Palestine Journal of Mathematics

Vol. 11(3)(2022) , 216–227

HARMONICITY AND DIFFERENTIAL EQUATIONS ACCORDING TO MEAN CURVATURE AND DARBOUX VECTOR OF INVOLUTE CURVE

Osman Çakır and Süleyman Şenyurt

Communicated by Zafar Ahsan

MSC 2010 Classifications: Primary 53A04, Secondary 34A30.

Keywords and phrases: Darboux vector, involute curve, mean curvature, biharmonic, differential equation, Laplace operator, Levi-Civita connection.

Acknowledgement: *The authors would like to thank to 3^{th} International Conference on Mathematical and Related Sciences, held online in 2020, for supporting our paper.*

Abstract In this paper, we write the differential equations and harmonicity of involute of a curve according to both Darboux and mean curvature vectors. In order to do these we make use of a new method, that is, covariant derivative with respect to N. Consequently we obtain all the characterizations of the involute curve in terms of the principal normal N and the Darboux vector W of the main curve.

1 Introduction

To assign a relation between the invariants of a curve and characterizations of it in $\mathbb{E}^3$ and other spaces is one of the main topics we encounter most widely among those of scholars particularly interested in geometry. There are so many papers explaining the close relevance between the characterizations and invariants of the curve pairs. We may recall some remarkable papers catching our attention: Ferrandez et al.[2] studied curves whose mean curvature vector field is in the kernel of certain elliptic differential operators. Bulca et al.[3] worked some characterizations of involute of a given curve in Euclidean space. Arslan et al.[4] examined weak biharmonic rotational surfaces on which parallel mean curvature vector field is weak biharmonic. Cakir and Senyurt [5] focused on the characterization of involute of a curve by means of the unit Darboux vector of the main curve. It is another interesting study on biharmonic curves that we may make a classification,[6]. In this way we can call curves satisfying the condition that the Laplace image of mean curvature is equal to zero as biharmonic, while some of them provided that the Laplace image of mean curvature is equal to non-zero real constant λ times mean curvature may be named as 1-type of harmonic. We use as a tool among many papers only some of them: Kocayigit et al.[7] studied 1-type curves by using the mean curvature vector field of the curve itself. They also studied the same topic by using the Darboux vector instead of mean curvature vector field,[8]. We may also mention that Senyurt and Cakir[9] studied biharmonic curves whose mean curvature vector field is the kernel of Laplace. Recently, Shaikh et. al.[10, 11, 12, 13, 14, 15] initiated the study of surface curves in a different way, especially, rectifying, osculating and normal curves on a surface by considering isometry and conformal map between two surfaces and investigated their invariancy under such maps. Throughout this work, we first take a unit speed curve which we call main curve and then we write an involute of this curve. After that we give all the characterizations of involute curve according to Darboux vector and mean curvature vector of this curve itself. It follows that we make use of the relations between the Frenet frames of involute curve and main curve. In this way we write all characterizations of involute curve in terms of the main curve. Finally we give an example to support our assertions. Now let us recall some basic concepts beginning with the Frenet-Serret formulas

$$T' = \vartheta\kappa N, \qquad N' = -\vartheta\kappa T + \vartheta\tau B\,, \qquad B' = -\vartheta\tau N. \tag{1.1}$$

Frenet vectors T, N, B form a Frenet frame and every Frenet frame moves along a rotation axis which is called a Darboux vector and given by

$$W = \tau T + \kappa B, \tag{1.2}$$

see [16].
Given that α is a differentiable curve with the unit tangent T and β is another differentiable curve. β is called the involute of α on condition that the unit tangent T is perpendicular to the tangent vector of β at the corresponding points of these curves. It is clear from this statement

$$\beta(s) = \alpha(s) + \lambda(s)T(s)\,, \quad \lambda(s) = c - s\,, \quad c \in \mathbb{R}. \tag{1.3}$$

The relation between the Frenet vectors of α and β is

$$T_\beta = N\,, \qquad N_\beta = \frac{-\kappa T + \tau B}{\sqrt{\kappa^2+\tau^2}}\,, \qquad B_\beta = \frac{\tau T + \kappa B}{\sqrt{\kappa^2+\tau^2}}\,. \tag{1.4}$$

Also the relation between the curvatures of α and β is given as

$$\kappa_\beta(s) = \frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\,, \qquad \tau_\beta(s) = \frac{(\frac{\tau}{\kappa})'\kappa}{\lambda(\kappa^2+\tau^2)}\,, \tag{1.5}$$

see [16].
The mapping defined as follows along a differentiable curve α

$$\Delta : \chi^{\perp}(\alpha(I)) \to \chi(\alpha(I))\,, \qquad \Delta H = -D_T^2 H \tag{1.6}$$

is called a Laplace operator and also the mapping

$$\Delta_T^{\perp} X = -D_T^{\perp} D_T^{\perp} X, \;\; \forall X \in \chi^{\perp}(\alpha(I)) \tag{1.7}$$

is called the normal Laplace operator, where H is mean curvature of α and D is Levi-Civita connection, see [6].

Theorem 1.1. *Let α be a unit speed curve with principal normal N and its involute curve is denoted by β. Then we can give the covariant derivatives on the curve β w.r. to Levi-Civita connection D as*

$$D_N T = \kappa N, \quad D_N N = -\kappa T + \tau B, \quad D_N B = -\tau N, \tag{1.8}$$

see [1].

Theorem 1.2. *Let α be a unit speed curve with the mean curvature vector H. Then the following propositions hold.*

i) If $\Delta H = 0$ then α is a biharmonic curve.

ii) If $\Delta H = \lambda H$, then α is a 1-type of harmonic curve.

iii) If $\Delta^{\perp} H = 0$ then α is a weak biharmonic curve.

iv) If $\Delta^{\perp} H = \lambda H$, then α is a 1-type of harmonic curve, provided that $\lambda \in \mathbb{R} - \{0\}$,

see [6].

Theorem 1.3. *Let α be a general helix with the mean curvature vector H, then we can write the differential equation of α w.r. to H*

$$\Delta H + \lambda_1 D_T H + \lambda_2 H = 0 \tag{1.9}$$

with the coefficients $\lambda_1 = 3(\frac{\kappa'}{\kappa})$ *and* $\lambda_2 = \frac{\kappa''}{\kappa} - 3(\frac{\kappa'}{\kappa})^2 - (\kappa^2+\tau^2)$, see [7].

