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U.P for Wavelet transform on the affine automorporphism group of quaternionic Heisenberg group

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Abstract In this paper, we will recall the main results of the Fourier transform on the quaternion Heisenberg group $\mathbb{H}_q$, then we introduce the notion of the wavelet transform on the affine automorphism group of $\mathbb{H}_q$. Finally, we will prove a certain number of uncertainty principles associated to this transform.

1 Introduction

The quaternionic Heisenberg group, is an example of a H-type group introduced by Kaplan [10], plays an important role in several branches of mathematics such as representation theory, harmonic analysis, several complex variables, partial differential equations and quantum mechanics. This group is a nilpotent Lie group with underlying manifold $\mathbb{R}^4 \times \mathbb{R}^3$ (see for example [12] for its precise definition).
Wavelet analysis on the Euclidean space $\mathbb{R}^n$ has many applications in pure and applied mathematics (see [4]). It is important to extend the theory of wavelet analysis to various cases (see [7, 9, 11]).

The wavelet theory was born from the analysis of operators (differential and functional space equations). The characterization of the fine oscillatory structure of these spaces led to simpler atomic decompositions and, ultimately, to smooth the orthonormal bases which captured this structure. A mathematical theory has therefore developed which has a rich range of applications in signal and image processing.

The aim of this paper is to show a certain number of uncertainty principles associated with the wavelet transform on the quaternionic Heisenberg group.

The plan of this article is as follows. In section 2, a summary of the Fourier transform on the quaternionic Heisenberg group . The section 3 deals with the wavelet transform on the affine automorphism group of $\mathbb{H}_q$. In the section 4, we will prove a certain number of uncertainty principles associated to this transform.

2 Fourier transform on the quaternionic Heisenberg group

Invented for the first by Sir W.R.Hamilton in 1843, quaternion algebra $\mathbb{H}$ is a non-commutative, associative and division algebra.
The basis $(1, i, j, k)$ of $\mathbb{H}$ satisfies:

$$i^2 = j^2 = k^2 = i.j.k = -1.$$

Let $x = x_0 + x_1.i + x_2.j + x_3.k$ a quaternion where x_0, x_1, x_2, x_3 are real numbers.

- $Re(x) := x_0$ is called the real part of x.
- x is called a pure quaternion if $Re(x) = 0$.
- $\text{Im}(x) := x_1.i + x_2.j + x_3.k$ is called the imaginary part of x. We will identify $\text{Im}(x)$ with the element (x_1, x_2, x_3) of $\mathbb{R}^3$.
- $\overline{x} = x_0 - x_1.i - x_2.j - x_3.k$ is the conjugate and $|x| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$ is the module (norm) of x and we have

$$\forall x, y \in \mathbb{H} : \ \ |x| = |\overline{x}|, \ \ \overline{x.y} = \overline{y}.\overline{x}, \ \ x.\overline{x} = \overline{x}.x = |x|^2.$$

- Let $Sp(1) = \{k \in \mathbb{H} / |k| = 1\}$ the subgroup (of $(\mathbb{H} - \{0\}, \times)$) of the quaternions of module 1. It is isomorphic to the $SO(3)$ group [2].

Let $\mathbb{H}_q := \mathbb{H} \times \mathbb{R}^3 = \{(x, t) \ / \ x \in \mathbb{H}, \ t \in \mathbb{R}^3\}$ equipped with the following product

$$(x, t).(y, s) = (x + y, t + s - 2\text{Im}(\overline{y}x)).$$

Then $\mathbb{H}_q$ becomes a non-commutative Lie group, called the quaternionic Heisenberg group. Let $dxdt$ be the Lebesgue measure defined on $\mathbb{H}_q$, the inner product of $L^2(\mathbb{H}_q, dxdt)$ is defined as follows :

$$\forall f, g \in L^2(\mathbb{H}_q, dxdt) : \ \ \langle f, g\rangle = \int_{\mathbb{H}_q} f(x, t)\overline{g(x, t)}dxdt.$$

Note that the pair $(\mathbb{H}_q, Sp(1))$ is a Gelfand pair [6].

- We define the translation operators by

$$T_{(x,t)} : (x', t') \longmapsto (x + x', t + t' - 2Im(\overline{x'}x)) = (x, t).(x', t')$$

  Since $\mathbb{H}$ is noncommutative, we define left translation and right translation which are different. Here translation means left translation.

- The dilation operators is defined by

$$T_\rho : (x', t') \longmapsto (\sqrt{\rho}x', \rho t'), \ \ \rho > 0$$

  This dilation plays the role of a homothety.

