HYPERSPHERE SATISFYING $\Delta x = Ax$ IN 4-SPACE

Erhan Güler and Kübra Yılmaz

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Abstract We consider hypersphere $x = x(u,v,w)$ in the four dimensional Euclidean space. We calculate the Gauss map, and the curvatures of it. Moreover, we obtain the Laplace-Beltrami operator the hypersphere satisfying $\Delta x = Ax$, where $A \in \text{Mat}(4,4)$.

1 Introduction

With the works of Chen [10, 11, 12, 13], the studies of submanifolds of finite type whose immersion into $\mathbb{E}^m$ (or $\mathbb{E}_n^m$) by using a finite number of eigenfunctions of their Laplacian have been studied for almost a half century.

Takahashi [49] gave that a connected Euclidean submanifold is of 1-type, iff it is either minimal in $\mathbb{E}^m$ or minimal in some hypersphere of $\mathbb{E}^m$. Submanifolds of finite type closest in simplicity to the minimal ones are the 2-type spherical submanifolds (where spherical means into a sphere). Some results of 2-type spherical closed submanifolds were given by [7, 8, 11].

Garay studied [25] an extension of Takahashi’s theorem in $\mathbb{E}^m$. Cheng and Yau [16] introduced hypersurfaces with constant scalar curvature; Chen and Piccinni [14] focused submanifolds with finite type Gauss map in $\mathbb{E}^m$. Lawson [37] gave minimal submanifolds in the lecture notes. Dursun [20] considered hypersurfaces with pointwise 1-type Gauss map in $\mathbb{E}_n^m$.

In $\mathbb{E}^3$; Takahashi [49] proved that minimal surfaces and spheres are the only surfaces satisfying the condition $\Delta r = \lambda r$, $\lambda \in \mathbb{R}$; Ferrandez, Garay, and Lucas [22] found that the surfaces satisfying $\Delta H = AH$, $A \in \text{Mat}(3,3)$ are either minimal, or an open piece of sphere or of a right circular cylinder; Choi and Kim [17] classified the minimal helicoid in terms of pointwise 1-type Gauss map of the first kind; Garay [24] studied a certain class of finite type surfaces of revolution; Dillen, Pas and Verstraelen [18] obtained that the only surfaces satisfying $\Delta r = Ar + B$, $A \in \text{Mat}(3,3)$, $B \in \text{Mat}(3,1)$ are the minimal surfaces, the spheres and the circular cylinders; Stamatakis and Zoubi [48] focused surfaces of revolution satisfying $\Delta^{III} r = Ax$; Senoussi and Bekkar [41] gave helicoidal surfaces $M^2$ which are of finite type with respect to the fundamental forms $I$, $II$ and $III$, i.e., their position vector field $r(u,v)$ satisfies the condition $\Delta^J r = Ar$, $J = I, II, III$, where $A \in \text{Mat}(3,3)$; Kim, Kim, and Kim [34] introduced Cheng-Yau operator and Gauss map of surfaces of revolution. Recently, Shaikh et. al [42, 43, 44, 45, 46, 47] initiated the study of surface curves in a different way, especially, rectifying, osculating and normal curves on a surface by considering isometry and conformal map between two surfaces and investigated their invariancy under such maps.

In Minkowski 4-space $\mathbb{E}^4_1$, Ganchev and Milousheva [23] studied analogue of surfaces of [39, 40]; Arvanitoyeorgos, Kaimakamis, and Magid [6] indicated that if the mean curvature vector field of $M^3_1$ satisfies the equation $\Delta H = \alpha H$ ($\alpha$ a constant), then $M^3_1$ has CMC; Arslan and Milousheva introduced meridians surfaces of elliptic or hyperbolic type with pointwise 1-type Gauss map; Turgay considered some classifications of Lorentzian surfaces with finite type Gauss map; Dursun and Turgay worked space-like surfaces in with pointwise 1-type Gauss map. Aksoyak and Yaylı [33] gave general rotational surfaces with pointwise 1-type Gauss map in $\mathbb{E}^2_2$. Bektaş, Canfes, and Dursun [9] obtained surfaces in a pseudo-sphere with 2-type pseudospherical Gauss map in $\mathbb{E}^2_5$.