Theorem 1.4. *Let α be a differentiable curve with the Darboux vector W, then we can write the differential equation of α w. r. to W*

$$c_1 D_T^3 W + c_2 D_T^2 W + c_3 D_T W + c_4 W = 0 \tag{1.10}$$

with the coefficients c_1 , c_2 , c_3 , c_4

$$\begin{aligned} c_1 &= \vartheta\big(\kappa\tau' - \kappa'\tau\big)^2, \quad c_2 = \big(\vartheta\kappa''\tau - \vartheta\kappa\tau'' - (\vartheta\kappa\tau' - \vartheta\kappa'\tau)'\big)\big(\kappa\tau' - \kappa'\tau\big)\,, \\ c_3 &= \big(\kappa'''\tau - \kappa\tau''' + \vartheta^2(\kappa\tau' - \kappa'\tau)(\kappa^2 + \tau^2)\big)\big(\vartheta\kappa\tau' - \vartheta\kappa'\tau\big) \\ &\quad + \big(\vartheta\kappa''\tau - \vartheta\kappa\tau'' - (\vartheta\kappa\tau' - \vartheta\kappa'\tau)'\big)\big(\kappa''\tau - \kappa\tau''\big)\,, \\ c_4 &= \big(\kappa'\tau''' - \kappa'''\tau' - \vartheta^2(\kappa\kappa' + \tau\tau')(\kappa\tau' - \kappa'\tau)\big)\big(\vartheta\kappa\tau' - \vartheta\kappa'\tau\big) + \big(\vartheta\kappa''\tau - \vartheta\kappa\tau'' \\ &\quad - (\vartheta\kappa\tau' - \vartheta\kappa'\tau)'(\kappa'\tau'' - \kappa''\tau')\big), \end{aligned}$$

see [8].

Theorem 1.5. *Let α be a differentiable curve with normal Darboux vector $W^{\perp}$, then we can write the differential equation of α w.r. to $W^{\perp}$*

$$\lambda_2 D_T^{\perp} D_T^{\perp} W^{\perp} + \lambda_1 D_T^{\perp} W^{\perp} + \lambda_0 W^{\perp} = 0 \tag{1.11}$$

with the coefficients λ_0 , λ_1 *and* λ_2

$$\begin{aligned} \lambda_0 &= \kappa'\big((\vartheta\kappa\tau)' + \vartheta\kappa'\tau\big) - \vartheta\kappa\tau\big(\kappa'' - \vartheta^2\kappa\tau^2\big)\,, \\ \lambda_1 &= -\kappa\big((\vartheta\kappa\tau)' + \vartheta\kappa'\tau\big) \;\; \textit{and} \;\; \lambda_2 = \vartheta\kappa^2\tau\,, \end{aligned}$$

see [17].

2 Calculations of harmonicity and differential equations of involute of a curve according to mean curvature and Darboux vector

It is worth noting at the beginning that throughout this study we use the Frenet elements given in the set $\{T, N, B, \kappa, \tau\}$ for the curve α and $\{T_\beta, N_\beta, B_\beta, \kappa_\beta, \tau_\beta\}$ for the curve β. We also express W, H to denote the Darboux vector and mean curvature vector of α and we use W_β, H_β for the curve β. It is quite clear that $\vartheta = \| \frac{d}{ds}\beta(s) \|$.
We show that the Frenet formulas given in Theorem 1.1, that is, a new type of covariant derivative with respect to N, has a quiet appropriate practise through the Darboux vector W and mean curvature vector H.

Theorem 2.1. *Let α be a differentiable curve with principal normal N, Darboux vector W and β is the involute of α. Then we can write the differential equation of the curve β with respect to connection D as*

$$\omega_1 D_N^3 W + \omega_2 D_N^2 W + \omega_3 D_N W + \omega_4 W + \omega_5 D_N^3 N + \omega_6 D_N^2 N + \omega_7 D_N N + \omega_8 N = 0$$

with the coefficients $\omega_1,\ \omega_2,\ \omega_3,\ \omega_4,\ \omega_5,\ \omega_6,\ \omega_7,\ \omega_8$

$$\omega_1 = \frac{c_1}{\lambda\kappa}\,,$$

$$\omega_2 = 3c_1(\frac{1}{\lambda\kappa})' + \frac{c_2}{\lambda\kappa}\,,$$

$$\omega_3 = 3c_1(\frac{1}{\lambda\kappa})'' + 2c_2(\frac{1}{\lambda\kappa})' + \frac{c_3}{\lambda\kappa},$$

$$\omega_4 = c_1(\frac{1}{\lambda\kappa})''' + c_2(\frac{1}{\lambda\kappa})'' + c_3(\frac{1}{\lambda\kappa})' + \frac{c_4}{\lambda\kappa},$$

$$\omega_5 = \frac{c_1\varphi'}{\lambda\kappa},$$

$$\omega_6 = 3c_1(\frac{\varphi'}{\lambda\kappa})' + \frac{c_2\varphi'}{\lambda\kappa},$$

$$\omega_7 = 3c_1(\frac{\varphi'}{\lambda\kappa})'' + 2c_2(\frac{\varphi'}{\lambda\kappa})' + \frac{c_3\varphi'}{\lambda\kappa},$$

$$\omega_8 = c_1(\frac{\varphi'}{\lambda\kappa})''' + c_2(\frac{\varphi'}{\lambda\kappa})'' + c_3(\frac{\varphi'}{\lambda\kappa})' + \frac{c_4\varphi'}{\lambda\kappa}$$

and $c_1,\ c_2,\ c_3,\ c_4$

$$c_1 = \left(\frac{\kappa^2+\tau^2}{(\lambda\kappa)^{\frac{3}{2}}}\left(\frac{\kappa\tau'-\kappa'\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\right)'\right)^2,$$

$$c_2 = \left(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\left(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\right)'' - \sqrt{\kappa^2+\tau^2}\left(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\right)''\right.$$

$$\left.+\left(\frac{\kappa^2+\tau^2}{\lambda\kappa}\left(\frac{\kappa'\tau-\kappa\tau'}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\right)'\right)'\right)\left(\frac{\kappa^2+\tau^2}{(\lambda\kappa)^2}\left(\frac{\kappa\tau'-\kappa'\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\right)'\right),$$

