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On Dominions and Closed Varieties of Semigroups

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Abstract In this paper, first we have shown that all the homotypical varieties defined by the identities axy=xyax, axy=xxya and axy=yaxx are closed. Further, we partially generalize a result of Isbell on semigroup dominions from the class of commutative semigroups to some classes of permutative semigroups by showing that dominions of such semigroups belongs to the same class.

1 Introduction

Let U be a subsemigroup of a semigroup S. Following Isbell [5], we say that U dominates an element d of S if for every semigroup T and for all homomorphisms $\beta, \gamma : S \longrightarrow T$ and $u\beta$=$u\gamma$ for every u in U implies $d\beta$=$d\gamma$. The set of all elements of S dominated by U is called $dominion$ of U in S and we denote it by $Dom(U, S)$. It can be easily verified that $Dom(U, S)$ is a subsemigroup of S containing U. A subsemigroup U of semigroup S is called closed if $Dom(U, S) = U$. A semigroup is called absolutely closed if it is closed in every containing semigroup. Let $\mathcal{C}$ be a class of semigroups. A semigroup U is said to be $\mathcal{C}$-closed if $Dom(U, S) = U$ for all $S \in \mathcal{C}$ such that $U \subseteq S$. Let $\mathcal{B}$ and $\mathcal{C}$ be classes of semigroups such that $\mathcal{B}$ is a subclass of $\mathcal{C}$. We say that $\mathcal{B}$ is $\mathcal{C}$-closed if every member of $\mathcal{B}$ is $\mathcal{C}$-closed. A class $\mathcal{C}$ of semigroups is said to be closed if $Dom(U, S) = U$ for all $U, S \in \mathcal{C}$ with U as a subsemigroup of S. Let $\mathcal{A}$ and $\mathcal{D}$ be two categories of semigroups with $\mathcal{A}$ is a subcategory of $\mathcal{D}$. Then it can be easily verified that a semigroup U is $\mathcal{A}$ closed if it is $\mathcal{D}$ closed. For any word u, the content of u (necessarily finite) is the set of all distinct variables appearing in u and is denoted by $C(u)$. the identity $u = v$ is said to be homotypical if $C(u) = C(v)$; otherwise heterotypical. A variety $\mathcal{V}$ of semigroups is said to be homotypical if it admits a homotypical identity.

The following theorem provided by Isbell [5], known as Isbell's zigzag theorem, is a most useful characterization of semigroup dominions and is of basic importance to our investigations.

Theorem 1.1. *([5], Theorem 2.3) Let U be a subsemigroup of a semigroup S and let $d \in S$. Then $d \in Dom(U, S)$ if and only if $d \in U$ or there exists a series of factorizations of d as follows:*

$$d = a_0t_1 = y_1a_1t_1 = y_1a_2t_2 = y_2a_3t_2 = \cdots = y_ma_{2m-1}t_m = y_ma_{2m} \tag{1.1}$$

where $m \geq 1,\ a_i \in U\ (i = 0, 1, \ldots, 2m),\ y_i, t_i \in S\ (i = 1, 2, \ldots, m)$, and

$$a_0 = y_1a_1, \qquad a_{2m-1}t_m = a_{2m},$$
$$a_{2i-1}t_i = a_{2i}t_{i+1}, \qquad y_ia_{2i} = y_{i+1}a_{2i+1} \qquad (1 \leq i \leq m-1).$$

Such a series of factorization is called a $zigzag$ in S over U with value d, length m and spine $a_0, a_1, \ldots, a_{2m}$.

The following result is from Khan [6] and is also necessary for our investigations.

Theorem 1.2. *([6], Result 3) Let U and S be semigroups with U as a subsemigroup of S. Take any $d \in S\backslash U$ such that $d \in Dom(U, S)$. Let (1) be a zigzag of shortest possible length m over U with value d. Then $t_j, y_j \in S\backslash U$ for all $j = 1, 2, \ldots, m$.*

Definition 1.3. Following Nagy [8], a semigroup S is called $externally\ commutative$ semigroup if it satisfies the identity $x_1x_2x_3 = x_3x_2x_1$.

Definition 1.4. Following Nagy [8], a semigroup S is called $right\ commutative$ semigroup if it satisfies the identity $x_1x_2x_3 = x_1x_3x_2$.

Definition 1.5. Following Nagy [8], a semigroup S is called $left\ commutative$ semigroup if it satisfies the identity $x_1x_2x_3 = x_2x_1x_3$.

Definition 1.6. A semigroup S is called $dual\ cyclic\ commutative$ semigroup if it satisfies the identity $x_1x_2x_3 = x_3x_1x_2$.

Definition 1.7. A semigroup S is called $right\ para\ externally\ commutative$ semigroup if it satisfies the identity $x_1x_2x_3x_4 = x_3x_4x_2x_1$.

Definition 1.8. A semigroup S is called $left\ para\ externally\ commutative$ semigroup if it satisfies the identity $x_1x_2x_3x_4 = x_4x_3x_1x_2$.

Definition 1.9. A semigroup S is called $right\ cyclic\ commutative$ semigroup if it satisfies the identity $x_1x_2x_3x_4 = x_1x_3x_4x_2$.

Definition 1.10. A semigroup S is called $left\ dual\ cyclic\ commutative$ semigroup if it satisfies the identity $x_1x_2x_3x_4 = x_3x_1x_2x_4$.

The semigroup theoretic notations and conventions of Clifford and Preston [2] and Howie [4] will be used throughout without explicit mention.

2 Closedness and Varieties of semigroups

Higgins [3, Chapter 4] had shown that not all varieties of semigroups are absolutely closed by giving examples of a rectangular band and a normal band etc. that are not absolutely closed. So, it is worthy of attention to find out the varieties which are closed in itself. In this section, we have been able to show that some homotypical varieties are closed in itself, but to find out the complete list of varieties that are closed in itself still remains an open problem.

