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HYPERSPHERE SATISFYING $\Delta \mathbf{x} = \mathcal{A}\mathbf{x}$ IN 4-SPACE

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Abstract We consider hypersphere $\mathbf{x} = \mathbf{x}(u, v, w)$ in the four dimensional Euclidean space. We calculate the Gauss map, and the curvatures of it. Moreover, we obtain the Laplace-Beltrami operator the hypersphere satisfying $\Delta \mathbf{x} = \mathcal{A}\mathbf{x}$, where $\mathcal{A} \in Mat\,(4, 4)$.

1 Introduction

With the works of Chen [10, 11, 12, 13], the studies of submanifolds of finite type whose immersion into $\mathbb{E}^m$ (or $\mathbb{E}^m_\nu$) by using a finite number of eigenfunctions of their Laplacian have been studied for almost a half century.

Takahashi [49] gave that a connected Euclidean submanifold is of 1-type, iff it is either minimal in $\mathbb{E}^m$ or minimal in some hypersphere of $\mathbb{E}^m$. Submanifolds of finite type closest in simplicity to the minimal ones are the 2-type spherical submanifolds (where spherical means into a sphere). Some results of 2-type spherical closed submanifolds were given by [7, 8, 11]. Garay studied [25] an extension of Takahashi's theorem in $\mathbb{E}^m$. Cheng and Yau [16] introduced hypersurfaces with constant scalar curvature; Chen and Piccinni [14] focused submanifolds with finite type Gauss map in $\mathbb{E}^m$. Lawson [37] gave minimal submanifolds in the lecture notes. Dursun [20] considered hypersurfaces with pointwise 1-type Gauss map in $\mathbb{E}^{n+1}$.

In $\mathbb{E}^3$; Takahashi [49] proved that minimal surfaces and spheres are the only surfaces satisfying the condition $\Delta r = \lambda r$, $\lambda \in \mathbb{R}$; Ferrandez, Garay, and Lucas [22] found that the surfaces satisfying $\Delta H = AH$, $A \in Mat(3, 3)$ are either minimal, or an open piece of sphere or of a right circular cylinder; Choi and Kim [17] classified the minimal helicoid in terms of pointwise 1-type Gauss map of the first kind; Garay [24] studied a certain class of finite type surfaces of revolution; Dillen, Pas and Verstraelen [18] obtained that the only surfaces satisfying $\Delta r = Ar + B$, $A \in Mat(3, 3)$, $B \in Mat(3, 1)$ are the minimal surfaces, the spheres and the circular cylinders; Stamatakis and Zoubi [48] focused surfaces of revolution satisfying $\Delta^{III} x = Ax$; Senoussi and Bekkar [41] gave helicoidal surfaces M^2 which are of finite type with respect to the fundamental forms I, II and III, i.e., their position vector field $r(u, v)$ satisfies the condition $\Delta^J r = Ar$, $J = I, II, III$, where $A \in Mat(3, 3)$; Kim, Kim, and Kim [34] introduced Cheng-Yau operator and Gauss map of surfaces of revolution. Recently, Shaikh et. al [42, 43, 44, 45, 46, 47] initiated the study of surface curves in a different way, especially, rectifying, osculating and normal curves on a surface by considering isometry and conformal map between two surfaces and investigated their invariancy under such maps.

In $\mathbb{E}^4$; Moore [39, 40] considered general rotational surfaces; Hasanis and Vlachos [31] studied hypersurfaces with harmonic mean curvature vector field; Cheng and Wan [15] gave complete hypersurfaces with CMC; Kim and Turgay [35] worked surfaces with L_1-pointwise 1-type Gauss map; Arslan et. al. [3] introduced Vranceanu surface with pointwise 1-type Gauss map; Arslan et. al. [4] worked generalized rotational surfaces; Aksoyak and Yaylı [32] studied flat rotational surfaces with pointwise 1-type Gauss map; Güler, Magid, and Yaylı [29] introduced helicoidal hypersurfaces; Güler, Hacısalihoğlu, and Kim [28] worked Gauss map and the third Laplace-Beltrami operator of the rotational hypersurface; Güler and Turgay [30] worked Cheng-Yau operator and Gauss map of rotational hypersurfaces; Altın, Kazan, and Karadağ [2] studied Monge hypersurfaces with density; Güler [27] obtained rotational hypersurfaces satisfying $\Delta^I R = AR$, where $A \in Mat(4, 4)$. He [26] also worked fundamental form *IV* and curvature formulas of the hypersphere.