- For $u, v \in Sp(1)$, we define the operator $T_{u,v}$ on $\mathbb{H}_q$ by

$$T_{u,v} : (x', t') \longmapsto (ux'\overline{v}, vt'\overline{v}).$$

  $T_{u,v}$ can be considered as the rotation transformation on $\mathbb{H}_q$ [8].

- Let $G = \{(x, t, \rho, u, v) \ / \ (x, t) \in \mathbb{H}_q, \rho > 0, u, v \in Sp(1)\}$ with the product

$$u_1.u_2 = (T_{(x_1,t_1)}T_{\rho_1}T_{u_1,v_1}(x_2, t_2), \rho_1\rho_2, u_1u_2, v_1v_2),$$

  where $u_1 = (x_1, t_1, \rho_1, u_1, v_1)$ and $u_2 = (x_2, t_2, \rho_2, u_2, v_2)$.

  This group G act on $\mathbb{H}_q$ by

$$\begin{aligned}(x, t, \rho, u, v)(x', t') &= (x + \sqrt{\rho}ux'\overline{v}, t + \rho vt'\overline{v} - 2\sqrt{\rho}Im(v\overline{x'}\overline{u}x)) \\ &= T_{(x,t)}T_\rho T_{u,v}(x', t').\end{aligned}$$

  Then (x, t, ρ, u, v) will be identified with $T_{(x,t)}T_\rho T_{u,v}$, so we will consider G instead of the affine automorphism group of $\mathbb{H}_q$.

- G is locally compact non-unimodular group with the left Haar measure

$$dm(x,t,\rho,u,v) = \frac{dxdtd\rho dudv}{\rho^6},$$

where du and dv are the normalized Haar measures of group $Sp(1)$.

Consider the unitary representation π of G on $L^2(\mathbb{H}_q)$ defined by ([8])

$$\pi(x,t,\rho,u,v)F = \rho^{-5/2}.F.T_{\frac{1}{\rho}}.T_{\overline{u},\overline{v}}.T_{(-x,-t)}$$

precisely we have

$$\pi(x,t,\rho,u,v)F(x',t') = \rho^{-5/2}F\left(\frac{\overline{u}(x'-x)v}{\sqrt{\rho}}, \frac{\overline{v}(t'-t+2Im(\overline{x'}x))v}{\rho}\right).$$

Given a non-zero pure quaternion λ ($\lambda \in \mathbb{R}^3\backslash\{0\}$), then the map $J_\lambda : q \longmapsto q\,\widetilde{\lambda}$ define a complex structure of $\mathbb{H}$, where $\widetilde{\lambda} = \frac{\lambda}{|\lambda|}$, [14]. Consider the Fock space $\mathcal{F}_\lambda$ of all holomorphic functions F on $(\mathbb{H}, J_\lambda) \approx \mathbb{C}^2$ such that

$$\|F\|^2 = \int_{\mathbb{H}} |F(q)|^2 e^{-2|\lambda||q|^2} dq < \infty.$$

Every irreducible, infinite dimensional, unitary representation of $\mathbb{H}_q$ is unitarily equivalent to one and only one of the following unitary representation $\pi_\lambda(x,t)$ of $\mathbb{H}_q$ on $\mathcal{F}_\lambda$ defined by

$$\forall F \in \mathcal{F}_\lambda,\ \ \pi_\lambda(x,t)F(q) := e^{i\langle\lambda,t\rangle}\ e^{-|\lambda|\left(|x|^2+2\langle q,x\rangle-2i\langle q\widetilde{\lambda},x\rangle\right)}F(q+x).$$

We associate to a integrable and square integrable function f on $\mathbb{H}_q$, the Fourier transform

$$\widehat{f}(\lambda) := \int_{\mathbb{H}_q} f(x,t)\ \pi_\lambda(x,t)dxdt.$$

$\widehat{f}(\lambda)$ is a bounded operator on $\mathcal{F}_\lambda$ and $\|\widehat{f}(\lambda)\|_\infty \leqslant \|f\|_1$. Moreover, $\widehat{f}(\lambda)$ is a Hilbert-Schmidt operator.

Let S_2 denote the Hilbert space of Hilbert-Schmidt operators on $L^2(\mathbb{H}_q, dxdt)$ with the inner product $\langle T, S\rangle = Trace(T.S^*)$ (where S^* is the adjoint operator of S) and $d\sigma(\lambda)$ be the measure defined on $\mathbb{R}^3\backslash\{0\}$ by $d\sigma(\lambda) = \frac{|\lambda|^2}{2\pi^5}d\lambda$ [3].