Theorem 2.1. *Let $\mathcal{V}$ be a variety admitting an identity of the form $[axy = xyax]$ is closed.*

Proof. Take any $U, S \in \mathcal{V}$ with U a subsemigroup of S and let $d \in Dom(U, S)\backslash U$. Suppose that d has zigzag equations of type (1) in S over U with value d of length m. Now

$$\begin{aligned}
d &= a_0t_1 \text{ (by zigzag equations)}\\
&= y_1a_1t_1 \text{ (by zigzag equations)}\\
&= a_1t_1y_1a_1 \text{ (since } S \text{ satisfies the identity } axy = xyax)\\
&= a_1(t_1y_1a_1)\\
&= a_1(y_1a_1t_1y_1) \text{ (since } S \text{ satisfies the identity } axy = xyax)\\
&= a_1y_1(a_1t_1)y_1\\
&= a_1y_1(a_2t_2)y_1 \text{ (by zigzag equations)}\\
&= a_1(y_1a_2)t_2y_1
\end{aligned}$$

$= a_1(y_2a_3)t_2y_1$ (by zigzag equations)

$= a_1(y_2a_3t_2)y_1$

$= a_1(a_3t_2y_2a_3)y_1$ (since S satisfies the identity $axy = xyax$)

$= a_1a_3t_2(y_2a_3)y_1$

$= a_1a_3t_2(y_1a_2)y_1$ (by zigzag equations)

$= (a_1(a_3t_2)y_1)a_2y_1$

$= ((a_3t_2)y_1a_1(a_3t_2))a_2y_1$ (since S satisfies the identity $axy = xyax$)

$= ((a_3t_2y_1a_1a_3t_2)a_2y_1)$

$= (a_2y_1(a_3t_2y_1a_1a_3t_2)a_2)$ (since S satisfies the identity $axy = xyax$)

$= a_2y_1((a_3t_2)y_1a_1(a_3t_2))a_2$

$= a_2y_1(a_1(a_3t_2)y_1)a_2$ (since S satisfies the identity $axy = xyax$)

$= a_2(y_1a_1(a_3t_2)y_1)a_2$

$= a_2((a_3t_2)y_1a_1)a_2$ (since S satisfies the identity $axy = xyax$)

$= a_2a_3t_2(y_1a_1)a_2$

$= a_2a_3t_2(a_0)a_2$ (by zigzag equations)

$= (a_2(a_3t_2)a_0a_2)$

$= a_0a_2a_3t_2$ (since S satisfies the identity $axy = xyax$)

$= (\prod_{i=0}^{1} a_{2i})(a_3t_2)$

$\vdots$

$= (\prod_{i=0}^{m-2} a_{2i})(a_{2m-3}t_{m-1})$

$= a_0a_2a_4 \cdots a_{2m-4}a_{2m-2}t_m$ (by zigzag equations)

$= y_1(a_1a_2(a_4 \cdots a_{2m-4}a_{2m-2}))t_m$ (by zigzag equations)

$= y_1(a_2(a_4 \cdots a_{2m-4}a_{2m-2})a_1a_2)t_m$ (since S satisfies the identity $axy = xyax$)

$= (y_1a_2)a_4 \cdots a_{2m-4}a_{2m-2}a_1a_2t_m$

$= (y_2a_3)a_4 \cdots a_{2m-4}a_{2m-2}a_1a_2t_m$ (by zigzag equations)

$= y_2(a_3a_4(a_6 \cdots a_{2m-4}a_{2m-2}a_1))a_2t_m$

$= y_2(a_4(a_6 \cdots a_{2m-4}a_{2m-2}a_1)a_3a_4)a_2t_m$ (since S satisfies the identity $axy = xyax$)

$= (y_2a_4)a_6 \cdots a_{2m-4}a_{2m-2}a_1a_3a_4a_2t_m$

$= (y_3a_5)a_6 \cdots a_{2m-4}a_{2m-2}a_1a_3a_4a_2t_m$ (by zigzag equations)

$= y_3(a_5a_6(a_8 \cdots a_{2m-4}a_{2m-2}a_1a_3))a_4a_2t_m$

$= y_3(a_6(a_8 \cdots a_{2m-4}a_{2m-2}a_1a_3)a_5a_6)a_4a_2t_m$
(since S satisfies the identity $axy = xyax$)

$\vdots$

$= (y_{m-1}a_{2m-2})a_1a_3a_5 \cdots a_{2m-3}a_{2m-2} \cdots a_6a_4a_2t_m$

$= (y_ma_{2m-1})a_1a_3a_5 \cdots a_{2m-3}a_{2m-2} \cdots a_6a_4a_2t_m$
(by zigzag equations)

$= (y_ma_{2m-1}(a_1a_3a_5 \cdots a_{2m-3}a_{2m-2} \cdots a_6a_4a_2))t_m$

$= (a_{2m-1}(a_1a_3a_5 \cdots a_{2m-3}a_{2m-2} \cdots a_6a_4a_2)y_ma_{2m-1})t_m$
(since S satisfies the identity $axy = xyax$)