We consider hypersphere in the four dimensional Euclidean geometry $\mathbb{E}^4_1$. In Section 2, we give some basic notions of the four dimensional Euclidean geometry. We consider curvature formulas of a hypersurface in $\mathbb{E}^4_1$, in Section 3. In Section 4, we define hypersphere. Finally, we study hypersphere satisfying $\Delta x = Ax$ for some $4 \times 4$ matrix $A$ in $\mathbb{E}^4_1$ in the last section.

2 Preliminaries

In this section, giving some of basic facts and definitions, we describe notations used whole paper. Let $\mathbb{E}^m$ denote the Euclidean $m$-space with the canonical Euclidean metric tensor given by $\bar{g} = (\cdot, \cdot) = \sum_{i=1}^{m} dx_i^2$, where $(x_1, x_2, \ldots, x_m)$ is a rectangular coordinate system in $\mathbb{E}^m$. Consider an $m$-dimensional Riemannian submanifold of the space $\mathbb{E}^m$. We denote the Levi-Civita connections of $\mathbb{E}^m$ and $M$ by $\bar{\nabla}$ and $\nabla$, respectively. We shall use letters $X, Y, Z, W$ (resp., $\xi, \eta$) to denote vectors fields tangent (resp., normal) to $M$. The Gauss and Weingarten formulas are given, respectively, by

\[
\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}_X \xi = -A_\xi(X) + D_X \xi,
\]

where $h$, $D$ and $A$ are the second fundamental form, the normal connection and the shape operator of $M$, respectively.

For each $\xi \in T^*_p M$, the shape operator $A_\xi$ is a symmetric endomorphism of the tangent space $T_p M$ at $p \in M$. The shape operator and the second fundamental form are related by

\[
\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle.
\]

The Gauss and Codazzi equations are given, respectively, by

\[
\langle R(X, Y, Z, W) = \langle h(Y, Z), h(X, W) \rangle - \langle h(X, Z), h(Y, W) \rangle, \quad (\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z),
\]

where $R$, $R^D$ are the curvature tensors associated with connections $\nabla$ and $D$, respectively, and $\nabla h$ is defined by

\[
(\nabla_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).
\]

2.1 Hypersurfaces of Euclidean space

Now, let $M$ be an oriented hypersurface in the Euclidean space $\mathbb{E}^{n+1}_1$, $S$ its shape operator (i.e. Weingarten map) and $x$ its position vector. We consider a local orthonormal frame field $\{e_1, e_2, \ldots, e_n\}$ of consisting of principal directions of $M$ corresponding from the principal curvature $k_i$ for $i = 1, 2, \ldots, n$. Let the dual basis of this frame field be $\{\theta_1, \theta_2, \ldots, \theta_n\}$. Then the first structural equation of Cartan is

\[
d\theta_i = \sum_{j=1}^{n} \theta_j \wedge \omega_{ij}, \quad i, j = 1, 2, \ldots, n,
\]
where $\omega_{ij}$ denotes the connection forms corresponding to the chosen frame field. We denote the Levi-Civita connection of $M$ and $E^{n+1}$ by $\nabla$ and $\nabla$, respectively. Then, from the Codazzi equation (2.3), we have

$$e_i(k_j) = \omega_{ij}(e_j)(k_i - k_j),$$

(2.6)

$$\omega_{ij}(e_i)(k_i - k_j) = \omega_{il}(e_j)(k_i - k_l),$$

(2.7)

for distinct $i, j, l = 1, 2, \ldots, n$.

We put $s_j = \sigma_j(k_1, k_2, \ldots, k_n)$, where $\sigma_j$ is the $j$-th elementary symmetric function given by

$$\sigma_j(a_1, a_2, \ldots, a_n) = \sum_{1 \leq i_1 < i_2 < \ldots < i_j \leq n} a_{i_1}a_{i_2} \ldots a_{i_j}.$$ We use following notation

$$r_i^j = \sigma_j(k_1, k_2, \ldots, k_{i-1}, k_{i+1}, k_{i+2}, \ldots, k_n).$$

By the definition, we have $r_0^j = 1$ and $s_{n+1} = s_{n+2} = \cdots = 0$. We call the function $s_k$ as the $k$-th mean curvature of $M$. We would like to note that functions $H = \frac{1}{n}s_1$ and $K = s_n$ are called the mean curvature and Gauss-Kronecker curvature of $M$, respectively. In particular, $M$ is said to be $j$-minimal if $s_j = 0$ on $M$.