The Fourier transform can be extended to an isometric isomorphism between $L^2(\mathbb{H}_q)$ and $L^2(\mathbb{R}^3\backslash\{0\}, S_2, d\sigma)$, the space of functions on $\mathbb{R}^3\backslash\{0\}$ taking values in S_2 and square integrable with respect to measure $d\sigma$. Furthermore we have (see [3, 13])

- The Plancherel theorem for the Heisenberg group $\mathbb{H}_q$ is given by

$$\forall f \in L^2(\mathbb{H}_q)\cap L^1(\mathbb{H}_q),\quad \int_{\mathbb{H}_q} |f(x,t)|^2\, dxdt = \int_{\mathbb{R}^3\backslash\{0\}} \|\widehat{f}(\lambda)\|^2_{H.S} d\sigma(\lambda),$$

where $\|\ \|_{H.S}$ is a Hilbert-Schmidt norm.

- The inversion Fourier transform for the Heisenberg group $\mathbb{H}_q$ is given by

$$\forall f \in S(\mathbb{H}_q),\ \ f(x,t) = \int_{\mathbb{R}^3\backslash\{0\}} Trace(\pi^*_\lambda(x,t)\widehat{f}(\lambda))d\sigma(\lambda),$$

where $S(\mathbb{H}_q)$ is the Schwartz space on $\mathbb{H}_q$.

- The convolution product of two integrable functions f and g on $\mathbb{H}_q$ is defined by

$$f * g(x,t) = \int_{\mathbb{H}_q} f(y,s)g((-y,-s).(x,t))dyds.$$

Let Δ be the sub-Laplacian on $\mathbb{H}_q$ (see [6] for its definition) and for $m \in \mathbb{N}$, we consider

$$H_m = \left\{ f \in L^2(\mathbb{H}_q) \,/\, \widehat{\Delta f}(\lambda) = -8(m+1)|\lambda| \widehat{f}(\lambda) \right\}$$

Theorem 2.1. *([8]) H_m is an irreducible invariant closed subspace of $L^2(\mathbb{H}_q)$ under the unitary representation π of G, and we have*

$$L^2(\mathbb{H}_q) = \bigoplus_{m=0}^{+\infty} H_m. \tag{2.1}$$

3 Continuous wavelet transform

Let $\phi \in L^2(\mathbb{H}_q)$, by the decomposition (2.1), we have

$$\phi = \sum_{m=0}^{+\infty} \phi_m, \quad \phi_m \in H_m$$

If there exists a constant C_ϕ, which is independent of m, such that

$$\frac{1}{m+1} \int_{\mathbb{R}^3 \setminus \{0\}} \|\widehat{\phi_m}(\lambda)\|^2_{H.S} \frac{d\lambda}{|\lambda|^3} = C_\phi < +\infty \;\text{ for all } m \in \mathbb{N},$$

we say that ϕ is an admissible wavelet in $L^2(\mathbb{H}_q)$ for π [8].

Example 3.1. For $r > 0$, let $\eta(r) = \frac{e^{-r}-1}{r^2+1}$ and consider the function ϕ defined by

$$\widehat{\phi}(\lambda) = \sum_{m=0}^{+\infty} \eta((m+1)|\lambda|) P_{\lambda,m},$$

where $P_{\lambda,m}$ is the orthogonal projection operator from $\mathcal{F}_\lambda$ to $\mathcal{F}_{\lambda,m}$,the subspace of $\mathcal{F}_\lambda$ which consists of all homogeneous polynomials of degree m in $q \in \mathbb{C}^2$.
We show that ϕ is admissible wavelet, moreover ϕ is a radial function (see [8, p.12-13] and with here $\eta(r) = \frac{e^{-r}-1}{r^2+1}$).

Definition 3.2. (Wavelet transform, [8]) We define the continuous wavelet transform of $f \in L^2(\mathbb{H}_q)$ by

$$W_\phi f(x,t,\rho,u,v) = \langle f, \pi(x,t,\rho,u,v)\phi \rangle_{L^2(\mathbb{H}_q)}. \tag{3.1}$$

Remark 3.3.

- The continuous wavelet transform is a voice transform on $L^2(\mathbb{H}_q)$.
- For $(x,t,\rho,u,v) \in G$, we have

$$|W_\phi f(x,t,\rho,u,v)| \leqslant \|f\|_2 \, \|\phi\|_2. \tag{3.2}$$

  Then $W_\phi f \in L^\infty(G)$ and $\|W_\phi f\|_\infty \leqslant \|f\|_2 \, \|\phi\|_2$.