$= a_{2m-1}a_1a_3a_5 \cdots a_{2m-3}a_{2m-2} \cdots a_6a_4a_2(y_ma_{2m-1}t_m)$

$= a_{2m-1}a_1a_3a_5 \cdots a_{2m-3}a_{2m-2} \cdots a_6a_4a_2(a_{2m-1}t_my_ma_{2m-1})$
(since S satisfies the identity $axy = xyax$)

$$= a_{2m-1}a_1a_3a_5\cdots a_{2m-3}a_{2m-2}\cdots a_6a_4a_2(a_{2m-1}t_m)y_ma_{2m-1}$$

$$= a_{2m-1}a_1a_3a_5\cdots a_{2m-3}a_{2m-2}\cdots a_6a_4a_2(a_{2m})y_ma_{2m-1}$$
(by zigzag equations)

$$= (a_{2m-1}(a_1a_3a_5\cdots a_{2m-3}a_{2m-2}\cdots a_6a_4a_2a_{2m})y_ma_{2m-1})$$

$$= (y_ma_{2m-1}(a_1a_3a_5\cdots a_{2m-3}a_{2m-2}\cdots a_6a_4a_2a_{2m}))$$
(since S satisfies the identity $axy = xyax$)

$$= (y_ma_{2m-1})a_1a_3a_5\cdots a_{2m-3}a_{2m-2}\cdots a_6a_4a_2a_{2m}$$

$$= (y_{m-1}a_{2m-2})a_1a_3a_5\cdots a_{2m-3}a_{2m-2}\cdots a_6a_4a_2a_{2m}$$
(by zigzag equations)

$$= y_{m-1}(a_{2m-2}(a_1a_3a_5\cdots a_{2m-5})a_{2m-3}a_{2m-2})a_{2m-4}\cdots a_6a_4a_2a_{2m}$$

$$= y_{m-1}(a_{2m-3}a_{2m-2}(a_1a_3a_5\cdots a_{2m-5}))a_{2m-4}\cdots a_6a_4a_2a_{2m}$$
(since S satisfies the identity $axy = xyax$)

$$= (y_{m-1}a_{2m-3})a_{2m-2}a_1a_3a_5\cdots a_{2m-5}a_{2m-4}\cdots a_6a_4a_2a_{2m}$$

$$= (y_{m-2}a_{2m-4})a_{2m-2}a_1a_3a_5\cdots a_{2m-5}a_{2m-4}\cdots a_6a_4a_2a_{2m}$$
(by zigzag equations)

$$= y_{m-2}(a_{2m-4}(a_{2m-2}a_1a_3a_5\cdots a_{2m-7})a_{2m-5}a_{2m-4})a_{2m-6}\cdots a_6a_4a_2a_{2m}$$

$$= y_{m-2}(a_{2m-5}a_{2m-4}(a_{2m-2}a_1a_3a_5\cdots a_{2m-7}))a_{2m-6}\cdots a_6a_4a_2a_{2m}$$
(since S satisfies the identity $axy = xyax$)

$$= (y_{m-2}a_{2m-5})a_{2m-4}a_{2m-2}a_1a_3a_5\cdots a_{2m-7}a_{2m-6}\cdots a_6a_4a_2a_{2m}$$

$$= (y_{m-3}a_{2m-6})a_{2m-4}a_{2m-2}a_1a_3a_5\cdots a_{2m-7}a_{2m-6}\cdots a_6a_4a_2a_{2m}$$
(by zigzag equations)

$$\vdots$$

$$= y_2(a_4(a_6\cdots a_{2m-6}a_{2m-4}a_{2m-2}a_1)a_3a_4)a_2a_{2m}$$

$$= y_2(a_3a_4(a_6 \cdots a_{2m-6}a_{2m-4}a_{2m-2}a_1))a_2a_{2m}$$
(since S satisfies the identity $axy = xyax$)

$$= (y_2a_3)a_4a_6 \cdots a_{2m-6}a_{2m-4}a_{2m-2}a_1a_2a_{2m}$$

$$= (y_1a_2)a_4a_6 \cdots a_{2m-6}a_{2m-4}a_{2m-2}a_1a_2a_{2m}$$
(by zigzag equations)

$$= y_1(a_2(a_4a_6 \cdots a_{2m-6}a_{2m-4}a_{2m-2})a_1a_2)a_{2m}$$

$$= y_1(a_1a_2(a_4a_6 \cdots a_{2m-6}a_{2m-4}a_{2m-2}))a_{2m}$$
(since S satisfies the identity $axy = xyax$)

$$= a_0a_2a_4a_6 \cdots a_{2m-6}a_{2m-4}a_{2m-2}a_{2m} \text{ (by zigzag equations)}$$
$$\in U$$

$\Rightarrow Dom(U, S) = U$.
Thus the proof of the theorem is completed. □

Theorem 2.2. *Let $\mathcal{V}$ be a variety admitting an identity of the form $[axy = xxya]$ is closed.*

Proof. Take any U, $S \in \mathcal{V}$ with U a subsemigroup of S and let $d \in Dom(U, S)\backslash U$ has zigzag equations of type (1) in S over U with value d of length m. Now

$$d = a_0t_1 \text{ (by zigzag equations)}$$

$$= y_1a_1t_1 \text{ (by zigzag equations)}$$

$$= a_1a_1t_1y_1 \text{ (since } S \text{ satisfies the identity } axy = xxya)$$

$$= a_1(a_1t_1y_1)$$

$$= a_1(t_1t_1y_1a_1) \text{ (since } S \text{ satisfies the identity } axy = xxya)$$

$$= (a_1t_1)t_1y_1a_1$$

$$= (a_2t_2)t_1y_1a_1 \text{ (by zigzag equations)}$$

$$= (a_2(t_2t_1)y_1)a_1$$

$$= ((t_2t_1)(t_2t_1)y_1a_2)a_1 \text{ (since } S \text{ satisfies the identity } axy = xxya)$$

$$= t_2t_1t_2t_1(y_1a_2)a_1$$

$= t_2t_1t_2t_1(y_2a_3)a_1$ (by zigzag equations)

$= t_2t_1t_2t_1(y_2a_3a_1)$

$= t_2t_1t_2t_1(a_3a_3a_1y_2)$ (since S satisfies the identity $axy = xxya$)