In $E^{n+1}$, to find the $i$-th curvature formulas $C_i$ (Curvature formulas sometimes are represented as mean curvature $H_i$, and sometimes as Gaussian curvature $K_i$ by different writers, such as [1] and [36]). We will call it just $i$-th curvature $C_i$ in this paper.), where $i = 0, \ldots, n$, firstly, we use the characteristic polynomial of $S$:

$$P_S(\lambda) = 0 = \det(S - \lambda I_n) = \sum_{k=0}^{n} (-1)^k s_k \lambda^{n-k},$$

(2.8)

where $i = 0, \ldots, n$, $I_n$ denotes the identity matrix of order $n$. Then, we get curvature formulas $(^n_i)C_i = s_i$. That is, $(^0_0)C_0 = s_0 = 1$ (by definition), $(^1_1)C_1 = s_1, \ldots, (^n_n)C_n = s_n = K$.

$k$-th fundamental form of $M$ is defined by $I(S^{k-1}(X), Y) = \langle S^{k-1}(X), Y \rangle$. So, we have

$$\sum_{i=0}^{n} (-1)^i (^n_i)C_i I(S^{n-i}(X), Y) = 0.$$ In particular, one can get classical result $C_0III - 2C_1II + C_2I = 0$ of surface theory for $n = 2$. See [36] for details.

For a Euclidean submanifold $x: M \to E^n$, the immersion $(M, x)$ is called finite type, if $x$ can be expressed as a finite sum of eigenfunctions of the Laplacian $\Delta$ of $(M, x)$, i.e. $x = x_0 + \sum_{i=1}^{k} x_i$, where $x_0$ is a constant map, $x_1, \ldots, x_k$ non-constant maps, and $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}, i = 1, \ldots, k$. If $\lambda_i$ are different, $M$ is called $k$-type. See [11] for details.

### 2.2 Rotational hypersurfaces

We will obtain a rotational hypersurface (rot-hypface for short) in Euclidean 4-space. Before we proceed, we would like to note that the definition of rot-hypfaces in Riemannian space forms were defined in [19]. A rot-hypface $M \subset E^{n+1}$ generated by a curve $C$ around an axis $\gamma$ that does not meet $C$ is obtained by taking the orbit of $C$ under those orthogonal transformations of $E^{n+1}$ that leaves $x$ pointwise fixed (See [19, Remark 2.3]).

Throughout the paper, we shall identify a vector $(a, b, c, d)$ with its transpose. Consider the case $n = 3$, and let $C$ be the curve parametrized by

$$\gamma(w) = (f(w), 0, 0, \varphi(w)).$$

(2.10)

If $x$ is the $x_4$-axis, then an orthogonal transformations of $E^{n+1}$ that leaves $x$ pointwise fixed has the form

$$Z(v, w) = \begin{pmatrix}
\cos u \cos v & -\sin u & -\cos u \sin v & 0 \\
\sin u \cos v & \cos u & -\sin u \sin v & 0 \\
\sin v & 0 & \cos v & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, u, v \in \mathbb{R}.$$
Therefore, the parametrization of the rot-hypface generated by a curve $C$ around an axis $r$ is given by

$$x(u, v, w) = Z(u, v)\gamma(w). \quad (2.12)$$

**Definition 2.1.** Let $x = x(u, v, w)$ be an immersion from $M^3 \subset \mathbb{E}^3$ to $\mathbb{E}^4$. In $\mathbb{E}^4$, inner product is defined by

$$\langle \vec{x}, \vec{y} \rangle = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4,$$

and triple vector product is given by

$$\vec{x} \times \vec{y} \times \vec{z} = \text{det} \begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix},$$

where $\vec{x} = (x_1, x_2, x_3, x_4)$, $\vec{y} = (y_1, y_2, y_3, y_4)$, $\vec{z} = (z_1, z_2, z_3, z_4)$.

**Definition 2.2.** For a hypface $x$ in 4-space, we have

$$I = \begin{pmatrix} E & F & A \\ F & G & B \\ A & B & C \end{pmatrix}, \quad II = \begin{pmatrix} L & M & P \\ M & N & T \\ P & T & V \end{pmatrix}, \quad (2.13)$$

and

$$\text{det} I = (EG - F^2)C - EB^2 + 2FAB - GA^2,$$

$$\text{det} II = (LN - M^2) V - LT^2 + 2MPT - NP^2,$$

where $I$ and $II$ are the first and the second fundamental form matrices, respectively, where $E = x_u \cdot x_u$, $F = x_u \cdot x_v$, $G = x_u \cdot x_w$, $A = x_u \cdot x_v$, $B = x_u \cdot x_w$, $C = x_w \cdot x_w$, $L = x_uu \cdot e$, $M = x_uv \cdot e$, $N = x_vv \cdot e$, $P = x_uw \cdot e$, $T = x_wu \cdot e$, $V = x_vw \cdot e$. Here,

$$e = \frac{x_u \times x_v \times x_w}{\|x_u \times x_v \times x_w\|}. \quad (2.14)$$

is unit normal (i.e. the Gauss map) of hypface $x$.