Theorem 3.4. *(Plancherel formula, [8]) Suppose ϕ is an admissible wavelet, then*

$$\|W_\phi f\|_{L^2(G,dm)} = C_\phi^{1/2} \|f\|_{L^2(\mathbb{H}_q)}, \quad f \in L^2(\mathbb{H}_q). \tag{3.3}$$

Theorem 3.5. *(Inversion formula,[8]) Suppose ϕ is an admissible wavelet and $f \in L^2(\mathbb{H}_q)$, then*

$$f(x,t) = \frac{1}{C_\phi} \int_G W_\phi f(y,s,\rho,u,v) \pi(y,s,\rho,u,v)\phi(x,t) dm(y,s,\rho,u,v). \tag{3.4}$$

4 Uncertainty principles for wavelet transform

Theorem 4.1 (Lieb inequality). *Let ψ and ϕ be two admissible wavelets in $L^2(\mathbb{H}_q)$ for π. For $p \in [1;+\infty[$ and f,g in $L^2(\mathbb{H}_q)$, the function*

$$(x,t,\rho,u,v) \longmapsto W_\phi f(x,t,\rho,u,v) W_\psi g(x,t,\rho,u,v)$$

belong to $(L^p(G), dm)$ and

$$\|W_\phi f W_\psi g\|_{(L^p(G),dm)} \leqslant (C_\phi C_\psi)^{\frac{1}{2p}} \left(\|\phi\|_2 \|\psi\|_2\right)^{\frac{p-1}{p}} \|f\|_2 \|g\|_2. \tag{4.1}$$

Proof.

- According to Cauchy-Schwartz inequality and Plancherel theorem (3.3) for the continuous wavelet transform W_ϕ and W_ψ and for every f,g in $L^2(\mathbb{H}_q)$, we get

$$\begin{aligned}\int_G |W_\phi f(x,t,\rho,u,v) W_\psi g(x,t,\rho,u,v)| dm(x,t,\rho,u,v) &\leqslant \|W_\phi f\|_2 \, \|W_\psi g\|_2 \\ &\leqslant \sqrt{C_\phi . C_\psi} \, \|f\|_2 \, \|g\|_2.\end{aligned}$$

- According to Cauchy-Schwartz inequality and (3.2), for every $(x,t,\rho,u,v) \in G$ we have

$$|W_\phi f(x,t,\rho,u,v) W_\psi g(x,t,\rho,u,v)| \leqslant \|\phi\|_2 \, \|\psi\|_2 \, \|f\|_2 \, \|g\|_2.$$

- For $p \in [1;+\infty[$, we have

$$\begin{aligned}&\int_G |W_\phi f(x,t,\rho,u,v) W_\psi g(x,t,\rho,u,v)|^p dm(x,t,\rho,u,v) \\ \leqslant\ & (\|\phi\|_2 \, \|\psi\|_2 \, \|f\|_2 \, \|g\|_2)^{p-1} \int_G |W_\phi f(x,t,\rho,u,v) W_\psi g(x,t,\rho,u,v)| dm(x,t,\rho,u,v) \\ \leqslant\ & (\|\phi\|_2 \, \|\psi\|_2 \, \|f\|_2 \, \|g\|_2)^{p-1} \sqrt{C_\phi . C_\psi} \, \|f\|_2 \, \|g\|_2 \\ \leqslant\ & (\|\phi\|_2 \, \|\psi\|_2)^{p-1} \sqrt{C_\phi . C_\psi} \, (\|f\|_2 \, \|g\|_2)^p,\end{aligned}$$

  then

$$\|W_\phi f W_\psi g\|_{(L^p(G),dm)} \leqslant (C_\phi C_\psi)^{\frac{1}{2p}} \left(\|\phi\|_2 \|\psi\|_2\right)^{\frac{p-1}{p}} \|f\|_2 \|g\|_2.$$

□

Proposition 4.2. *Let ϕ be an admissible wavelet for π in $L^2(\mathbb{H}_q)$ and $p \in [2;+\infty[\cup \{+\infty\}$. For every f in $L^2(\mathbb{H}_q)$, the function $W_\phi f$ belong to $L^p(G)$ and*

$$\|W_\phi f\|_p \leqslant C_\phi^{\frac{1}{p}} \|\phi\|_2^{1-\frac{2}{p}} \|f\|_2. \tag{4.2}$$