In Minkowski 4-space $\mathbb{E}^4_1$; Ganchev and Milousheva [23] studied analogue of surfaces of [39, 40]; Arvanitoyeorgos, Kaimakamais, and Magid [6] indicated that if the mean curvature vector field of M^3_1 satisfies the equation $\Delta H = \alpha H$ (α a constant), then M^3_1 has CMC; Arslan and Milousheva introduced meridian surfaces of elliptic or hyperbolic type with pointwise 1-type Gauss map; Turgay considered some classifications of Lorentzian surfaces with finite type Gauss map; Dursun and Turgay worked space-like surfaces in with pointwise 1-type Gauss map. Aksoyak and Yaylı [33] gave general rotational surfaces with pointwise 1-type Gauss map in $\mathbb{E}^4_2$. Bektaş, Canfes, and Dursun [9] obtained surfaces in a pseudo-sphere with 2-type pseudo-spherical Gauss map in $\mathbb{E}^5_2$.

We consider hypersphere in the four dimensional Euclidean geometry $\mathbb{E}^4$. In Section 2, we give some basic notions of the four dimensional Euclidean geometry. We consider curvature formulas of a hypersurface in $\mathbb{E}^4$, in Section 3. In Section 4, we define hypersphere. Finally, we study hypersphere satisfying $\Delta \mathbf{x} = \mathcal{A}\mathbf{x}$ for some 4×4 matrix $\mathcal{A}$ in $\mathbb{E}^4$ in the last section.

2 Preliminaries

In this section, giving some of basic facts and definitions, we describe notations used whole paper. Let $\mathbb{E}^m$ denote the Euclidean m-space with the canonical Euclidean metric tensor given by $\widetilde{g} = \langle\ ,\ \rangle = \sum_{i=1}^{m} dx_i^2$, where $(x_1, x_2, \ldots, x_m)$ is a rectangular coordinate system in $\mathbb{E}^m$. Consider an m-dimensional Riemannian submanifold of the space $\mathbb{E}^m$. We denote the Levi-Civita connections of $\mathbb{E}^m$ and M by $\widetilde{\nabla}$ and ∇, respectively. We shall use letters X, Y, Z, W (resp., ξ, η) to denote vectors fields tangent (resp., normal) to M. The Gauss and Weingarten formulas are given, respectively, by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\widetilde{\nabla}_X \xi = -A_\xi(X) + D_X \xi, \tag{2.2}$$

where h, D and A are the second fundamental form, the normal connection and the shape operator of M, respectively.

For each $\xi \in T_p^\perp M$, the shape operator A_ξ is a symmetric endomorphism of the tangent space T_pM at $p \in M$. The shape operator and the second fundamental form are related by

$$\langle h(X, Y), \xi\rangle = \langle A_\xi X, Y\rangle .$$

The Gauss and Codazzi equations are given, respectively, by

$$\langle R(X, Y,)Z, W\rangle = \langle h(Y, Z), h(X, W)\rangle - \langle h(X, Z), h(Y, W)\rangle, \tag{2.3}$$

$$(\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(X, Z), \tag{2.4}$$

where R, R^D are the curvature tensors associated with connections ∇ and D, respectively, and $\bar{\nabla} h$ is defined by

$$(\bar{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

2.1 Hypersurfaces of Euclidean space

Now, let M be an oriented hypersurface in the Euclidean space $\mathbb{E}^{n+1}$, $\mathbf{S}$ its shape operator (i.e. Weingarten map) and x its position vector. We consider a local orthonormal frame field $\{e_1, e_2, \ldots, e_n\}$ of consisting of principal directions of M corresponding from the principal curvature k_i for $i = 1, 2, \ldots n$. Let the dual basis of this frame field be $\{\theta_1, \theta_2, \ldots, \theta_n\}$. Then the first structural equation of Cartan is