**Definition 2.3.** Product matrices $I^{-1}, II$ gives the matrix of the shape operator $S$ of hypface $x$ in 4-space as follows

$$S = \frac{1}{\text{det} I} \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix}, \quad (2.15)$$

where

$$\text{det} I = (EG - F^2)C - A^2G + 2ABF - B^2E,$$

$$s_{11} = ABM - CFM - AGP + BFP + CGL - B^2L,$$

$$s_{12} = ABN - CFN - AGT + BFT + CGM - B^2M,$$

$$s_{13} = ABT - CFT - AGV + BFV + CGP - B^2P,$$

$$s_{21} = ABL - CFL + AFP - BPE + CME - A^2M,$$

$$s_{22} = ABM - CFM + AFT - BTE + CNE - A^2N,$$

$$s_{23} = ABP - CFP + AVF - BVE + CTE - A^2T,$$

$$s_{31} = -AGL + BFL + AFM - BME + GPE - F^2P,$$

$$s_{32} = -AGM + BFM + AFN - BNE + GTE - F^2T,$$

$$s_{33} = -AGB + BFP + AFT - BTE + GVE - F^2V.$$

See [28, 29, 30] for details.
3 i-th Curvatures

To compute the i-th mean curvature formula \( \mathcal{C}_i \), where \( i = 0, \ldots, 3 \), we use characteristic polynomial \( P_S(\lambda) = a\lambda^3 + b\lambda^2 + c\lambda + d = 0 \):

\[
P_S(\lambda) = \det(S - \lambda I_3) = 0.
\]

Then, obtain \( \mathcal{C}_0 = 1 \) (by definition), \( \binom{i}{1}\mathcal{C}_1 = \binom{i}{1} H = -\frac{b}{3}, \binom{i}{2}\mathcal{C}_2 = \frac{c}{3}, \binom{i}{3}\mathcal{C}_3 = K = -\frac{d}{3} \).

Therefore, we find i-th curvature folmulas depends on the coefficients of the fundamental forms \( \Pi \) and \( \Pi' \) in 4-space. See [26] for details.

**Theorem 3.1.** Any hypface \( x \) in \( \mathbb{E}^4 \) has following curvature formulas, \( \mathcal{C}_0 = 1 \) (by definition),

\[
\begin{align*}
\mathcal{C}_1 &= \frac{\{ EN + GL - 2FM\}C + (EG - F^2)V - LB^2 - NA^2}{3 \left[ (EG - F^2)C - EB^2 + 2FAB - GA^2 \right]}, \\
\mathcal{C}_2 &= \frac{\{ EN + GL - 2FM\} V + (LN - M^2)C - ET^2 - GP^2}{3 \left[ (EG - F^2)C - EB^2 + 2FAB - GA^2 \right]}, \\
\mathcal{C}_3 &= \frac{(LN - M^2) V - LT^2 + 2MPT - NP^2}{(EG - F^2)C - EB^2 + 2FAB - GA^2}.
\end{align*}
\]

**Proof.** Solving \( \det(S - \lambda I_3) = 0 \) with some algebraic computations, we obtain coefficients \( a, b, c, d \) of polynomial \( P_S(\lambda) \).

A hypersurface \( x \) in \( \mathbb{E}^4 \) is \( \mathcal{C}_i \)-minimal, when \( \mathcal{C}_i = 0 \) identically on \( x \).

4 Hypersphere

In this section, we define hypersphere, then find its differential geometric properties in \( \mathbb{E}^4 \).

For an open interval \( I \subset \mathbb{R} \), let \( \gamma : I \rightarrow \Pi \) be a curve in a plane \( \Pi \) in \( \mathbb{E}^4 \), and let \( \ell \) be a straight line in \( \Pi \).