$$c_3 = \left(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\left(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\right)''' - \frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\left(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\right)'''\right.$$

$$+\left(\frac{\kappa\tau'-\kappa'\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\right)'(\kappa^2+\tau^2)\left(\frac{\kappa^2+\tau^2}{(\lambda\kappa)^2}+\left(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\right)^2\right)\bigg)\left(\frac{\kappa^2+\tau^2}{\lambda\kappa}\left(\frac{\kappa\tau'-\kappa'\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\right)'\right)$$

$$+\left(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\left(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\right)'' - \sqrt{\kappa^2+\tau^2}\left(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\right)'' + \left(\left(\frac{\kappa'\tau-\kappa\tau'}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\right)'\frac{\kappa^2+\tau^2}{\lambda\kappa}\right)'\right)$$

$$\cdot\left(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\left(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\right)'' - \frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\left(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\right)''\right),$$

$$
\begin{aligned}
c_4 \quad = \quad & \Big((\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})' \Big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\Big)''' - (\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})''' \Big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\Big)' \\
& + \Big(\frac{\kappa'\tau-\kappa\tau'}{\kappa^2+\tau^2}\Big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\Big) - \sqrt{\kappa^2+\tau^2}\,(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})''\Big) \cdot \\
& \frac{\kappa^2+\tau^2}{\lambda\kappa}\Big(\frac{\kappa\tau'-\kappa'\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\Big)'\Big)\Big(\frac{\kappa^2+\tau^2}{\lambda\kappa}\Big(\frac{\kappa\tau'-\kappa'\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\Big)'\Big) \\
& + \Big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\Big(\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\Big)'' - \sqrt{\kappa^2+\tau^2}\,\Big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\Big)'' \\
& + \Big(\Big(\frac{\kappa'\tau-\kappa\tau'}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\Big)' \frac{\kappa^2+\tau^2}{\lambda\kappa}\Big)' \cdot \\
& \Big((\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})' \Big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\Big)'' - (\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'' \Big(\frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\Big)'\Big)\Big).
\end{aligned}
$$

Proof. From (1.10) we can write

$$
c_{\beta 1} D^3_{T_\beta} W_\beta + c_{\beta 2} D^2_{T_\beta} W_\beta + c_{\beta 3} D_{T_\beta} W_\beta + c_{\beta 4} W_\beta = 0 \tag{2.1}
$$

with the coefficients $c_{\beta 1}$, $c_{\beta 2}$, $c_{\beta 3}$, $c_{\beta 4}$

$$
c_{\beta 1} \quad = \quad \vartheta\Big(\kappa_\beta(\tau_\beta)' - (\kappa_\beta)'\tau_\beta\Big)^2,
$$

$$
c_{\beta 2} \quad = \quad \Big(\vartheta(\kappa_\beta)''\tau_\beta - \vartheta\kappa_\beta(\tau_\beta)'' - \big(\vartheta\kappa_\beta(\tau_\beta)' - \vartheta(\kappa_\beta)'\tau_\beta\big)'\Big)\Big(\kappa_\beta(\tau_\beta)' - (\kappa_\beta)'\tau_\beta\Big),
$$

$$
\begin{aligned}
c_{\beta 3} \quad = \quad & \Big((\kappa_\beta)'''\tau_\beta - \kappa_\beta(\tau_\beta)''' + \vartheta^2(\kappa_\beta(\tau_\beta)' - (\kappa_\beta)'\tau_\beta)((\kappa_\beta)^2 + (\tau_\beta)^2)\Big)\Big(\vartheta\kappa_\beta(\tau_\beta)' - \vartheta(\kappa_\beta)'\tau_\beta\Big) \\
& + \Big(\vartheta(\kappa_\beta)''\tau_\beta - \vartheta\kappa_\beta(\tau_\beta)'' - \big(\vartheta\kappa_\beta(\tau_\beta)' - \vartheta(\kappa_\beta)'\tau_\beta\big)'\Big)\Big((\kappa_\beta)''\tau_\beta - \kappa_\beta(\tau_\beta)''\Big),
\end{aligned}
$$

$$
\begin{aligned}
c_{\beta 4} \quad = \quad & \Big((\kappa_\beta)'(\tau_\beta)''' - (\kappa_\beta)'''(\tau_\beta)' - \vartheta^2(\kappa_\beta(\kappa_\beta)' + \tau_\beta(\tau_\beta)')(\kappa_\beta(\tau_\beta)' - (\kappa_\beta)'\tau_\beta)\Big) \\
& \Big(\vartheta\kappa_\beta(\tau_\beta)' - \vartheta(\kappa_\beta)'\tau_\beta\Big) \\
& + \Big(\vartheta(\kappa_\beta)''\tau_\beta - \vartheta\kappa_\beta(\tau_\beta)'' - \big(\vartheta\kappa_\beta(\tau_\beta)' - \vartheta(\kappa_\beta)'\tau_\beta\big)'\big((\kappa_\beta)'(\tau_\beta)'' - (\kappa_\beta)''(\tau_\beta)'\big)\Big).
\end{aligned}
$$

By making use of equalities (1.4) and (1.5) we can write equivalents of coefficients $c_{\beta 1}$, $c_{\beta 2}$, $c_{\beta 3}$, $c_{\beta 4}$ and W_β as c_1, c_2, c_3, c_4 and

$$W_\beta = \frac{1}{\lambda\kappa}W + \frac{\varphi'}{\lambda\kappa}N.$$

Applying the formulas given in (1.8) we may write the counterparts of $D_{T_\beta}W_\beta$, $D^2_{T_\beta}W_\beta$, $D^3_{T_\beta}W_\beta$ as in the following form:

$$D_{T_\beta}W_\beta = \frac{1}{\lambda\kappa}D_N W + (\frac{1}{\lambda\kappa})'W + \frac{\varphi'}{\lambda\kappa}D_N N + (\frac{\varphi'}{\lambda\kappa})'N,$$

$$D^2_{T_\beta}W_\beta = \frac{1}{\lambda\kappa}D^2_N W + 2(\frac{1}{\lambda\kappa})'D_N W + (\frac{1}{\lambda\kappa})''W + \frac{\varphi'}{\lambda\kappa}D^2_N N + 2(\frac{\varphi'}{\lambda\kappa})'D_N N + (\frac{\varphi'}{\lambda\kappa})''N,$$

$$\begin{aligned} D^3_{T_\beta}W_\beta &= \frac{1}{\lambda\kappa}D^3_N W + 3(\frac{1}{\lambda\kappa})'D^2_N W + 3(\frac{1}{\lambda\kappa})''D_N W + (\frac{1}{\lambda\kappa})'''W \\ &+ \frac{\varphi'}{\lambda\kappa}D^3_N N + 3(\frac{\varphi'}{\lambda\kappa})'D^2_N N + 3(\frac{\varphi'}{\lambda\kappa})''D_N N + (\frac{\varphi'}{\lambda\kappa})'''N\,. \end{aligned}$$