$$d\theta_i = \sum_{i=1}^{n} \theta_j \wedge \omega_{ij}, \quad i, j = 1, 2, \ldots, n, \tag{2.5}$$

where ω_{ij} denotes the connection forms corresponding to the chosen frame field. We denote the Levi-Civita connection of M and $\mathbb{E}^{n+1}$ by ∇ and $\widetilde{\nabla}$, respectively. Then, from the Codazzi equation (2.3), we have

$$e_i(k_j) = \omega_{ij}(e_j)(k_i - k_j), \tag{2.6}$$
$$\omega_{ij}(e_l)(k_i - k_j) = \omega_{il}(e_j)(k_i - k_l) \tag{2.7}$$

for distinct $i, j, l = 1, 2, \ldots, n$.

We put $s_j = \sigma_j(k_1, k_2, \ldots, k_n)$, where σ_j is the j-th elementary symmetric function given by

$$\sigma_j(a_1, a_2, \ldots, a_n) = \sum_{1 \leq i_1 < i_2 < \ldots, i_j \leq n} a_{i_1} a_{i_2} \ldots a_{i_j}.$$

We use following notation

$$r_i^j = \sigma_j(k_1, k_2, \ldots, k_{i-1}, k_{i+1}, k_{i+2}, \ldots, k_n).$$

By the definition, we have $r_i^0 = 1$ and $s_{n+1} = s_{n+2} = \cdots = 0$. We call the function s_k as the k-th mean curvature of M. We would like to note that functions $H = \frac{1}{n} s_1$ and $K = s_n$ are called the mean curvature and Gauss-Kronecker curvature of M, respectively. In particular, M is said to be j-minimal if $s_j \equiv 0$ on M.

In $\mathbb{E}^{n+1}$, to find the i-th curvature formulas $\mathfrak{C}_i$ (Curvature formulas sometimes are represented as mean curvature H_i, and sometimes as Gaussian curvature K_i by different writers, such as [1] and [36]. We will call it just i-th curvature $\mathfrak{C}_i$ in this paper.), where $i = 0, .., n$, firstly, we use the characteristic polynomial of $\mathbf{S}$:

$$P_{\mathbf{S}}(\lambda) = 0 = \det(\mathbf{S} - \lambda I_n) = \sum_{k=0}^{n} (-1)^k s_k \lambda^{n-k}, \tag{2.8}$$

where $i = 0, .., n$, I_n denotes the identity matrix of order n. Then, we get curvature formulas $\binom{n}{i}\mathfrak{C}_i = s_i$,. That is, $\binom{n}{0}\mathfrak{C}_0 = s_0 = 1$ (by definition), $\binom{n}{1}\mathfrak{C}_1 = s_1, \ldots, \binom{n}{n}\mathfrak{C}_n = s_n = K$.

k-th fundamental form of M is defined by $I\left(\mathbf{S}^{k-1}(X), Y\right) = \left\langle \mathbf{S}^{k-1}(X), Y \right\rangle$. So, we have

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} \mathfrak{C}_i I\left(\mathbf{S}^{n-i}(X), Y\right) = 0. \tag{2.9}$$

In particular, one can get classical result $\mathfrak{C}_0 III - 2\mathfrak{C}_1 II + \mathfrak{C}_2 I = 0$ of surface theory for $n = 2$. See [36] for details.

For a Euclidean submanifold $x\colon M \longrightarrow \mathbb{E}^m$, the immersion (M, x) is called *finite type,* if x can be expressed as a finite sum of eigenfunctions of the Laplacian Δ of (M, x), i.e. $x = x_0 + \sum_{i=1}^{k} x_i$, where x_0 is a constant map, $x_1, \ldots, x_k$ non-constant maps, and $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, k$. If λ_i are different, M is called *k-type*. See [11] for details.

