3}\quad H_{\alpha\beta}=\tfrac{\beta^3-3\alpha\beta^2}{(\beta-\alpha)^3}\quad H_{\beta\beta}=\tfrac{2\beta^3-6\alpha\beta^3+6\alpha^2\beta}{(\beta-\alpha)^3}$$
$$H_{\alpha\alpha\alpha}=\tfrac{6\beta^3}{(\beta-\alpha)^4}\quad H_{\alpha\alpha\beta}=\tfrac{-6\alpha\beta^2}{(\beta-\alpha)^4}\quad H_{\beta\beta\beta}=\tfrac{-6\alpha^3}{(\beta-\alpha)^4}\quad H_{\alpha\beta\beta}=\tfrac{6\alpha^2\beta}{(\beta-\alpha)^4}$$

Using equation (3.15) and (3.17) we have following invariants

$$\rho_1=\frac{2\beta^3-3\alpha\beta^3}{2(\beta-\alpha)^2},\quad \rho=\frac{\beta^3}{2(\beta-\alpha)^2},\quad \rho_0=\frac{\beta^3-3\alpha\beta^2+3\alpha^2\beta}{(\beta-\alpha)^3}, \tag{5.7}$$

$$\rho_{-1}=\frac{\beta^3-3\alpha\beta^2}{2\alpha(\beta-\alpha)^3}\quad \rho_{-2}=\frac{\beta^3(3\alpha-\beta)}{2\alpha^3(\beta-\alpha)^3}$$

Proposition 5.6. *The invariants ρ never vanishes in a Cartan space ℓ^n equipped with deformed infinite series metric function $H(\alpha,\beta)=\frac{\beta^3}{\beta-\alpha}$ -metric on $\widetilde{T^*M}$. Converesely, we have $H_\alpha\neq 0$ on $\widetilde{T^*M}$.*

Again using equations (3.20) and (3.23) we have following invariants

$$r_{-1}=\frac{-3\alpha^3}{(\beta-\alpha)^4},\qquad r_{-2}=\frac{3\alpha\beta}{(\beta-\alpha)^4}, \tag{5.8}$$

$$r_{-3}=\frac{4\alpha\beta^3-\beta^4-9\alpha^2\beta^2}{2\alpha^3(\beta-\alpha)^4}\quad r_{-4}=\frac{15\alpha^2\beta^3+3\beta^5-12\alpha\beta^4}{2\alpha^5(\beta-\alpha)^4}$$

Proposition 5.7. *The invariants of Cartan tensor C^{ijk} in Cartan space ℓ^n which equipped with deformed infinite series metric function $H(\alpha,\beta)=\frac{\beta^3}{\beta-\alpha}$ is given by (5.7) and (5.8).*

The invariants of equations (5.7) and (5.8) satisfy the following relations

$$\alpha^2\rho_{-1}+\beta\rho_0=\rho_1,\quad \alpha^2\rho_{-2}+\beta\rho_{-1}=0, \tag{5.9}$$

$$\alpha^2r_{-2}+\beta r_{-1}=0,\quad \alpha^2r_{-3}+\beta r_{-2}=-\rho_{-1},$$

$$\alpha^2r_{-4}+\beta r_{-3}=-2\rho_{-2},$$

Theorem 5.8. *The Cartan space ℓ^n equipped with deformed infinite series metric function $H(\alpha,\beta) = \frac{\beta^3}{\beta-\alpha}$ has the invariants in equations (5.7) and (5.8) satify the relations in (5.9).*

From equation (5.7) and (5.8), the fundamental tensor $g^{ij}(x,y)$ is of the form

$$g^{ij}(x,y) = \frac{\beta^3}{2(\beta-\alpha)^2}a^{ij} + \frac{\beta^3 - 3\alpha\beta^2 + 3\alpha^2\beta}{(\beta-\alpha)^3}b^i b^j \tag{5.10}$$

$$+\frac{\beta^3 - 3\alpha\beta^2}{2(\beta-\alpha)^3}(b^i Y^j + b^j Y^i) + \frac{\beta^3(3\alpha-\beta)}{2\alpha^3(\beta-\alpha)^3}Y^i Y^j$$

Corollary 5.9. *The fundamental tensor $g^{ij}(x,y)$ of the space ℓ^n endoewd with the metric function $H(\alpha,\beta) = \frac{\beta^3}{\beta-\alpha}$ is given by the equation (5.10).*