**Definition 4.1.** A rotational hypersurface in \( \mathbb{E}^4 \) is called hypersphere, when a curve

\[
\gamma(w) = (r \cos w, 0, 0, r \sin w)
\]

rotates by \( (2.11) \) around a line \( \ell = (0, 0, 0, 1) \) (these are called the profile curve and the axis, respectively).

So, the hypersphere which is spanned by the vector \( \ell \), is as follows

\[
x(u, v, w) = Z(u, v)\gamma(w)
\]

in \( \mathbb{E}^4 \), where \( u, v, w \in [0, 2\pi] \). Therefore, more clear form of (4.1) is as follows

\[
x(u, v, w) = \begin{pmatrix} r \cos u \cos v \cos w \\ r \sin u \cos v \cos w \\ r \sin v \cos w \\ r \sin w \end{pmatrix},
\]

where \( r \in \mathbb{R}\setminus\{0\} \) and \( 0 \leq u, v, w \leq 2\pi \). When \( w = 0 \), we have a sphere in \( \mathbb{E}^4 \).

Next, we obtain the curvatures and the Gaussian curvature of the hypersphere (4.2). We get the first differentials of (4.2) with respect to \( u, v, w \), respectively,

\[
x_u = \begin{pmatrix} -r \sin u \cos v \cos w \\ r \cos u \cos v \cos w \\ 0 \\ 0 \end{pmatrix},
\]

\[
x_v = \begin{pmatrix} -r \cos u \sin v \cos w \\ -r \cos u \sin v \cos w \\ r \cos v \cos w \\ 0 \end{pmatrix}.
\]
and

\[ x_w = \begin{pmatrix} -r \cos u \cos v \sin w \\ -r \sin u \cos v \sin w \\ -r \sin v \sin w \\ r \cos w \end{pmatrix}. \]

The first quantities of (4.2) are as follows

\[ I = \begin{pmatrix} r^2 \cos^2 v \cos^2 w & 0 & 0 \\ 0 & r^2 \cos^2 w & 0 \\ 0 & 0 & r^2 \end{pmatrix}. \] (4.3)

We have \( \det I = r^6 \cos^2 v \cos^4 w \). The line element of the hypersphere is given by

\[ ds^2 = r^2 \left[ (\cos^2 v du^2 + dv^2) \cos^2 w + dw^2 \right]. \]

Using (2.14), we get the Gauss map of the hypersphere (4.2) as follows

\[ e = \begin{pmatrix} \cos u \cos v \cos w \\ \sin u \cos v \cos w \\ \sin v \cos w \\ \sin w \end{pmatrix}. \] (4.4)

The second differentials of (4.2) with respect to \( u, v, w \), and the Gauss map (4.4) of the hypersphere (4.2), we have the second quantities as follows

\[ II = \begin{pmatrix} -r \cos^2 v \cos^2 w & 0 & 0 \\ 0 & -r \cos^2 w & 0 \\ 0 & 0 & -r \end{pmatrix}. \] (4.5)

So, we get \( \det II = -r^3 \cos^2 v \cos^4 w \).

We calculate the shape operator matrix of the hypersphere (4.2), using (2.15), as follows

\[ S = \begin{pmatrix} -\frac{1}{r} & 0 & 0 \\ 0 & -\frac{1}{r} & 0 \\ 0 & 0 & -\frac{1}{r} \end{pmatrix}. \]

Finally, using (3.1), (3.2) and (3.3), with (4.3), (4.5), respectively, we find the curvatures of the hypersphere (4.2) as follows:

**Corollary 4.2.** Let \( x : M^3 \rightarrow \mathbb{E}^4 \) be an immersion given by (4.2). Then \( M^3 \) has constant (mean) 1-curvature

\[ C_1 = H = -\frac{1}{r}. \]

**Corollary 4.3.** Let \( x : M^3 \rightarrow \mathbb{E}^4 \) be an immersion given by (4.2). Then \( M^3 \) has constant 2-curvature

\[ C_2 = \frac{1}{r^2}. \]

**Corollary 4.4.** Let \( x : M^3 \rightarrow \mathbb{E}^4 \) be an immersion given by (4.2). Then \( M^3 \) has negative constant (Gaussian) 3-curvature

\[ C_3 = K = -\frac{1}{r^3}. \]
5 Hypersphere satisfying $\Delta x = Ax$

In this section, we give the Laplace-Beltrami operator of a smooth function, then calculate it using hypersphere.