Setting the equivalents of coefficients and derivatives with respect to N into the equ.(2.1) we get desired result which completes the proof. □

Theorem 2.2. *Let α be a differentiable curve with principal normal N, Darboux vector W and β is the involute of α. We can give the differential equation of β w.r. to normal connection*

$$\delta_1 D^\perp_N D^\perp_N W + \delta_2 D^\perp_N W + \delta_3 W = 0$$

with the coefficients δ_1, δ_2, δ_3

$$\delta_1 = \frac{(\tau/\kappa)'}{\kappa\lambda^3}\,,$$

$$\delta_2 = 2\frac{(\tau/\kappa)'}{\lambda^2}(\frac{1}{\lambda\kappa})' - \frac{\sqrt{\kappa^2+\tau^2}}{(\lambda\kappa)^2}((\frac{(\tau/\kappa)'\kappa}{\lambda\sqrt{\kappa^2+\tau^2}})' + (\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2})\,,$$

$$\begin{aligned} \delta_3 &= \frac{(\tau/\kappa)'}{\lambda^2}(\frac{1}{\lambda\kappa})'' + \frac{(\sqrt{\kappa^2+\tau^2})'}{(\lambda\kappa)^2}((\frac{(\tau/\kappa)'\kappa}{\lambda\sqrt{\kappa^2+\tau^2}})' + (\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}) \\ &- \frac{(\tau/\kappa)'}{\lambda^2\sqrt{\kappa^2+\tau^2}}((\frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa})'' - \frac{((\tau/\kappa)')^2\kappa^3}{\lambda(\kappa^2+\tau^2)^{3/2}}). \end{aligned}$$

Proof. It is obvious from equation (1.11) we can write

$$\vartheta(\kappa_\beta)^2\tau_\beta D^{\perp}_{T_\beta} D^{\perp}_{T_\beta} W^{\perp}_\beta - \kappa_\beta\big((\vartheta\kappa_\beta\tau_\beta)' + \vartheta\kappa'_\beta\tau_\beta\big) D^{\perp}_{T_\beta} W^{\perp}_\beta$$

$$+\Big(\kappa'_\beta((\vartheta\kappa_\beta\tau_\beta)' + \vartheta\kappa'_\beta\tau_\beta) - \vartheta\kappa_\beta\tau_\beta(\kappa''_\beta - \vartheta^2\kappa_\beta(\tau_\beta)^2)\Big) W^{\perp}_\beta = 0. \tag{2.2}$$

Since the counterparts of $\kappa_\beta,\ \tau_\beta$ are clear from (1.5) we look at the equivalents of $W^{\perp}_\beta,\ D^{\perp}_{T_\beta} W^{\perp}_\beta$ and $D^{\perp}_{T_\beta} D^{\perp}_{T_\beta} W^{\perp}_\beta$ referring the equalities (1.4) and (1.5) as follows

$$W^{\perp}_\beta = \frac{1}{\lambda\kappa} W, \quad D^{\perp}_{T_\beta} W^{\perp}_\beta = (\frac{1}{\lambda\kappa})' W + \frac{1}{\lambda\kappa} D^{\perp}_N W \quad \text{and}$$

$$D^{\perp}_{T_\beta} D^{\perp}_{T_\beta} W^{\perp}_\beta = \frac{1}{\lambda\kappa} D^{\perp}_N D^{\perp}_N W + 2(\frac{1}{\lambda\kappa})' D^{\perp}_N W + (\frac{1}{\lambda\kappa})'' W.$$

Writing the equivalents of coefficients of (2.2) with the use of (1.4) , (1.5) and then the derivatives with respect to N into (2.2) we get desired result which completes the proof. □

Theorem 2.3. *Let β be an involute of a differentiable curve α. Then we can give the differential equation w.r. to connection characterizing the curve β by means of the mean curvature vector H_β as follows*

$$h_{\beta 1} D^3_{T_\beta} H_\beta + h_{\beta 2} D^2_{T_\beta} H_\beta + h_{\beta 3} D_{T_\beta} H_\beta + h_{\beta 4} H_\beta = 0 \tag{2.3}$$

with the coefficients $h_{\beta 1},\ h_{\beta 2},\ h_{\beta 3}$ and $h_{\beta 4}$

$$h_{\beta 1} = -\Big(\frac{\kappa_\beta}{\tau_\beta}\Big)'\Big(\vartheta^2\kappa_\beta\tau_\beta\Big)^2,$$

$$h_{\beta 2} = 3\vartheta^2\kappa_\beta\tau_\beta\Big(\vartheta\kappa_\beta(\vartheta\kappa_\beta)'\Big)' - (\vartheta\kappa_\beta)^2\Big(2\vartheta\tau_\beta(\vartheta\kappa_\beta)' + \vartheta\kappa_\beta(\vartheta\tau_\beta)'\Big)',$$

$$\begin{aligned} h_{\beta 3} &= \Big(\vartheta^4\kappa^2_\beta\big(\kappa^2_\beta + \tau^2_\beta\big) - \vartheta\kappa_\beta\big(\vartheta\kappa_\beta\big)'' - 3\big(\vartheta\kappa_\beta\big(\vartheta\kappa_\beta\big)'\big)'\Big)\Big(2\vartheta\tau_\beta\big(\vartheta\kappa_\beta\big)' + \vartheta\kappa_\beta\big(\vartheta\tau_\beta\big)'\Big) \\ &+\Big(\vartheta\tau_\beta\big(\vartheta\kappa_\beta\big)'' - \vartheta^4\kappa_\beta\tau_\beta\big(\kappa^2_\beta + \tau^2_\beta\big) + \Big(2\vartheta\tau_\beta(\vartheta\kappa_\beta)' + \vartheta\kappa_\beta(\vartheta\tau_\beta)'\Big)'\Big)\Big(3\vartheta\kappa_\beta\big(\vartheta\kappa_\beta\big)'\Big), \end{aligned}$$