Proof. For $p = +\infty$, the result is deduced from the formula (3.2).
Let $p \geqslant 2$, then $b = \frac{p}{2}$ is in $[1;+\infty[$. By taking in the inequality (4.1) of the previous theorem (Lieb inequality) $f = g$ and $\phi = \psi$, we obtain

$$\|(W_\phi f)^2\|_{(L^b(G),dm)} \leqslant C_\phi^{\frac{1}{b}} \|\phi\|_2^{2\frac{b-1}{b}} \|f\|_2^2,$$

therefore

$$\left(\int_G |W_\phi f(x,t,\rho,u,v)|^{2b} dm(x,t,\rho,u,v)\right)^{\frac{1}{b}} \leqslant C_\phi^{\frac{1}{b}} \|\phi\|_2^{2(1-1/b)} \|f\|_2^2,$$

we obtain

$$\|W_\phi f\|_{L^p(G)}^2 \leqslant C_\phi^{\frac{2}{p}} \|\phi\|_2^{2(1-2/p)} \|f\|_2^2,$$

so

$$\|W_\phi f\|_p \leqslant C_\phi^{\frac{1}{p}} \|\phi\|_2^{1-\frac{2}{p}} \|f\|_2.$$

□

Proposition 4.3. *If $\phi \in L^2(\mathbb{H}_q)$ is an admissible wavelet for π such that*

$$W_\phi(\phi) \in L^1(\mathbb{H}_q) \ \ and \ \ C_\phi = 1,$$

then for $f \in L^2(\mathbb{H}_q)$

$$W_\phi(f) * W_\phi(\phi) = W_\phi(f), \tag{4.3}$$

where "$*$" is the convolution product in $L^2(G)$.

Proof. Let y in G,

$$\begin{aligned}
W_\phi(f) * W_\phi(\phi)(y) &= \int_G W_\phi(f)(x) W_\phi(\phi)(x^{-1}y) dm(x) \\
&= \int_G \langle f, \pi(x)\phi\rangle \langle \phi, \pi(x^{-1}y)\phi\rangle dm(x) \\
&= \int_G \langle f, \pi(x)\phi\rangle \overline{\langle \pi(y)\phi, \pi(x)\phi\rangle} dm(x) \\
&= C_\phi^2 \langle f, \pi(y)\phi\rangle \ \ \text{(according to the formula (3.3))} \\
&= W_\phi(f)(y),
\end{aligned}$$

then $W_\phi(f) * W_\phi(\phi) = W_\phi(f)$. □

Remark 4.4. With the previous hypotheses of proposition 4.3, the set $S = \text{rang}(W_\phi)$ is a closed subspace of $L^2(G)$ and the above proposition identifies S as a reproducing kernel Hilbert space.

Definition 4.5. A function F in $L^2(G)$ is α-concentrated ($\alpha \geqslant 0$) on a measurable set $\Omega \subseteq G$ if

$$\left(\int_{G \setminus \Omega} |F(x,t,\rho,u,v)|^2 dm(x,t,\rho,u,v) \right)^{\frac{1}{2}} \leqslant \alpha \|F\|_2. \tag{4.4}$$

Theorem 4.6 (Donoho-Stark for W_ϕ). *Let $\alpha \geqslant 0$, ϕ be an admissible wavelet for π and let $f \in L^2(\mathbb{H}_q)$ such that $f \neq 0$. If $W_\phi f$ is α-concentrated on a measurable set $\Omega \subseteq G$ hence , we have*

$$\mu(\Omega) \geqslant C_\phi \frac{1-\alpha^2}{\|\phi\|_2^2}. \tag{4.5}$$

Where $\mu(\Omega)$ is the measure of Ω to respecte the measure dm.

Proof. We have

$$\|W_\phi f\|_2^2 = \|\chi_\Omega W_\phi f\|_2^2 + \|\chi_{G\setminus\Omega} W_\phi f\|_2^2,$$

where χ_E denote the indicator function of the set E.
Since $W_\phi f$ is α-concentrated on Ω and according the Plancherel formula (3.3), we get

$$\begin{aligned}
C_\phi \|f\|_2^2 &\leqslant \|\chi_\Omega W_\phi f\|_2^2 + \alpha^2 \|W_\phi f\|_2^2 \\
&\leqslant \|\chi_\Omega W_\phi f\|_2^2 + \alpha^2 . C_\phi \|f\|_2^2,
\end{aligned}$$

then

$$\begin{aligned}
(1-\alpha^2) C_\phi \|f\|_2^2 &\leqslant \int_G \chi_\Omega(x,t,\rho,u,v) |W_\phi f(x,t,\rho,u,v)|^2 dm(x,t,\rho,u,v) \\
&\leqslant \|W_\phi f\|_\infty^2 \int_G \chi_\Omega(x,t,\rho,u,v) dm(x,t,\rho,u,v) \\
&\leqslant \|f\|_2^2 \ \|\phi\|_2^2 \ \mu(\Omega), \ \text{(using the inequality (3.2))}.
\end{aligned}$$

thus, we deduce the desired result. □

Theorem 4.7 (Lieb uncertainty principle). *Let $\alpha \in [0;1]$, ϕ be an admissible wavelet for π and let $f \in L^2(\mathbb{H}_q)$ such that $f \neq 0$. If $W_\phi f$ is α-concentrated on a measurable set $\Omega \subseteq G$ hence for $p > 2$, we have*

$$\mu(\Omega) \geqslant C_\phi \frac{(1-\alpha^2)^{\frac{p}{p-2}}}{\|\phi\|_2^2}. \tag{4.6}$$