$= ((t_2t_1)(t_2t_1)a_3a_3)a_1y_2$

$= (a_3(t_2t_1)a_3)a_1y_2$ (since S satisfies the identity $axy = xxya$)

$= a_3t_2t_1(a_3a_1y_2)$

$= a_3t_2t_1(a_1a_1y_2a_3)$ (since S satisfies the identity $axy = xxya$)

$= a_3t_2t_1a_1a_1(y_2a_3)$

$= a_3t_2t_1a_1a_1(y_1a_2)$ (by zigzag equations)

$= a_3t_2(t_1a_1a_1)y_1a_2$

$= a_3t_2(a_1a_1a_1t_1)y_1a_2$ (since S satisfies the identity $axy = xxya$)

$= a_3t_2a_1a_1(a_1t_1)y_1a_2$

$= a_3t_2a_1a_1(a_2t_2)y_1a_2$ (by zigzag equations)

$= a_3t_2(a_1a_1a_2t_2)y_1a_2$

$= a_3t_2(t_2a_1a_2)y_1a_2$ (since S satisfies the identity $axy = xxya$)

$= a_3(t_2t_2a_1a_2)y_1a_2$

$= a_3(a_2t_2a_1)y_1a_2$ (since S satisfies the identity $axy = xxya$)

$= a_3a_2(t_2a_1(y_1a_2))$

$= a_3a_2(a_1a_1(y_1a_2)t_2)$ (since S satisfies the identity $axy = xxya$)

$= (a_3a_2(a_1a_1y_1a_2))t_2$

$= (a_2a_2(a_1a_1y_1a_2)a_3)t_2$ (since S satisfies the identity $axy = xxya$)

$= (a_2a_2(a_1a_1)y_1)a_2a_3t_2$

$= (y_1a_2(a_1a_1))a_2a_3t_2$ (since S satisfies the identity $axy = xxya$)

$= y_1a_2(a_1a_1a_2(a_3t_2))$

$= y_1a_2((a_3t_2)a_1a_2)$ (since S satisfies the identity $axy = xxya$)

$= ((y_1a_2a_3t_2)a_1a_2)$

$= (a_1a_1a_2(y_1a_2a_3t_2))$ (since S satisfies the identity $axy = xxya$)

$= (a_1a_1a_2y_1)a_2a_3t_2$

$= (y_1a_1a_2)a_2a_3t_2$ (since S satisfies the identity $axy = xxya$)

$= (y_1a_1)a_2a_2a_3t_2$

$= (a_0)a_2a_2a_3t_2$ (by zigzag equations)

$= (a_0a_2(a_2a_3t_2))$

$= (a_2a_2(a_2a_3t_2)a_0)$ (since S satisfies the identity $axy = xxya$)

$= a_2(a_2a_2(a_3t_2)a_0)$

$= a_2(a_0a_2(a_3t_2))$ (since S satisfies the identity $axy = xxya$)

$= ((a_2a_0)a_2(a_3t_2))$

$= (a_2a_2(a_3t_2)(a_2a_0))$ (since S satisfies the identity $axy = xxya$)

$= (a_2a_2(a_3t_2)a_2)a_0$

$= (a_2a_2(a_3t_2))a_0$ (since S satisfies the identity $axy = xxya$)

$= (a_2a_2(a_3t_2)a_0)$

$= a_0a_2a_3t_2$ (since S satisfies the identity $axy = xxya$)

$$= (\prod_{i=0}^{1} a_{2i})(a_3 t_2)$$

$\vdots$

$$= (\prod_{i=0}^{m-2} a_{2i})(a_{2m-3} t_{m-1})$$

$$= a_0 a_2 a_4 \cdots a_{2m-4} a_{2m-2} t_m \text{ (by zigzag equations)}$$

$$= y_1(a_1 a_2(a_4 \cdots a_{2m-4} a_{2m-2}))t_m \text{ (by zigzag equations)}$$

$$= y_1(a_2 a_2(a_4 \cdots a_{2m-4} a_{2m-2})a_1)t_m \text{ (since } S \text{ satisfies the identity } axy = xxya)$$

$$= (y_1 a_2)a_2 a_4 \cdots a_{2m-4} a_{2m-2} a_1 t_m$$

$$= (y_2 a_3)a_2 a_4 \cdots a_{2m-4} a_{2m-2} a_1 t_m \text{ (by zigzag equations)}$$

$$= y_2((a_3 a_2)a_4(a_6 \cdots a_{2m-4} a_{2m-2} a_1))t_m$$

$$= y_2(a_4 a_4(a_6 \cdots a_{2m-4} a_{2m-2} a_1)(a_3 a_2))t_m \text{ (since } S \text{ satisfies the identity } axy = xxya)$$

$$= (y_2 a_4)a_4 a_6 \cdots a_{2m-4} a_{2m-2} a_1 a_3 a_2 t_m$$

$$= (y_3 a_5)a_4 a_6 \cdots a_{2m-4} a_{2m-2} a_1 a_3 a_2 t_m \text{ (by zigzag equations)}$$

$$= y_3((a_5 a_4)a_6(a_8 \cdots a_{2m-4} a_{2m-2} a_1 a_3))a_2 t_m$$

$$= y_3(a_6 a_6(a_8 \cdots a_{2m-4} a_{2m-2} a_1 a_3)(a_5 a_4))a_2 t_m \text{ (since } S \text{ satisfies the identity } axy = xxya)$$