2.2 Rotational hypersurfaces

We will obtain a rotational hypersurface (rot-hypface for short) in Euclidean 4-space. Before we proceed, we would like to note that the definition of rot-hypfaces in Riemannian space forms were defined in [19]. A rot-hypface $M \subset \mathbb{E}^{n+1}$ generated by a curve $\mathcal{C}$ around an axis $\mathcal{C}$ that does not meet $\mathcal{C}$ is obtained by taking the orbit of $\mathcal{C}$ under those orthogonal transformations of $\mathbb{E}^{n+1}$ that leaves $\mathfrak{r}$ pointwise fixed (See [19, Remark 2.3]).

Throughout the paper, we shall identify a vector (a, b, c, d) with its transpose. Consider the case $n = 3$, and let $\mathcal{C}$ be the curve parametrized by

$$\gamma(w) = (f(w), 0, 0, \varphi(w)). \tag{2.10}$$

If $\mathfrak{r}$ is the x_4-axis, then an orthogonal transformations of $\mathbb{E}^{n+1}$ that leaves $\mathfrak{r}$ pointwise fixed has the form

$$\mathbf{Z}(v, w) = \begin{pmatrix} \cos u \cos v & -\sin u & -\cos u \sin v & 0 \\ \sin u \cos v & \cos u & -\sin u \sin v & 0 \\ \sin v & 0 & \cos v & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad u, v \in \mathbb{R}. \tag{2.11}$$

Therefore, the parametrization of the rot-hypface generated by a curve $\mathcal{C}$ around an axis $\mathfrak{r}$ is given by

$$\mathbf{x}(u,v,w)=\mathbf{Z}(u,v)\gamma(w). \tag{2.12}$$

Definition 2.1. Let $\mathbf{x}=\mathbf{x}(u,v,w)$ be an immersion from $M^3\subset\mathbb{E}^3$ to $\mathbb{E}^4$. In $\mathbb{E}^4$, inner product is defined by

$$\langle\overrightarrow{x},\overrightarrow{y}\rangle=x_1y_1+x_2y_2+x_3y_3+x_4y_4,$$

and triple vector product is given by

$$\overrightarrow{x}\times\overrightarrow{y}\times\overrightarrow{z}=\det\begin{pmatrix}e_1&e_2&e_3&e_4\\x_1&x_2&x_3&x_4\\y_1&y_2&y_3&y_4\\z_1&z_2&z_3&z_4\end{pmatrix},$$

where $\overrightarrow{x}=(x_1,x_2,x_3,x_4)$, $\overrightarrow{y}=(y_1,y_2,y_3,y_4)$, $\overrightarrow{z}=(z_1,z_2,z_3,z_4)$.

Definition 2.2. For a hypface $\mathbf{x}$ in 4-space, we have

$$\mathbf{I}=\begin{pmatrix}E&F&A\\F&G&B\\A&B&C\end{pmatrix},\ \mathbf{II}=\begin{pmatrix}L&M&P\\M&N&T\\P&T&V\end{pmatrix}, \tag{2.13}$$

and

$$\begin{aligned}\det\mathbf{I}&=(EG-F^2)C-EB^2+2FAB-GA^2,\\\det\mathbf{II}&=\left(LN-M^2\right)V-LT^2+2MPT-NP^2,\end{aligned}$$

where I and II are the first and the second fundamental form matrices, respectively, where $E=\mathbf{x}_u\cdot\mathbf{x}_u$, $F=\mathbf{x}_u\cdot\mathbf{x}_v$, $G=\mathbf{x}_v\cdot\mathbf{x}_v$, $A=\mathbf{x}_u\cdot\mathbf{x}_v$, $B=\mathbf{x}_v\cdot\mathbf{x}_w$, $C=\mathbf{x}_w\cdot\mathbf{x}_w$, $L=\mathbf{x}_{uu}\cdot e$, $M=\mathbf{x}_{uv}\cdot e$, $N=\mathbf{x}_{vv}\cdot e$, $P=\mathbf{x}_{uw}\cdot e$, $T=\mathbf{x}_{vw}\cdot e$, $V=\mathbf{x}_{ww}\cdot e$. Here,

$$e=\frac{\mathbf{x}_u\times\mathbf{x}_v\times\mathbf{x}_w}{\|\mathbf{x}_u\times\mathbf{x}_v\times\mathbf{x}_w\|} \tag{2.14}$$

is unit normal (i.e. the Gauss map) of hypface $\mathbf{x}$.