Proof. Let $I = \int_G \chi_\Omega(x,t,\rho,u,v)|W_\phi f(x,t,\rho,u,v)|^2 dm(x,t,\rho,u,v)$, using the Hôlder inequality, we have

$$\begin{aligned}
I &\leqslant \left(\int_G |W_\phi f(x,t,\rho,u,v)|^p dm(x,t,\rho,u,v)\right)^{\frac{2}{p}} \left(\int_G \chi_\Omega(x,t,\rho,u,v)^{\frac{p}{p-2}} dm(x,t,\rho,u,v)\right)^{\frac{p-2}{p}} \\
&\leqslant \|W_\phi f\|_{L^p(G)}^2 (\mu(\Omega))^{\frac{p-2}{p}} \\
&\leqslant \left(C_\phi^{\frac{1}{p}} \|\phi\|_2^{1-\frac{2}{p}} \|f\|_2\right)^2 (\mu(\Omega))^{\frac{p-2}{p}} \quad \text{(by (4.2))} \\
&\leqslant C_\phi^{\frac{2}{p}} \|\phi\|_2^{2-\frac{4}{p}} \|f\|_2^2 (\mu(\Omega))^{\frac{p-2}{p}}.
\end{aligned}$$

However, we have already shown in the previous proof that

$$(1-\alpha^2) C_\phi \|f\|_2^2 \leqslant \|\chi_\Omega W_\phi f\|_2^2,$$

then

$$(1-\alpha^2) C_\phi \leqslant C_\phi^{\frac{2}{p}} \|\phi\|_2^{2-\frac{4}{p}} (\mu(\Omega))^{\frac{p-2}{p}},$$

we then obtain

$$\mu(\Omega) \geqslant C_\phi \frac{(1-\alpha^2)^{\frac{p}{p-2}}}{\|\phi\|_2^2}.$$

□

In the following, we will present theorems of concentration of wavelet transform on subsets of G of finite measure, similar to those proven in [16] and [5].

Let $\phi \in L^2(\mathbb{H}_q)$ be an admissible wavelet for π and P_ϕ is the orthogonal projection on $S = \text{rang}(W_\phi)$ (i.e. the base projection $S = \text{rang}(W_\phi)$ and direction the orthogonal to $S = \text{rang}(W_\phi)$).

Let Ω be a measurable subset of G such that $0 < \mu(\Omega) < +\infty$, and consider the operator P_Ω defined by $P_\Omega F = \chi_\Omega F$ for $F \in L^2(G)$.

The operator norm of $P_\Omega P_\phi$ is defined by

$$\|P_\Omega P_\phi\|_{op} = \sup\left\{\|P_\Omega P_\phi F\|_{L^2(G)} \;/\; F \in L^2(G) \text{ and } \|F\|_2 \leqslant 1\right\}$$

Theorem 4.8. *If $\|P_\Omega P_\phi\|_{op} < 1$ then for every f in $L^2(\mathbb{H}_q)$, we have*

$$\sqrt{C_\phi}\|f\|_2 \leqslant \frac{1}{\sqrt{1-\|P_\Omega P_\phi\|_{op}^2}} \|\chi_{G\setminus\Omega} W_\phi f\|_{L^2(G)} \tag{4.7}$$

Proof. We have

$$\|W_\phi f\|_2^2 = \|\chi_\Omega W_\phi f\|_2^2 + \|\chi_{G\setminus\Omega} W_\phi f\|_2^2,$$

and $\chi_\Omega W_\phi f = P_\Omega P_\phi (W_\phi f)$, then

$$\begin{aligned}
\|\chi_\Omega W_\phi f\|_2^2 &\leqslant \|P_\Omega P_\phi\|_{op}^2 \|W_\phi f\|_2^2 \\
&\leqslant \|P_\Omega P_\phi\|_{op}^2 C_\phi \|f\|_2^2,
\end{aligned}$$

so

$$\begin{aligned}\|\chi_{G\setminus\Omega}W_\phi f\|_2^2 &= \|W_\phi f\|_2^2 - \|\chi_\Omega W_\phi f\|_2^2\\ &\geqslant C_\phi\|f\|_2^2 - \|P_\Omega P_\phi\|_{op}^2 C_\phi\|f\|_2^2\\ &= C_\phi\|f\|_2^2(1-\|P_\Omega P_\phi\|_{op}^2).\end{aligned}$$