$\vdots$

$$= (y_{m-1} a_{2m-2})a_{2m-2} a_1 a_3 a_5 \cdots a_{2m-3} a_{2m-4} \cdots a_4 a_2 t_m$$

$$= (y_m a_{2m-1})a_{2m-2} a_1 a_3 a_5 \cdots a_{2m-3} a_{2m-4} \cdots a_4 a_2 t_m \text{ (by zigzag equations)}$$

$$= (y_m a_{2m-1}(a_{2m-2} a_1 a_3 a_5 \cdots a_{2m-3} a_{2m-4} \cdots a_4 a_2))t_m$$

$$= (a_{2m-1} a_{2m-1}(a_{2m-2} a_1 a_3 a_5 \cdots a_{2m-3} a_{2m-4} \cdots a_4 a_2)y_m)t_m$$
(since S satisfies the identity $axy = xxya$)

$$= (a_{2m-1} a_{2m-1}(a_{2m-2} a_1 a_3 a_5 \cdots a_{2m-3} a_{2m-4} \cdots a_4 a_2 y_m))t_m$$

$= (a_{2m-1}a_{2m-1}(a_{2m-2}a_1a_3a_5\cdots a_{2m-3}a_{2m-4}\cdots a_4a_2y_m)a_{2m-1})t_m$
(since S satisfies the identity $axy = xxya$)

$= (a_{2m-1}a_{2m-1}(a_{2m-2}a_1a_3a_5\cdots a_{2m-3}a_{2m-4}\cdots a_4a_2)y_m)a_{2m-1}t_m$

$= (y_m a_{2m-1}(a_{2m-2}a_1a_3a_5\cdots a_{2m-3}a_{2m-4}\cdots a_4a_2))a_{2m-1}t_m$
(since S satisfies the identity $axy = xxya$)

$= (y_m a_{2m-1})a_{2m-2}a_1a_3a_5\cdots a_{2m-3}a_{2m-4}\cdots a_4a_2a_{2m-1}t_m$

$= (y_{m-1}a_{2m-2})a_{2m-2}a_1a_3a_5\cdots a_{2m-3}a_{2m-4}\cdots a_4a_2a_{2m-1}t_m$
(by zigzag equations)

$= y_{m-1}a_{2m-2}a_{2m-2}a_1a_3a_5\cdots a_{2m-3}a_{2m-4}\cdots a_4a_2(a_{2m-1}t_m)$

$= y_{m-1}a_{2m-2}a_{2m-2}a_1a_3a_5\cdots a_{2m-3}a_{2m-4}\cdots a_4a_2(a_{2m})$
(by zigzag equations)

$= y_{m-1}(a_{2m-2}a_{2m-2}(a_1a_3a_5\cdots a_{2m-5})(a_{2m-3}a_{2m-4}))a_{2m-6}\cdots a_4a_2a_{2m}$

$= y_{m-1}((a_{2m-3}a_{2m-4})a_{2m-2}(a_1a_3a_5\cdots a_{2m-5}))a_{2m-6}\cdots a_4a_2a_{2m}$
(since S satisfies the identity $axy = xxya$)

$= (y_{m-1}a_{2m-3})a_{2m-4}a_{2m-2}a_1a_3a_5\cdots a_{2m-5}a_{2m-6}\cdots a_4a_2a_{2m}$

$= (y_{m-2}a_{2m-4})a_{2m-4}a_{2m-2}a_1a_3a_5\cdots a_{2m-5}a_{2m-6}\cdots a_4a_2a_{2m}$
(by zigzag equations)

$= y_{m-2}(a_{2m-4}a_{2m-4}(a_{2m-2}a_1a_3a_5\cdots a_{2m-7})(a_{2m-5}a_{2m-6}))a_{2m-8}\cdots a_4a_2a_{2m}$

$= y_{m-2}((a_{2m-5}a_{2m-6})a_{2m-4}(a_{2m-2}a_1a_3a_5\cdots a_{2m-7}))a_{2m-8}\cdots a_4a_2a_{2m}$
(since S satisfies the identity $axy = xxya$)

$= (y_{m-2}a_{2m-5})a_{2m-6}a_{2m-4}a_{2m-2}a_1a_3a_5\cdots a_{2m-7}a_{2m-8}\cdots a_4a_2a_{2m}$

$= (y_{m-3}a_{2m-6})a_{2m-6}a_{2m-4}a_{2m-2}a_1a_3a_5\cdots a_{2m-7}a_{2m-8}\cdots a_4a_2a_{2m}$
(by zigzag equations)

$= y_{m-3}(a_{2m-6}a_{2m-6}(a_{2m-4}a_{2m-2}a_1a_3a_5\cdots a_{2m-9})(a_{2m-7}a_{2m-8}))a_{2m-10}\cdots a_4a_2a_{2m}$

$= y_{m-3}((a_{2m-7}a_{2m-8})a_{2m-6}(a_{2m-4}a_{2m-2}a_1a_3a_5\cdots a_{2m-9}))a_{2m-10}\cdots a_4a_2a_{2m}$
(since S satisfies the identity $axy = xxya$)

$= (y_{m-3}a_{2m-7})a_{2m-8}a_{2m-6}a_{2m-4}a_{2m-2}a_1a_3a_5\cdots a_{2m-9}a_{2m-10}\cdots a_4a_2a_{2m}$

$= (y_{m-4}a_{2m-8})a_{2m-8}a_{2m-6}a_{2m-4}a_{2m-2}a_1a_3a_5\cdots a_{2m-9}a_{2m-10}\cdots a_4a_2a_{2m}$
(by zigzag equations)

$\vdots$

$= y_2(a_4a_4(a_6\cdots a_{2m-8}a_{2m-6}a_{2m-4}a_{2m-2}a_1)(a_3a_2))a_{2m}$

$= y_2((a_3a_2)a_4(a_6\cdots a_{2m-8}a_{2m-6}a_{2m-4}a_{2m-2}a_1))a_{2m}$
(since S satisfies the identity $axy = xxya$)