The inverse of the matrix

\[
(g_{ij}) = \begin{pmatrix}
g_{11} & g_{12} & g_{13} \\
g_{21} & g_{22} & g_{23} \\
g_{31} & g_{32} & g_{33}
\end{pmatrix}
\]

is as follows

\[
\frac{1}{g} \begin{pmatrix}
g_{22}g_{33} - g_{23}g_{32} & -(g_{12}g_{33} - g_{13}g_{32}) & g_{12}g_{23} - g_{13}g_{22} \\
-(g_{21}g_{33} - g_{23}g_{31}) & g_{11}g_{33} - g_{13}g_{31} & -(g_{11}g_{23} - g_{13}g_{21}) \\
g_{21}g_{32} - g_{22}g_{31} & -(g_{11}g_{22} - g_{12}g_{21}) & g_{11}g_{22} - g_{12}g_{21}
\end{pmatrix},
\]

where

\[
g = \det(g_{ij}) = g_{11}g_{22}g_{33} - g_{11}g_{23}g_{32} + g_{12}g_{31}g_{32} - g_{12}g_{31}g_{33} + g_{13}g_{21}g_{32} - g_{13}g_{21}g_{33}.
\]

**Definition 5.1.** The Laplace-Beltrami operator of a smooth function $\phi = \phi(x^1, x^2, x^3)$ \(|D| (D \subset \mathbb{R}^3)\) of class $C^3$ with respect to the first fundamental form of a hypersurface $M$ is the operator $\Delta$ which is defined by as follows

\[
\Delta \phi = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{3} \frac{\partial}{\partial x^i} \left( \sqrt{g} g^{ij} \frac{\partial \phi}{\partial x^j} \right),
\]

where $(g^{ij}) = (g_{ij})^{-1}$ and $g = \det(g_{ij})$.

Clearly, we can write (5.1) as follows

\[
\Delta \phi = \frac{1}{\sqrt{g}} \begin{pmatrix}
\frac{\partial}{\partial x^1} \left( \sqrt{g} g^{11} \frac{\partial \phi}{\partial x^1} \right) - \frac{\partial}{\partial x^2} \left( \sqrt{g} g^{12} \frac{\partial \phi}{\partial x^2} \right) + \frac{\partial}{\partial x^3} \left( \sqrt{g} g^{13} \frac{\partial \phi}{\partial x^3} \right) \\
+ \frac{\partial}{\partial x^1} \left( \sqrt{g} g^{21} \frac{\partial \phi}{\partial x^1} \right) - \frac{\partial}{\partial x^2} \left( \sqrt{g} g^{22} \frac{\partial \phi}{\partial x^2} \right) + \frac{\partial}{\partial x^3} \left( \sqrt{g} g^{23} \frac{\partial \phi}{\partial x^3} \right) \\
+ \frac{\partial}{\partial x^1} \left( \sqrt{g} g^{31} \frac{\partial \phi}{\partial x^1} \right) - \frac{\partial}{\partial x^2} \left( \sqrt{g} g^{32} \frac{\partial \phi}{\partial x^2} \right) + \frac{\partial}{\partial x^3} \left( \sqrt{g} g^{33} \frac{\partial \phi}{\partial x^3} \right)
\end{pmatrix}.
\]

So, we get the inverse of (4.3) as follows

\[
\Gamma^{-1} = \frac{1}{\det \mathbf{I}} \begin{pmatrix}
CG - B^2 & AB - CF & BF - AG \\
AB - CF & CE - A^2 & AF - BE \\
BF - AG & AF - BE & EG - F^2
\end{pmatrix},
\]

where $\det \mathbf{I} = (EG - F^2)C - A^2G + 2ABF - B^2E$. Hence, more clear notation of (5.2) for a smooth function $\phi = \phi(u, v, w)$ is as follows

\[
\Delta \phi = \frac{1}{\sqrt{|\det \mathbf{I}|}} \begin{pmatrix}
\frac{\partial}{\partial u} \left( \sqrt{|\det \mathbf{I}|} \left( CG - B^2 \phi_u - (AB - CF) \phi_v + (BF - AG) \phi_w \right) \right) \\
- \frac{\partial}{\partial v} \left( \sqrt{|\det \mathbf{I}|} \left( AB - CF \phi_u - (CE - A^2) \phi_v + (AF - BE) \phi_w \right) \right) \\
+ \frac{\partial}{\partial w} \left( \sqrt{|\det \mathbf{I}|} \left( BF - AG \phi_u - (AF - BE) \phi_v + (EG - F^2) \phi_w \right) \right)
\end{pmatrix}.
\]

We continue our calculations to find the Laplace-Beltrami operator $\Delta x$ of the hypersphere $x$ using (4.2) and (5.3).