$$\begin{aligned} h_{\beta 4} &= \Big(\vartheta^4\kappa^2_\beta\big(\kappa^2_\beta + \tau^2_\beta\big) - \vartheta\kappa_\beta\big(\vartheta\kappa_\beta\big)'' - 3\big(\vartheta\kappa_\beta(\vartheta\kappa_\beta)'\big)'\Big) \\ &\quad \Big(\vartheta\tau_\beta\big(\vartheta\kappa_\beta\big)'' - \vartheta^4\kappa_\beta\tau_\beta\big(\kappa^2_\beta + \tau^2_\beta\big) - \frac{(\vartheta\kappa_\beta)'}{\vartheta\kappa_\beta}\Big(2\vartheta\tau_\beta(\vartheta\kappa_\beta)' + \vartheta\kappa_\beta(\vartheta\tau_\beta)'\Big)\Big) \\ &+ \Big(\Big(\big(\vartheta\kappa_\beta\big)'' - \vartheta^3\kappa_\beta\big(\kappa^2_\beta + \tau^2_\beta\big)\Big)' - 3\big(\vartheta\kappa_\beta\big)^2\big(\vartheta\kappa_\beta\big)' - 2\big(\vartheta\tau_\beta\big)^2\big(\vartheta\kappa_\beta\big)' - \vartheta^2\kappa_\beta\tau_\beta\big(\vartheta\tau_\beta\big)'\Big) \\ &\quad \Big((\frac{\kappa_\beta}{\tau_\beta})'\vartheta^3\kappa_\beta\tau^2_\beta\Big) + \Big(\big(\vartheta\tau_\beta\big(\vartheta\kappa_\beta\big)'' - \vartheta^4\kappa_\beta\tau_\beta\big(\kappa^2_\beta + \tau^2_\beta\big) + \big(2\vartheta\tau_\beta\big(\vartheta\kappa_\beta\big)' + \vartheta\kappa_\beta\big(\vartheta\tau_\beta\big)'\big)'\Big) \\ &\quad \Big(\vartheta\kappa_\beta\big(\vartheta\kappa_\beta\big)'' - \vartheta^4\kappa^2_\beta\big(\kappa^2_\beta + \tau^2_\beta\big) - 3\big(\big(\vartheta\kappa_\beta\big)'\big)^2\Big). \end{aligned}$$

Proof. From equ.(1.1) we have $H_\beta = \vartheta\kappa_\beta N_\beta$. Taking the derivatives w. r. to T_β we get

$$D_{T_\beta} H_\beta = -\vartheta^2\kappa_\beta^2 T_\beta + (\vartheta\kappa_\beta)' N_\beta + \vartheta^2\kappa_\beta\tau_\beta B_\beta\,, \tag{2.4}$$

$$\begin{aligned} D^2_{T_\beta} H_\beta &= \big(-3\vartheta\kappa_\beta(\vartheta\kappa_\beta)'\big)T_\beta + \big((\vartheta\kappa_\beta)'' - (\vartheta\kappa_\beta)^3 - \vartheta^3\kappa_\beta\tau_\beta^2\big)N_\beta \\ &\quad + \big(2\vartheta\tau_\beta(\vartheta\kappa_\beta)' + \vartheta\kappa_\beta(\vartheta\tau_\beta)'\big)B_\beta, \end{aligned} \tag{2.5}$$

$$\begin{aligned} D^3_{T_\beta} H_\beta &= \Big(\vartheta^4\kappa_\beta^2(\kappa_\beta^2+\tau_\beta^2) - \vartheta\kappa_\beta(\vartheta\kappa_\beta)'' - 3\big(\vartheta\kappa_\beta(\vartheta\kappa_\beta)'\big)'\Big)T_\beta \\ &\quad + \Big(\big((\vartheta\kappa_\beta)'' - \vartheta^3\kappa_\beta(\kappa_\beta^2+\tau_\beta^2)\big)' - 3(\vartheta\kappa_\beta)^2(\vartheta\kappa_\beta)' \\ &\quad - 2(\vartheta\tau_\beta)^2(\vartheta\kappa_\beta)' - \vartheta^2\kappa_\beta\tau_\beta(\vartheta\tau_\beta)'\Big)N_\beta \\ &\quad + \Big(\vartheta\tau_\beta(\vartheta\kappa_\beta)'' - \vartheta^4\kappa_\beta\tau_\beta(\kappa_\beta^2+\tau_\beta^2) + \big(2\vartheta\tau_\beta(\vartheta\kappa_\beta)' + \vartheta\kappa_\beta(\vartheta\tau_\beta)'\big)'\Big)B_\beta\,. \end{aligned} \tag{2.6}$$

From equ.(2.4) and equ.(2.5) we can write the vectors T_β and B_β as

$$\begin{aligned} T_\beta &= \frac{\vartheta\tau_\beta(\vartheta\kappa_\beta)'' - \vartheta^4\kappa_\beta\tau_\beta(\kappa_\beta^2+\tau_\beta^2) - \frac{(\vartheta\kappa_\beta)'}{\vartheta\kappa_\beta}\big(2\vartheta\tau_\beta(\vartheta\kappa_\beta)' + \vartheta\kappa_\beta(\vartheta\tau_\beta)'\big)}{(\frac{\kappa_\beta}{\tau_\beta})'(\vartheta^2\kappa_\beta\tau_\beta)^2} H_\beta \\ &\quad + \frac{2\vartheta\tau_\beta(\vartheta\kappa_\beta)' + \vartheta\kappa_\beta(\vartheta\tau_\beta)'}{(\frac{\kappa_\beta}{\tau_\beta})'(\vartheta^2\kappa_\beta\tau_\beta)^2} D_{T_\beta}H_\beta - \frac{1}{(\frac{\kappa_\beta}{\tau_\beta})'(\vartheta^2\kappa_\beta\tau_\beta)} D^2_{T_\beta}H_\beta\,, \end{aligned}$$