$= (y_2a_3)a_2a_4a_6\cdots a_{2m-8}a_{2m-6}a_{2m-4}a_{2m-2}a_1a_{2m}$

$= (y_1a_2)a_2a_4a_6\cdots a_{2m-8}a_{2m-6}a_{2m-4}a_{2m-2}a_1a_{2m}$
(by zigzag equations)

$= y_1(a_2a_2(a_4\cdots a_{2m-8}a_{2m-6}a_{2m-4}a_{2m-2})a_1)a_{2m}$

$= y_1(a_1a_2(a_4\cdots a_{2m-8}a_{2m-6}a_{2m-4}a_{2m-2}))a_{2m}$
(since S satisfies the identity $axy = xxya$)

$= a_0a_2a_4\cdots a_{2m-8}a_{2m-6}a_{2m-4}a_{2m-2}a_{2m}$ (by zigzag equations)
$\in U$

$\Rightarrow Dom(U, S) = U$.
Thus the proof of the theorem is completed. □

Dually, we can prove the following result:

Theorem 2.3. *Let $\mathcal{V}$ be a variety admitting an identity of the form $[axy = yaxx]$ is closed.*

3 Dominions and some generalized classes of commutative semigroups

Isbell [5, Corollary 2.5] showed that the dominion of a commutative semigroup is commutative. But Khan [7] gave a counter-example to show that this stronger result is false for each (non-trivial) permutation identity other than commutativity. Recently Alam, Higgins and Khan [1] generalized Isbell's result from commutative semigroups to $\mathcal{H}$-commutative semigroups. We, now, find some generalized classes of commutative semigroups for which this stronger result is true in some weaker form.

Theorem 3.1. *Let U be an externally commutative subsemigroup of a right para externally commutative semigroup S. Then $Dom(U, S)$ is also externally commutative semigroup.*

Proof. Let U be an externally commutative subsemigroup of a right para externally commutative semigroup S. Then we have to show that $Dom(U,S)$ is also externally commutative semigroup.

Case (i): If $d_1, d_2, d_3 \in U$, then the result holds trivially.

Case (ii): $d_1 \in Dom(U,S)\backslash U$ and $d_2, d_3 \in U$.
Then, by Theorem 1.1, d_1 has zigzag equations of type (1) in S over U of length m.
Now

$$\begin{aligned}
d_1d_2d_3 &= a_0t_1d_2d_3 \text{ (by zigzag equations)}\\
&= d_2d_3t_1a_0 \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_3x_4x_2x_1)\\
&= t_1a_0d_3d_2 \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_3x_4x_2x_1)\\
&= d_3d_2a_0t_1 \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_3x_4x_2x_1)\\
&= d_3d_2(a_0t_1)\\
&= d_3d_2d_1 \text{ (by zigzag equations)},
\end{aligned}$$

as required.
Case (iii): $d_1, d_2 \in Dom(U,S)\backslash U$ and $d_3 \in U$.
Then, by Theorem 1.1, d_2 has zigzag equations of type (1) in S over U of length m.
Now

$$\begin{aligned}
d_1d_2d_3 &= d_1y_ma_{2m}d_3 \text{ (by zigzag equations)}\\
&= a_{2m}d_3y_md_1 \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_3x_4x_2x_1)\\
&= y_md_1d_3a_{2m} \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_3x_4x_2x_1)\\
&= d_3a_{2m}d_1y_m \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_3x_4x_2x_1)\\
&= (d_3a_{2m}d_1)y_m\\
&= (d_1a_{2m}d_3)y_m \text{ (by Case (ii))}\\
&= d_3y_ma_{2m}d_1 \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_3x_4x_2x_1)\\
&= d_3(y_ma_{2m})d_1\\
&= d_3d_2d_1 \text{ (by zigzag equations)},
\end{aligned}$$

as required.

Case (iv): $d_1, d_2, d_3 \in Dom(U,S)\backslash U$.
Then, by Theorem 1.1, d_3 has zigzag equations of type (1) in S over U of length m.
Now

$$\begin{aligned} d_1d_2d_3 &= d_1d_2y_ma_{2m} \text{ (by zigzag equations)} \\ &= y_ma_{2m}d_2d_1 \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_3x_4x_2x_1) \\ &= (y_ma_{2m})d_2d_1 \\ &= d_3d_2d_1 \text{ (by zigzag equations)}, \end{aligned}$$

as required. Thus the proof of the theorem is completed. □

Theorem 3.2. *Let U be an externally commutative subsemigroup of a left para externally commutative semigroup S. Then $Dom(U,S)$ is also externally commutative semigroup.*

Proof. Let U be an externally commutative subsemigroup of a left para externally commutative semigroup S. Then we have to show that $Dom(U,S)$ is also externally commutative semigroup.

Case (i): If $d_1, d_2, d_3 \in U$, then the result holds trivially.

Case (ii): $d_1 \in Dom(U,S)\backslash U$ and $d_2, d_3 \in U$.
Then, by Theorem 1.1, d_1 has zigzag equations of type (1) in S over U of length m.
Now

$$\begin{aligned} d_1d_2d_3 &= a_0t_1d_2d_3 \text{ (by zigzag equations)} \\ &= d_3d_2a_0t_1 \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_4x_3x_1x_2) \\ &= d_3d_2(a_0t_1) \\ &= d_3d_2d_1 \text{ (by zigzag equations)}, \end{aligned}$$

as required.
Case (iii): $d_1, d_2 \in Dom(U,S)\backslash U$ and $d_3 \in U$.
Then, by Theorem 1.1, d_2 has zigzag equations of type (1) in S over U of length m.
Now

$$\begin{aligned} d_1d_2d_3 &= d_1y_ma_{2m}d_3 \text{ (by zigzag equations)} \\ &= d_3a_{2m}d_1y_m \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_4x_3x_1x_2) \\ &= (d_3a_{2m}d_1)y_m \\ &= (d_1a_{2m}d_3)y_m \text{ (by Case (ii))} \end{aligned}$$