Definition 2.3. Product matrices $\mathbf{I}^{-1}\cdot\mathbf{II}$ gives the matrix of the shape operator $\mathbf{S}$ of hypface $\mathbf{x}$ in 4-space as follows

$$\mathbf{S}=\frac{1}{\det\mathbf{I}}\begin{pmatrix}s_{11}&s_{12}&s_{13}\\s_{21}&s_{22}&s_{23}\\s_{31}&s_{32}&s_{33}\end{pmatrix}, \tag{2.15}$$

where

$$\det\mathbf{I}=(EG-F^2)C-A^2G+2ABF-B^2E,$$

$$\begin{aligned}s_{11}&=ABM-CFM-AGP+BFP+CGL-B^2L,\\s_{12}&=ABN-CFN-AGT+BFT+CGM-B^2M,\\s_{13}&=ABT-CFT-AGV+BFV+CGP-B^2P,\\s_{21}&=ABL-CFL+AFP-BPE+CME-A^2M,\\s_{22}&=ABM-CFM+AFT-BTE+CNE-A^2N,\\s_{23}&=ABP-CFP+AFV-BVE+CTE-A^2T,\\s_{31}&=-AGL+BFL+AFM-BME+GPE-F^2P,\\s_{32}&=-AGM+BFM+AFN-BNE+GTE-F^2T,\\s_{33}&=-AGP+BFP+AFT-BTE+GVE-F^2V.\end{aligned}$$

See [28, 29, 30] for details.

3 i-th Curvatures

To compute the i-th mean curvature formula $\mathfrak{C}_i$, where $i = 0, .., 3$, we use characteristic polynomial $P_{\mathbf{S}}(\lambda) = a\lambda^3 + b\lambda^2 + c\lambda + d = 0$:

$$P_{\mathbf{S}}(\lambda) = \det(\mathbf{S} - \lambda I_3) = 0.$$

Then, obtain $\mathfrak{C}_0 = 1$ (by definition), $\binom{3}{1}\mathfrak{C}_1 = \binom{3}{1}H = -\frac{b}{a}$, $\binom{3}{2}\mathfrak{C}_2 = \frac{c}{a}$, $\binom{3}{3}\mathfrak{C}_3 = K = -\frac{d}{a}$.

Therefore, we find i-th curvature folmulas depends on the coefficients of the fundamental forms **I** and **II** in 4-space. See [26] for details.

Theorem 3.1. *Any hypface* $\mathbf{x}$ *in* $\mathbb{E}^4$ *has following curvature formulas,* $\mathfrak{C}_0 = 1$ *(by definition),*

$$\mathfrak{C}_1 = \frac{\left\{\begin{array}{c}(EN + GL - 2FM)C + (EG - F^2)V - LB^2 - NA^2 \\ -2(APG - BPF - ATF + BTE - ABM)\end{array}\right\}}{3\left[(EG - F^2)C - EB^2 + 2FAB - GA^2\right]}, \tag{3.1}$$

$$\mathfrak{C}_2 = \frac{\left\{\begin{array}{c}(EN + GL - 2FM)\,V + \left(LN - M^2\right)C - ET^2 - GP^2 \\ -2\,(APN - BPM - ATM + BTL - PTF)\end{array}\right\}}{3\left[(EG - F^2)C - EB^2 + 2FAB - GA^2\right]}, \tag{3.2}$$

$$\mathfrak{C}_3 = \frac{\left(LN - M^2\right)V - LT^2 + 2MPT - NP^2}{(EG - F^2)C - EB^2 + 2FAB - GA^2}. \tag{3.3}$$

Proof. Solving $\det(\mathbf{S} - \lambda I_3) = 0$ with some algebraic computations, we obtain coefficients a, b, c, d of polynomial $P_{\mathbf{S}}(\lambda)$. □

A hypersurface $\mathbf{x}$ in $\mathbb{E}^4$ is $\mathfrak{C}_i$−minimal, when $\mathfrak{C}_i = 0$ identically on $\mathbf{x}$.