$$\begin{aligned} B_\beta &= \frac{\vartheta\kappa_\beta(\vartheta\kappa_\beta)'' - \vartheta^4\kappa_\beta^2(\kappa_\beta^2+\tau_\beta^2) - 3((\vartheta\kappa_\beta)')^2}{(\frac{\kappa_\beta}{\tau_\beta})'(\vartheta^2\kappa_\beta\tau_\beta)^2} H_\beta + \frac{3\vartheta\kappa_\beta(\vartheta\kappa_\beta)'}{(\frac{\kappa_\beta}{\tau_\beta})'(\vartheta^2\kappa_\beta\tau_\beta)^2} D_{T_\beta}H_\beta \\ &\quad - \frac{1}{(\frac{\kappa_\beta}{\tau_\beta})'(\vartheta\tau_\beta)^2} D^2_{T_\beta}H_\beta\,. \end{aligned}$$

Putting the equivalents of vectors T_β and B_β into the equ.(2.6) results the desired equation which completes the proof. □

Theorem 2.4. *Let β be an involute of the curve α. Then we can give the differential equation w. r. to connection characterizing the curve β by means of the mean curvature vector $D_N N$ as follows*

$$h_4 D^4_N N + h_3 D^3_N N + h_2 D^2_N N + h_1 D_N N = 0$$

with the coefficients $h_1,\ h_2,\ h_3,\ h_4$

$$
h_1 = \frac{(\kappa\tau'-\kappa'\tau)^2}{\kappa^2+\tau^2}\Big(\frac{(\kappa^2+\tau^2)^{3/2}}{\kappa'\tau-\kappa\tau'}\Big)',
$$

$$
\begin{aligned}
h_2 = {} & 3\frac{\kappa\tau'-\kappa'\tau}{\sqrt{\kappa^2+\tau^2}}\Big(\sqrt{\kappa^2+\tau^2}(\sqrt{\kappa^2+\tau^2})'\Big)' - (\kappa^2+\tau^2)\Big(2\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}(\sqrt{\kappa^2+\tau^2})' \\
& +\sqrt{\kappa^2+\tau^2}(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2})'\Big)',
\end{aligned}
$$

$$
\begin{aligned}
h_3 = {} & \Big((\kappa^2+\tau^2)\Big(\kappa^2+\tau^2+(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2})^2\Big) - \sqrt{\kappa^2+\tau^2}(\sqrt{\kappa^2+\tau^2})'' \\
& -3\Big(\sqrt{\kappa^2+\tau^2}(\sqrt{\kappa^2+\tau^2})')\Big)'\Big)\Big(2\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}(\sqrt{\kappa^2+\tau^2})' + \sqrt{\kappa^2+\tau^2}\Big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\Big)'\Big) \\
& +\Big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}(\sqrt{\kappa^2+\tau^2})'' - \frac{\kappa\tau'-\kappa'\tau}{\sqrt{\kappa^2+\tau^2}}\Big(\kappa^2+\tau^2+(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2})^2\Big) \\
& +\Big(2\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}(\sqrt{\kappa^2+\tau^2})' + \sqrt{\kappa^2+\tau^2}(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2})')\Big)'\Big)\Big(3\sqrt{\kappa^2+\tau^2}(\sqrt{\kappa^2+\tau^2})'\Big),
\end{aligned}
$$

$$
\begin{aligned}
h_4 = {} & \Big((\kappa^2+\tau^2)^2 + \frac{(\kappa\tau'-\kappa'\tau)^2}{\kappa^2+\tau^2} - \sqrt{\kappa^2+\tau^2}(\sqrt{\kappa^2+\tau^2})'' - 3(\sqrt{\kappa^2+\tau^2}(\sqrt{\kappa^2+\tau^2})')'\Big) \\
& \Big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}(\sqrt{\kappa^2+\tau^2})'' - \frac{\kappa\tau'-\kappa'\tau}{\sqrt{\kappa^2+\tau^2}}\Big(\kappa^2+\tau^2+(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2})^2\Big) \\
& -\frac{(\sqrt{\kappa^2+\tau^2})'}{\sqrt{\kappa^2+\tau^2}}\Big(2\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\sqrt{\kappa^2+\tau^2}' + \sqrt{\kappa^2+\tau^2}(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2})')\Big)\Big) \\
& +\Big(\Big((\sqrt{\kappa^2+\tau^2})'' - \sqrt{\kappa^2+\tau^2}(\kappa^2+\tau^2+(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2})^2)\Big)' - 3(\kappa^2+\tau^2)(\sqrt{\kappa^2+\tau^2})' \\
& -2(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2})^2(\sqrt{\kappa^2+\tau^2})' - \frac{\kappa\tau'-\kappa'\tau}{\sqrt{\kappa^2+\tau^2}}(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2})'\Big)\Big((\frac{(\kappa^2+\tau^2)^{3/2}}{\kappa\tau'-\kappa'\tau})'\frac{(\kappa\tau'-\kappa'\tau)^2}{(\kappa^2+\tau^2)^{3/2}}\Big) \\
& +\Big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}(\sqrt{\kappa^2+\tau^2})'' - \frac{\kappa\tau'-\kappa'\tau}{\sqrt{\kappa^2+\tau^2}}\Big(\kappa^2+\tau^2+(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2})^2\Big) \\
& +\Big(2\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}(\sqrt{\kappa^2+\tau^2})' + \sqrt{\kappa^2+\tau^2}(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2})')\Big)'\Big)\Big(\sqrt{\kappa^2+\tau^2}(\sqrt{\kappa^2+\tau^2})'' \\
& -(\kappa^2+\tau^2)^2 - \frac{(\kappa\tau'-\kappa'\tau)^2}{\kappa^2+\tau^2} - 3((\sqrt{\kappa^2+\tau^2})')^2\Big).
\end{aligned}
$$

Proof. We can make use of equ.(1.4) and also equ.(1.8) in order to write the equivalents of derivatives $D^3_{T_\beta}H_\beta,\ D^2_{T_\beta}H_\beta,\ D_{T_\beta}H_\beta$ with respect to N. It follows that

$$H_\beta = D_{T_\beta}T_\beta = D_N N,\ \ D_{T_\beta}H_\beta = D^2_N N,\ \ D^2_{T_\beta}H_\beta = D^3_N N \text{ and } D^3_{T_\beta}H_\beta = D^4_N N.$$

By the same method from equ.(1.5) we can write the counterparts of coefficients $h_{\beta 1},\ h_{\beta 2},\ h_{\beta 3}$ and $h_{\beta 4}$ as $h_1,\ h_2,\ h_3$ and h_4.