$$= y_m d_3 d_1 a_{2m} \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_4x_3x_1x_2)$$

$$= a_{2m} d_1 y_m d_3 \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_4x_3x_1x_2)$$

$$= d_3 y_m a_{2m} d_1 \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_4x_3x_1x_2)$$

$$= d_3(y_m a_{2m})d_1$$

$$= d_3d_2d_1 \text{ (by zigzag equations)},$$

as required.
Case (iv): $d_1, d_2, d_3 \in Dom(U, S)\backslash U$.
Then, by Theorem 1.1, d_3 has zigzag equations of type (1) in S over U of length m.
Now

$$d_1d_2d_3 = d_1d_2y_m a_{2m} \text{ (by zigzag equations)}$$

$$= a_{2m} y_m d_1 d_2 \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_4x_3x_1x_2)$$

$$= d_2 d_1 a_{2m} y_m \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_4x_3x_1x_2)$$

$$= y_m a_{2m} d_2 d_1 \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_4x_3x_1x_2)$$

$$= (y_m a_{2m})d_2d_1$$

$$= d_3d_2d_1 \text{ (by zigzag equations)},$$

as required. Thus the proof of the theorem is completed. □

Theorem 3.3. *Let U be a right commutative subsemigroup of a right cyclic commutative semigroup S. Then $Dom(U, S)$ is also right commutative semigroup.*

Proof. Let U be a right commutative subsemigroup of a right cyclic commutative semigroup S. Then we have to show that $Dom(U, S)$ is also right commutative semigroup.

Case (i): If $d_1, d_2, d_3 \in U$, then the result holds trivially.

Case (ii): $d_1 \in Dom(U, S)\backslash U$ and $d_2, d_3 \in U$.
Then, by Theorem 1.1, d_1 has zigzag equations of type (1) in S over U of length m.

Now

$$\begin{aligned} d_1d_2d_3 &= y_m(a_{2m}d_2d_3) \text{ (by zigzag equations)} \\ &= y_m(a_{2m}d_3d_2) \text{ (since } U \text{ is right commutative)} \\ &= (y_m a_{2m})d_3d_2 \\ &= d_1d_3d_2 \text{ (by zigzag equations)}, \end{aligned}$$

as required.
Case (iii): If $d_1, d_2 \in Dom(U,S)\backslash U$ and $d_3 \in U$.
Then, by Theorem 1.1, d_2 has zigzag equations of type (1) in S over U of length m.
Now

$$\begin{aligned} d_1d_2d_3 &= d_1y_m a_{2m}d_3 \text{ (by zigzag equations)} \\ &= d_1a_{2m}d_3y_m \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_1x_3x_4x_2) \\ &= d_1d_3y_m a_{2m} \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_1x_3x_4x_2) \\ &= d_1d_3(y_m a_{2m}) \\ &= d_1d_3d_2 \text{ (by zigzag equations)}, \end{aligned}$$

as required.
Case (iv): $d_1, d_2, d_3 \in Dom(U,S)\backslash U$.
Then, by Theorem 1.1, d_3 has zigzag equations of type (1) in S over U of length m.
Now

$$\begin{aligned} d_1d_2d_3 &= d_1d_2y_m a_{2m} \text{ (by zigzag equations)} \\ &= d_1y_m a_{2m}d_2 \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_1x_3x_4x_2) \\ &= d_1(y_m a_{2m})d_2 \\ &= d_1d_3d_2 \text{ (by zigzag equations)}, \end{aligned}$$

as required. Thus the proof of the theorem is completed. □

Theorem 3.4. *Let U be a left commutative subsemigroup of a left dual-cyclic commutative semigroup S. Then $Dom(U,S)$ is also left commutative semigroup.*

Proof. Let U be a left commutative subsemigroup of a left dual-cyclic commutative semigroup S. Then we have to show that $Dom(U,S)$ is also left commutative semigroup.

Case (i): If $d_1, d_2, d_3 \in U$, then the result holds trivially.

Case (ii): $d_1 \in Dom(U,S) \backslash U$ and $d_2, d_3 \in U$.
Then, by Theorem 1.1, d_1 has zigzag equations of type (1) in S over U of length m.
Now

$$
\begin{aligned}
d_1d_2d_3 &= y_m(a_{2m}d_2d_3) \text{ (by zigzag equations)}\\
&= y_m(d_2a_{2m}d_3) \text{ (since } U \text{ is left commutative)}\\
&= a_{2m}y_md_2d_3 \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_3x_1x_2x_4)\\
&= d_2a_{2m}y_md_3 \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_3x_1x_2x_4)\\
&= d_2(a_{2m})y_md_3\\
&= d_2(a_{2m-1}t_m)y_md_3 \text{ (by zigzag equations)}\\
&= d_2(a_{2m-1}t_my_md_3)\\
&= d_2(y_ma_{2m-1}t_md_3) \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_3x_1x_2x_4)\\
&= d_2y_m(a_{2m-1}t_m)d_3\\
&= d_2y_m(a_{2m})d_3 \text{ (by zigzag equations)}\\
&= d_2(y_ma_{2m})d_3\\
&= d_2d_1d_3 \text{ (by zigzag equations),}
\end{aligned}
$$

as required.
Case (iii): $d_1, d_2 \in Dom(U,S) \backslash U$ and $d_3 \in U$.
Then, by Theorem 1.1, d_2 has zigzag equations of type (1) in S over U of length m.
Now

$$
\begin{aligned}
d_1d_2d_3 &= d_1y_ma_{2m}d_3 \text{ (by zigzag equations)}\\
&= a_{2m}d_1y_md_3 \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_3x_1x_2x_4)\\
&= y_ma_{2m}d_1d_3 \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_3x_1x_2x_4)
\end{aligned}
$$