The Laplace-Beltrami operator of the hypersphere (4.2) is given by

\[
\Delta x = \frac{1}{\sqrt{|\det \mathbf{I}|}} \left( \frac{\partial}{\partial u} U - \frac{\partial}{\partial v} V + \frac{\partial}{\partial w} W \right),
\]

where $U, V, W$ are given functions.
where

\[ \mathcal{U} = \frac{(CG - B^2)x_u - (AB - CF)x_v + (BF - AG)x_w}{\sqrt{|\det I|}}, \]
\[ \mathcal{V} = \frac{(AB - CF)x_u - (CE - A^2)x_v + (AF - BE)x_w}{\sqrt{|\det I|}}, \]
\[ \mathcal{W} = \frac{(BF - AG)x_u - (AF - BE)x_v + (EG - F^2)x_w}{\sqrt{|\det I|}}. \]

Here, \( A = B = F = 0 \). Hence, we briefly can write \( \mathcal{U}, \mathcal{V}, \mathcal{W} \), as follows

\[ \mathcal{U} = \frac{CG}{\sqrt{|\det I|}}x_u, \quad \mathcal{V} = -\frac{CE}{\sqrt{|\det I|}}x_v, \quad \mathcal{W} = \frac{EG}{\sqrt{|\det I|}}x_w. \]

Finally, substituting \( \frac{\partial}{\partial u} \mathcal{U}, \frac{\partial}{\partial v} \mathcal{V}, \frac{\partial}{\partial w} (\mathcal{W}) \) into (5.4), we get

\[ \Delta x = \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \end{pmatrix} = \begin{pmatrix} -\frac{r}{3} \cos u \cos v \cos w \\ -\frac{r}{3} \sin u \cos v \cos w \\ -\frac{r}{3} \sin v \cos w \\ -\frac{r}{3} \sin w \end{pmatrix}. \]

Therefore, we have following results:

**Corollary 5.2.** Let \( x : M^3 \rightarrow \mathbb{E}^4 \) be an immersion given by (4.2). Then \( x \) has

\[ \Delta x = 3\mathcal{C}_1 e, \]

where \( \mathcal{C}_1 \) and \( e \) are the mean 1-curvature and the Gauss map, respectively.

**Corollary 5.3.** Let \( x : M^3 \rightarrow \mathbb{E}^4 \) be an immersion given by (4.2). Then \( x \) has

\[ \Delta x = -3 (\mathcal{C}_2)^{1/2} e, \]

where \( \mathcal{C}_2 \) and \( e \) are the 2-curvature and the Gauss map, respectively.

**Corollary 5.4.** Let \( x : M^3 \rightarrow \mathbb{E}^4 \) be an immersion given by (4.2). Then \( x \) has

\[ \Delta x = 3 (\mathcal{C}_3)^{1/3} e, \]

where \( \mathcal{C}_3 \) and \( e \) are the 3-curvature and the Gauss map, respectively.

**Corollary 5.5.** Let \( x : M^3 \rightarrow \mathbb{E}^4 \) be an immersion given by (4.2). Then \( x \) has

\[ \Delta x = Ax, \]

where

\[ A = -3 (\mathcal{C}_1)^2 I_4 = -3\mathcal{C}_2 I_4 = -3 (\mathcal{C}_1)^{-1} \mathcal{C}_3 I_4 = -3 \left( (\mathcal{C}_2)^{-1} \mathcal{C}_3 \right)^2 I_4, \]

and \( A \in \text{Mat}(4, 4) \), \( I_4 = \text{diag}(1, 1, 1, 1) \).

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**References**


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**Author information**

Erhan Güler, Bartın University, Faculty of Sciences, Department of Mathematics, 74100 Bartın, Turkey.
E-mail: eguler@bartin.edu.tr, ergler@gmail.com
ORCID: https://orcid.org/0000-0003-3264-6239

Kübra Yılmaz, Bartın University, Graduate School of Natural and Applied Science, 74100 Bartın, Turkey.
E-mail: kbrylmz89@gmail.com
ORCID: https://orcid.org/0000-0003-1124-381X

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