We get the desired result. □

Proposition 4.9. *If $W_\phi(\phi)\in L^1(G)$ and $C_\phi = 1$, then*

$$\frac{\|P_\Omega P_\phi\|_{op}^2}{\|\phi\|^2} \leqslant \mu(\Omega). \tag{4.8}$$

Proof. From formula (4.3),then

$$W_\phi f(x',t',\rho',u',v') = \int_G W_\phi f(x,t,\rho,u,v) W_\phi\phi((x,t,\rho,u,v)^{-1}(x',t',\rho',u',v'))dm(x,t,\rho,u,v)$$

and

$$P_\Omega P_\phi(W_\phi f)(x',t',\rho',u',v') =$$

$$\int_G W_\phi f(x,t,\rho,u,v)\chi_\Omega(x',t',\rho',u',v') W_\phi\phi((x,t,\rho,u,v)^{-1}(x',t',\rho',u',v'))dm(x,t,\rho,u,v),$$

from this formula, $P_\Omega P_\phi$ is an integral operator and its Hilbert-Schmidt norm $N=\|P_\Omega P_\phi\|_{H.S}^2$ is given by

$$N = \int_G\int_G |\chi_\Omega(x',t',\rho',u',v')W_\phi\phi((x,t,\rho,u,v)^{-1}(x',t',\rho',u',v'))|^2 dm(x,t,\rho,u,v)dm(x',t',\rho',u',v')$$

$$= \int_G \chi_\Omega(x',t',\rho',u',v')\left(\int_G |W_\phi\phi((x,t,\rho,u,v)^{-1}(x',t',\rho',u',v'))|^2 dm(x,t,\rho,u,v)\right)dm(x',t',\rho',u',v')$$

$$= \mu(\Omega)\|W_\phi\phi\|_2^2$$

$$= \mu(\Omega)C_\phi\|\phi\|_2^2.$$

In particular and since $C_\phi = 1$, we have:

$$\|P_\Omega P_\phi\|_{op}^2 \leqslant \|P_\Omega P_\phi\|_{H.S}^2 \leqslant \mu(\Omega)\|\phi\|_2^2.$$

□

Remark 4.10. With the hypotheses and the notations of the two previous results, we find Donoho-Stark's theorem.

Definition 4.11 (Entropy). The entropy of a probability density function P on G is defined by

$$E(P) = -\int_G \ln(P(x,t,\rho,u,v))\; P(x,t,\rho,u,v)dm(x,t,\rho,u,v). \tag{4.9}$$

Remark 4.12. Entropy plays an important role in several areas of physics, to better understand its physical meaning, see [1]. Entropy represents an advantageous means of measuring the decrease of a function f, so it was very interesting to locate the entropy of a probability measure and one of its transforms.

Theorem 4.13 (Beckner's u.p in terms of entropy for W_ϕ). *Let ϕ be an admissible wavelet for π. Then for all $f \in L^2(\mathbb{H}_q)$ such that $f \neq 0$, we have*

$$E(|W_\phi f|^2) \geqslant C_\phi \, \|f\|^2 \, \ln\left(\frac{1}{\|\phi\|_2^2 \, \|f\|_2^2}\right). \tag{4.10}$$

Proof. Assume that $\|\phi\|_2 = \|f\|_2 = 1$, then by relation (3.2) we deduce that

$$\forall (x,t,\rho,u,v) \in G, \; |W_\phi f(x,t,\rho,u,v)| \leqslant \|\phi\|_2 \, \|f\|_2 = 1,$$

then $\ln(|W_\phi f(x,t,\rho,u,v)|) \leqslant 0$, in particular $E(|W_\phi f|) \geqslant 0$.

- Therefore if the entropy $E(|W_\phi f|) = +\infty$, then the inequality (4.10) holds trivially.
- Suppose now that the entropy $E(|W_\phi f|) < +\infty$.