$$= (y_m a_{2m})d_1 d_3$$

$$= d_2 d_1 d_3 \text{ (by zigzag equations)},$$

as required.
Case (iv): $d_1, d_2, d_3 \in Dom(U, S)\backslash U$.
Then, by Theorem 1.1, d_3 has zigzag equations of type (1) in S over U of length m.
Now

$$d_1 d_2 d_3 = (d_1 d_2 a_0)t_1 \text{ (by zigzag equations)}$$

$$= (d_2 d_1 a_0)t_1 \text{ (by Case (iii))}$$

$$= d_2 d_1 (a_0 t_1)$$

$$= d_2 d_1 d_3 \text{ (by zigzag equations)}$$

as required. Thus the proof of the theorem is completed. □

Theorem 3.5. *Let U be a dual cyclic commutative subsemigroup of a left dual-cyclic commutative semigroup S. Then $Dom(U, S)$ is also dual cyclic commutative semigroup.*

Proof. Let U be a dual cyclic commutative subsemigroup of a left dual-cyclic commutative semigroup S. Then we have to show that $Dom(U, S)$ is also dual cyclic commutative semigroup.

Case (i): If $d_1, d_2, d_3 \in U$, then the result holds trivially.

Case (ii): $d_1 \in Dom(U, S)\backslash U$ and $d_2, d_3 \in U$.
Then, by Theorem 1.1, d_1 has zigzag equations of type (1) in S over U of length m.
Now

$$d_1 d_2 d_3 = y_m(a_{2m} d_2 d_3) \text{ (by zigzag equations)}$$

$$= y_m(d_3 a_{2m} d_2) \text{ (since } U \text{ is dual cyclic commutative)}$$

$$= a_{2m} y_m d_3 d_2 \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_3x_1x_2x_4)$$

$$= d_3 a_{2m} y_m d_2 \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_3x_1x_2x_4)$$

$$= d_3(a_{2m})y_m d_2$$

$$= d_3(a_{2m-1}t_m)y_m d_2 \text{ (by zigzag equations)}$$

$$= d_3(a_{2m-1}t_m y_m d_2)$$

$$= d_3(y_m a_{2m-1} t_m d_2) \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_3x_1x_2x_4)$$

$$= d_3 y_m (a_{2m-1} t_m) d_2$$

$$= d_3 y_m (a_{2m}) d_2 \text{ (by zigzag equations)}$$

$$= d_3 (y_m a_{2m}) d_2$$

$$= d_3 d_1 d_2 \text{ (by zigzag equations)},$$

as required.
Case (iii): $d_1, d_2 \in Dom(U,S)\backslash U$ and $d_3 \in U$.
Then, by Theorem 1.1, d_2 has zigzag equations of type (1) in S over U of length m.
Now

$$d_1d_2d_3 = d_1 y_m a_{2m} d_3 \text{ (by zigzag equations)}$$

$$= a_{2m} d_1 y_m d_3 \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_3x_1x_2x_4)$$

$$= y_m(a_{2m} d_1 d_3) \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_3x_1x_2x_4)$$

$$= y_m(d_1 d_3 a_{2m}) \text{ (by Case (ii))}$$

$$= y_m d_1 d_3 (a_{2m})$$

$$= y_m d_1 d_3 (a_{2m-1} t_m) \text{ (by zigzag equations)}$$

$$= y_m (d_1 d_3 a_{2m-1} t_m)$$

$$= y_m (a_{2m-1} d_1 d_3 t_m) \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_3x_1x_2x_4)$$

$$= y_m (a_{2m-1} d_1 d_3 t_m)$$

$$= y_m (d_3 a_{2m-1} d_1 t_m) \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_3x_1x_2x_4)$$

$$= (y_m (d_3 a_{2m-1}) d_1 t_m)$$

$$= (d_1 y_m (d_3 a_{2m-1}) t_m)(\text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_3x_1x_2x_4)$$

$$= (d_1 y_m d_3 a_{2m-1}) t_m$$

$$= (d_3d_1y_ma_{2m-1})t_m \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_3x_1x_2x_4)$$

$$= d_3d_1y_m(a_{2m-1}t_m)$$

$$= d_3d_1y_m(a_{2m}) \text{ (by zigzag equations)}$$

$$= d_3d_1(y_ma_{2m})$$

$$= d_3d_1d_2 \text{ (by zigzag equations)},$$

as required.
Case (iv): $d_1, d_2, d_3 \in Dom(U, S)\setminus U$.
Then, by Theorem 1.1, d_3 has zigzag equations of type (1) in S over U of length m.
Now

$$d_1d_2d_3 = d_1d_2y_ma_{2m} \text{ (by zigzag equations)}$$

$$= y_m(d_1d_2a_{2m}) \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_3x_1x_2x_4)$$

$$= y_m(a_{2m}d_1d_2) \text{ (by Case (iii))}$$

$$= (y_ma_{2m})d_1d_2$$

$$= d_3d_1d_2 \text{ (by zigzag equations)},$$

as required. Thus the proof of the theorem is completed. □

In the view of results of section 3. The following problem still remains open for the research.

Problem Let U be a subsemigroup of S such that U and S satisfies nontrivial permutation identities $(x_1x_2 \cdots x_n = x_{i_1}x_{i_2} \cdots x_{i_n})$ and $(x_1x_2 \cdots x_m = x_{j_1}x_{j_2} \cdots x_{j_m})$ respectively, where, i is a permutation on the set $\{1, 2, \cdots n\}$ and j is a permutation on the set $\{1, 2, \cdots m\}$. Then $Dom(U, S)$ satisfies the identity $(x_1x_2 \cdots x_n = x_{i_1}x_{i_2} \cdots x_{i_n})$.

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