Converesely, we obtain

Theorem 5.10. *The Cartan space with (α,β)-metric which have the invariants such that (5.7)and (5.8) is the spaces ℓ^n with the fundamental function $H(\alpha,\beta) = \frac{\beta^3}{\beta-\alpha}$,i.e. deformed infinite series metric.*

6 Cartan Classes of Deformed Infinite Series Metric

In 2001 and 2003, S S Vasile and H. Shimada [17, 18]developed the concept of Classes of (α,β)-metrics and further they studied classes of Randers,Kropina and Matsumoto metric.In this section, we consider invariants defined in (5.3),(5.4),(5.7)and (5.8) and find the classes of deformed infinite series metric by direct integration.

Since

$$H_{\beta\beta\beta} = 2r_{-1} = \frac{-6\alpha^3}{(\beta-\alpha)^4} \tag{6.1}$$

Integrating equation (6.1) w.r.t. β,we have

$$H_{\beta\beta} = \frac{2\alpha^3}{(\beta-\alpha)^3} + f_1(\alpha) \tag{6.2}$$

Again integrating equation (6.2) w.r.t. β,we have

$$H_{\beta} = \frac{-\alpha^3}{(\beta-\alpha)^2} + f_1(\alpha)\beta + f_2(\alpha) \tag{6.3}$$

Again integrating equation (6.3) w.r.t. β,we have

$$H = \frac{\alpha^3}{(\beta-\alpha)} + f_1(\alpha)\frac{\beta^2}{2} + f_2(\alpha)\beta + f_3(\alpha) \tag{6.4}$$

Thus equation (6.4) written as

$$H(\alpha,\beta) = \frac{\alpha^3}{(\beta-\alpha)} + C_1\alpha^2 + 2C_2\alpha\beta + C_3\beta^2 \tag{6.5}$$

Theorem 6.1. *A Cartan Space with an (α,β)-metric satisfying*

$$r_{-1} = \frac{-3\alpha^3}{(\beta-\alpha)^4}$$

has the fundamental function of the form (6.5), where C_1, C_2 and C_3 are real contants.

Corollary 6.2. *If $C_1 = 1, C_2 = \frac{1}{2}$ and $C_3 = 0$, then equation (6.5) gives*

$$H(\alpha,\beta) = \frac{\alpha\beta^2}{(\beta-\alpha)} \tag{6.6}$$

i.e. product of Riemannian and infinite series metric.

Corollary 6.3. *If* $C_1 = 1, C_2 = \frac{1}{2}$ *and* $C_3 = 1$ *,then equation (6.5) gives*

$$H(\alpha,\beta) = \frac{\beta^3}{(\beta-\alpha)} \tag{6.7}$$

i.e. product of one form and infinite series metric.

Corollary 6.4. *If* $C_1 = 0, C_2 = 0$ *and* $C_3 = 0$ *,then equation (6.5) gives*

$$H(\alpha,\beta) = \frac{\alpha^3}{(\beta-\alpha)} = \frac{-\alpha^3}{(\alpha-\beta)} \tag{6.8}$$

i.e. product of Riemannian and Matsumoto metric with negative sign.

Definition 6.5. Thus, we can call the metric (6.5) as the Cartan deformed infinite series class of (α,β) metric.

7 Conclusions

In this work, we consider the infinite series (α,β)-metric, Riemannian metric and 1-form metric and we determine relations with the $\imath invariants$ which characterize the special classes in Cartan Finsler frames. Further, we determine Cartan class of (α,β)-metric for deformed infinite series metric and discuss some special cases as well. The authors [19] have developed a non-holonomic frames for Finsler space with infinite series of (α,β)-metric as an importnat work for present metric. In future, we can determine L dual relation between Finsler space and Lagrange Space for present metric. Moreover, hypersurfaces of nonholonomic frame for Finsler Space with deformed infinite series metric and some homogeneous frame can also be considered for future work. But, in Finsler geometry, there are many(α,β)-metrics, and such frames can be determined in future as well.

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Author information

[1]Brijesh Kumar Tripathi, Department of Mathematics, L.D. College of Engineering, Ahmedabad, Gujarat-380015., India.
E-mail: `brijeshkumartripathi4@gmail.com`

[2]V. K. Chaubey, Department of Applied Sciences,Buddha Institute of Technology, GIDA , Gorakhpur, U.P.-273209., India.
E-mail: `vkcoct@gmail.com`

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