Let $a \in [0;1]$. The study of the variations of the function $p \longmapsto \frac{a^2 - a^p}{p-2}$ over the interval $]2;3]$ gives

Lemma 4.14. *For all $a \in [0;1]$ and for all $p \in]2;3]$, we get*

$$0 \leqslant \frac{a^2 - a^p}{p-2} \leqslant -a^2 \ln(a).$$

Then, for all $(x,t,\rho,u,v) \in G$ and $p \in]2;3]$, we have

$$0 \leqslant \frac{|W_\phi f(x,t,\rho,u,v)|^2 - |W_\phi f(x,t,\rho,u,v)|^p}{p-2} \leqslant -|W_\phi f(x,t,\rho,u,v)|^2 \ln\left(|W_\phi f(x,t,\rho,u,v)|\right). \tag{4.11}$$

Let H be the function defined on $[2;3]$ by

$$H(p) = \int_G |W_\phi f(x,t,\rho,u,v)|^p dm(x,t,\rho,u,v) - C_\phi.$$

According to inequality (4.2), we have $H(p) \leqslant 0$. Moreover $H(2) = 0$, then $H(p) \leqslant H(2)$ and $\left(\frac{dH}{dp}\right)_{p=2^+} \leqslant 0$, whenever this derivative is well defined.

$$\begin{aligned}
\left(\frac{dH}{dp}\right)_{p=2^+} &= \lim_{\substack{p\to 2\\ p>2}} \int_G \frac{|W_\phi f(x,t,\rho,u,v)|^2 - |W_\phi f(x,t,\rho,u,v)|^p}{p-2} dm(x,t,\rho,u,v) \\
&\quad \text{by Lebesgue's dominated convergence theorem and (4.11)} \\
&= \int_G \lim_{\substack{p\to 2\\ p>2}} \frac{|W_\phi f(x,t,\rho,u,v)|^2 - |W_\phi f(x,t,\rho,u,v)|^p}{p-2} dm(x,t,\rho,u,v) \\
&= \frac{1}{2} \int_G |W_\phi f(x,t,\rho,u,v)|^2 \left|\ln\left(|W_\phi f(x,t,\rho,u,v)|\right)\right|^2 dm(x,t,\rho,u,v) \\
&= -\frac{1}{2} E(|W_\phi f|^2),
\end{aligned}$$

so we get $E(|W_\phi f|^2) \geqslant 0$.

For the general case, let $h = \frac{f}{\|f\|_2}$ and $\psi = \frac{\phi}{\|\phi\|_2}$, from where

$$W_\psi(h)(x,t,\rho,u,v) = \frac{W_\phi f(x,t,\rho,u,v)}{\|f\|_2 \, \|\phi\|_2},$$

and

$$\begin{aligned}
0 &\leqslant E(|W_\psi(h)|^2) \\
&= -\int_G \ln(|W_\psi(h)(x,t,\rho,u,v)|^2)\, |W_\psi(h)(x,t,\rho,u,v)|^2 dm(x,t,\rho,u,v) \\
&= -\int_G \ln\left(\frac{|W_\phi f(x,t,\rho,u,v)|^2}{\|f\|_2^2\, \|\phi\|_2^2}\right) \left|\frac{W_\phi f(x,t,\rho,u,v)}{\|f\|_2\, \|\phi\|_2}\right|^2 dm(x,t,\rho,u,v) \\
&= -\int_G \left(\ln(|W_\phi f(x,t,\rho,u,v)|^2) - \ln(\|f\|_2^2\, \|\phi\|_2^2)\right) \left|\frac{W_\phi f(x,t,\rho,u,v)}{\|f\|_2\, \|\phi\|_2}\right|^2 dm(x,t,\rho,u,v) \\
&= \frac{E(|W_\phi f|^2)}{\|f\|_2^2\, \|\phi\|_2^2} + \ln(\|f\|_2^2\, \|\phi\|_2^2)\frac{C_\phi}{\|\phi\|_2^2} \\
&= \frac{E(|W_\phi f|^2)}{\|f\|_2^2\, \|\phi\|_2^2} - \ln\left(\frac{1}{\|f\|_2^2\, \|\phi\|_2^2}\right)\frac{C_\phi}{\|\phi\|_2^2},
\end{aligned}$$

we deduce that

$$E(|W_\phi f|^2) \geqslant C_\phi\, \|f\|_2^2\, \ln\left(\frac{1}{\|\phi\|_2^2\, \|f\|_2^2}\right).$$

□

5 Conclusion

In this article, after recalling the definitions and fundamental properties of Fourier transform on the Heisenberg quaternion group $\mathbb{H}_q$, we introduced the wavelet transform on the affine automorphism group of $\mathbb{H}_q$ and its inversion formula,then we have proved some qualitatives uncertainty principles: Lieb inequality, Donoho-Stark's uncertainty principle....We soon hope to determine a version of Hardy and Beurling's theorem for this transform.

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