Setting the equivalents of coefficients and derivatives with respect to N into the equ.(2.3) we get desired result which completes the proof. □

Theorem 2.5. *Let β be an involute of α with the mean curvature H_β. According to connection, harmonicity (biharmonic or 1-type harmonic) of the curve β can be expressed through the Frenet elements of α as*

1. *Involute curve β is biharmonic if*

$$\frac{\kappa\big(\sqrt{\kappa^2+\tau^2}\,\big)''}{\sqrt{\kappa^2+\tau^2}}-\kappa\Big(\kappa^2+\tau^2+\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)^2\Big)-2\tau\frac{(\kappa\tau'-\kappa'\tau)(\sqrt{\kappa^2+\tau^2})'}{(\kappa^2+\tau^2)^{3/2}}-\tau\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)'=0\,,$$

$$\tau\Big(\kappa^2+\tau^2+\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)^2\Big)-\frac{\tau(\sqrt{\kappa^2+\tau^2})''}{\sqrt{\kappa^2+\tau^2}}-2\kappa\frac{(\kappa\tau'-\kappa'\tau)(\sqrt{\kappa^2+\tau^2})'}{(\kappa^2+\tau^2)^{3/2}}-\kappa\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)'=0\,,$$

and $\kappa\kappa'+\tau\tau'=0$.

2. *Involute curve β is 1-type of harmonic if*

$$\begin{aligned}\frac{\kappa\big(\sqrt{\kappa^2+\tau^2}\big)''}{\sqrt{\kappa^2+\tau^2}}-\kappa\Big(\kappa^2+\tau^2+\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)^2\Big) \quad - \quad & 2\tau\frac{(\kappa\tau'-\kappa'\tau)(\sqrt{\kappa^2+\tau^2})'}{(\kappa^2+\tau^2)^{3/2}}\\ & -\tau\Big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\Big)'=-\lambda\kappa,\end{aligned}$$

$$\begin{aligned}\tau\Big(\kappa^2+\tau^2+\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)^2\Big)-\frac{\tau(\sqrt{\kappa^2+\tau^2})''}{\sqrt{\kappa^2+\tau^2}} \quad - \quad & 2\kappa\frac{(\kappa\tau'-\kappa'\tau)(\sqrt{\kappa^2+\tau^2})'}{(\kappa^2+\tau^2)^{3/2}}\\ & -\kappa\Big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\Big)'=\lambda\tau,\end{aligned}$$

and $\kappa\kappa'+\tau\tau'=0\,,\ \lambda\in\mathbb{R}\,.$

Proof. From equ.(2.5) we have

$$\begin{aligned}\Delta H_\beta \;=\;& 3\vartheta\kappa_\beta(\vartheta\kappa_\beta)'T_\beta+\big((\vartheta\kappa_\beta)^3-(\vartheta\kappa_\beta)''+\vartheta^3\kappa_\beta\tau_\beta^2\big)N_\beta\\ & -\big(2\vartheta\tau_\beta(\vartheta\kappa_\beta)'+\vartheta\kappa_\beta(\vartheta\tau_\beta)'\big)B_\beta\end{aligned}$$

Taking the equations (1.4) and (1.5) together into consideration we get

$$\begin{aligned}\Delta D_N N \;=\;& \Bigg(\frac{\kappa\big(\sqrt{\kappa^2+\tau^2}\big)''}{\sqrt{\kappa^2+\tau^2}}-\kappa\Big(\kappa^2+\tau^2+\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)^2\Big)-2\tau\frac{(\kappa\tau'-\kappa'\tau)(\sqrt{\kappa^2+\tau^2})'}{(\kappa^2+\tau^2)^{3/2}}\\ & -\tau\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)'\Bigg)T+\Big(\sqrt{\kappa^2+\tau^2}(\sqrt{\kappa^2+\tau^2})'\Big)N+\Bigg(-\kappa\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)'\\ & +\tau\Big(\kappa^2+\tau^2+\big(\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}\big)^2\Big)-\frac{\tau(\sqrt{\kappa^2+\tau^2})''}{\sqrt{\kappa^2+\tau^2}}-2\kappa\frac{(\kappa\tau'-\kappa'\tau)(\sqrt{\kappa^2+\tau^2})'}{(\kappa^2+\tau^2)^{3/2}}\Bigg)B.\end{aligned}$$

If we consider the case $\Delta D_N N=0$ from (i) of Theorem 1.2 we obtain that the first proposition holds. By the same way if we consider the case $\Delta D_N N=\lambda D_N N$ from (ii) of Theorem 1.2 second proposition also holds. □

Corollary 2.6. *Let β be an involute of α. If α is a general helix then β is biharmonic with respect to connection.*

Proof. If α is a general helix then we have $\tau/\kappa = const$. First derivative of this equality is

$$\kappa\tau' - \kappa'\tau = 0 \Longrightarrow \frac{\kappa}{\tau} = \frac{\kappa'}{\tau'} = const.$$

From Theorem 2.5 we write

$$\begin{aligned} \sqrt{\kappa^2+\tau^2}'\sqrt{\kappa^2+\tau^2} = 0 \quad &\Longrightarrow \quad \kappa\kappa' + \tau\tau' = 0 \\ &\Longrightarrow \quad \frac{\kappa}{\tau} = \frac{\kappa'}{\tau'} = const. \end{aligned}$$

Again from Theorem 2.5 taking the equality $\kappa/\tau = const$ into account we get

$$\kappa\Big(\frac{(\sqrt{\kappa^2+\tau^2})''}{\sqrt{\kappa^2+\tau^2}} - (\kappa^2+\tau^2)\Big) = 0 \text{ and } -\tau\Big(\frac{(\sqrt{\kappa^2+\tau^2})''}{\sqrt{\kappa^2+\tau^2}} - (\kappa^2+\tau^2)\Big) = 0.$$

This system yields that

$$(\kappa-\tau)\Big(\frac{(\sqrt{\kappa^2+\tau^2})''}{\sqrt{\kappa^2+\tau^2}} - (\kappa^2+\tau^2)\Big) = 0$$

and we obtain $\kappa/\tau = const$. Hence the condition $\Delta D_N N = 0$ is satisfied. □

Corollary 2.7. *Let β be an involute of α. If α is a circular helix then β is 1-type of harmonic with respect to connection.*

Proof. If α is a circular helix then we have $\kappa = const$ and $\tau = const$. From theorem we have $-\kappa(\kappa^2+\tau^2) = -\lambda\kappa$ and $\tau(\kappa^2+\tau^2) = \lambda\tau$. If we consider the case $\Delta D_N N = \lambda D_N N$, it is clear that the involute curve β is of 1-type harmonic provided $\lambda = \kappa^2+\tau^2$. □

Example 2.8. Given that the curve $\alpha(s) = \frac{1}{\sqrt{2}}(coss, sins, s)$ be a circular helix. It is plain to write the involute curve $\beta(s)$ as

$$\beta(s) = \tfrac{1}{\sqrt{2}}(coss - (c-s)sins,\ sins + (c-s)coss,\ c),\ c \in \mathbb{R}.$$

It follows that $W_\beta = (\frac{\sqrt{2}}{\lambda})W$ and $H_\beta = \frac{1}{\sqrt{2}}(-T+B) = D_N N.$

Taking the above theorems into account we can write the characterizations of the curve β through the Frenet elements of α in the following cases.

$$i)\ D_N(\frac{1}{\lambda}W) - \frac{1}{\lambda}D_N W - (\frac{1}{\lambda})'W = 0\,,$$

$$ii)\ D_N^{\perp}(\frac{1}{\lambda}W) - \frac{1}{\lambda}D_N^{\perp}W - (\frac{1}{\lambda})'W = 0\,,$$

$$iii)\ D_N^3 N + D_N N = 0\,.$$

Conclusion Writing differential equations and defining the harmonicity of a curve in Euclidean space are well known. From this point of view, by taking the advantage of properties of connected curves we derive the necessary conditions and also clarify the differential equations of involute of a curve in terms of the Frenet apparatus of the main curve. We wish this work inspire the scholars to make scientific studies in non-Euclidean spaces.

References

[1] O. Çakır and S. Șenyurt, Harmonicity and Differential Equation of Involute of a Curve in E^3, *Thermal Science*, **23(6)**, 2119–2125 (2019).

[2] A. Ferrandez, P. Lucas and M. A. Merono, Biharmonic hopf cylinders, *Rocky Mountain Journal of Maths.*, **28(3)**, 957–975 (1998).

[3] B. Bulca, K. Arslan and G. Ozturk, A characterization of involutes and evolutes of a given curve in E^n, *Kyungpook Math. J.*, **58**, 117–135 (2018).

[4] M. Harmanli, K. Arslan and B. Bulca, On weak biharmonic generalized rotational surface in E^4, *Journal of Universal Mathematics*, **2(2)**, 166–174 (2019).

[5] O. Çakır and S. Șenyurt, Calculation of the differential equations and harmonicity of the involute curve according to unit Darboux vector with a new method, *Turkish Journal of Science*, **5(2)**, 62–77 (2020).

[6] B. Y. Chen and S. Ishikawa, Biharmonic Surface in Pseudo-Euclidean Spaces, *Mem. Fac. Sci. Kyushu Univ.Ser.A*, **45(2)**, 323–347 (1991).

[7] H. Kocayigit and H. H. Hacisalihoglu, 1-Type curves and biharmonic curves in Euclidean 3-space, *Int. Elect. Journ. of Geo.*, **4(1)**, 97–101 (2011).

[8] K. Arslan, H. Kocayigit and M. Onder, Characterizations of Space Curves with 1-type Darboux Instantaneous Rotation Vector, *Commun. Korean Math. Soc.*, **31(2)**, 379–388 (2016).

[9] S. Șenyurt and O. Çakır, Differential Equations for a Space Curve According to the Unit Darboux Vector, *Turk. J. Math. Comput. Sci.*, **9(1)**, 91–97 (2018).

[10] A. A. Shaikh and P. R. Ghosh, Rectifying curves on a smooth surfaces immersed in the Euclidean space, *Indian J. Pure Appl. Math.*, **50(4)**, 883–890 (2019).

[11] A. A. Shaikh and P. R. Ghosh, Rectifying and Osculating curves on a smooth surface, *Indian J. Pure Appl. Math.*, **51(1)**, 67–75 (2020).

[12] A. A. Shaikh, M. S. Lone and P. R. Ghosh, Conformal image of an osculating curves on a smooth immersed surface, *J. Geom. Phy.*, **151**, (2020).

[13] A. A. Shaikh and P. R. Ghosh, Curves on a smooth surface with position vectors lie in the tangent plane, *Indian J. Pure Appl. Math.*, **51(3)**, 1097–1104 (2020).

[14] Shaikh, A. A., Lone, M.S. and Ghosh, P. R., Normal curves on a smooth immersed surface, *Indian J. Pure Appl. Math.*, 51(4) (2020), p. 1343–1355.

[15] A. A. Shaikh, M. S. Lone and P. R. Ghosh, Rectifying curves under conformal transformation, *J. Geom. Phy.*, **163**, (2021).

[16] A. Sabuncuoglu, *Diferensiyel Geometri*, Nobel Akademik Yayincilik, Ankara, (2014).

[17] H. Kocayigit, M. Önder and H. H. Hacisalihoglu, Harmonic 1-type Curves and Weak Biharmonic Curves in Lorentzian 3-space, *An Alele Stiintifice Ale Universitatii "Al.I. Cuza" Din Iasi(S.N.) Matematica*, Tomul LX, **60(1)**, 109–124 (2014).

Author information

Osman Çakır and Süleyman Şenyurt, Department of Mathematics, Ordu University, 52200 – Ordu, TURKEY.
E-mail: `senyurtsuleyman52@gmail.com`

Received: May 5, 2021
Accepted: November 9, 2021