4 Hypersphere

In this section, we define hypersphere, then find its differential geometric properties in $\mathbb{E}^4$.

For an open interval $I \subset \mathbb{R}$, let $\gamma : I \longrightarrow \Pi$ be a curve in a plane Π in $\mathbb{E}^4$, and let ℓ be a straight line in Π.

Definition 4.1. A *rotational hypersurface* in $\mathbb{E}^4$ is called hypersphere, when a curve

$$\gamma(w) = (r\cos w, 0, 0, r\sin w)$$

rotates by (2.11) around a line $\ell = (0, 0, 0, 1)$ (these are called the *profile curve* and the *axis*, respectively).

So, the hypersphere which is spanned by the vector ℓ, is as follows

$$\mathbf{x}(u, v, w) = Z(u, v)\gamma(w) \tag{4.1}$$

in $\mathbb{E}^4$, where $u, v, w \in [0, 2\pi]$. Therefore, more clear form of (4.1) is as follows

$$\mathbf{x}(u, v, w) = \begin{pmatrix} r\cos u\cos v\cos w \\ r\sin u\cos v\cos w \\ r\sin v\cos w \\ r\sin w \end{pmatrix}, \tag{4.2}$$

where $r \in \mathbb{R}\backslash\{0\}$ and $0 \leq u, v, w \leq 2\pi$. When $w = 0$, we have a sphere in $\mathbb{E}^4$.

Next, we obtain the curvatures and the Gaussian curvature of the hypersphere (4.2).

We get the first differentials of (4.2) with respect to u, v, w, respectively,

$$\mathbf{x}_u = \begin{pmatrix} -r\sin u\cos v\cos w \\ r\cos u\cos v\cos w \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{x}_v = \begin{pmatrix} -r\cos u\sin v\cos w \\ -r\cos u\sin v\cos w \\ r\cos v\cos w \\ 0 \end{pmatrix},$$

and

$$\mathbf{x}_w = \begin{pmatrix} -r\cos u\cos v\sin w \\ -r\sin u\cos v\sin w \\ -r\sin v\sin w \\ r\cos w \end{pmatrix}.$$

The first quantities of (4.2) are as follows

$$\mathbf{I} = \begin{pmatrix} r^2\cos^2 v\cos^2 w & 0 & 0 \\ 0 & r^2\cos^2 w & 0 \\ 0 & 0 & r^2 \end{pmatrix}. \tag{4.3}$$

We have $\det \mathbf{I} = r^6\cos^2 v\cos^4 w$. The line element of the hypersphere is given by

$$ds^2 = r^2\left[\left(\cos^2 v du^2 + dv^2\right)\cos^2 w + dw^2\right].$$

Using (2.14), we get the Gauss map of the hypersphere (4.2) as follows

$$e = \begin{pmatrix} \cos u\cos v\cos w \\ \sin u\cos v\cos w \\ \sin v\cos w \\ \sin w \end{pmatrix}. \tag{4.4}$$

The second differentials of (4.2) with respect to u, v, w, and the Gauss map (4.4) of the hypersphere (4.2), we have the second quantities as follows

$$\mathbf{II} = \begin{pmatrix} -r\cos^2 v\cos^2 w & 0 & 0 \\ 0 & -r\cos^2 w & 0 \\ 0 & 0 & -r \end{pmatrix}. \tag{4.5}$$

So, we get $\det \mathbf{II} = -r^3\cos^2 v\cos^4 w$.

We calculate the shape operator matrix of the hypersphere (4.2), using (2.15), as follows

$$\mathbf{S} = \begin{pmatrix} -\frac{1}{r} & 0 & 0 \\ 0 & -\frac{1}{r} & 0 \\ 0 & 0 & -\frac{1}{r} \end{pmatrix}.$$

Finally, using (3.1), (3.2) and (3.3), with (4.3), (4.5), respectively, we find the curvatures of the hypersphere (4.2) as follows:

Corollary 4.2. *Let $\mathbf{x} : M^3 \longrightarrow \mathbb{E}^4$ be an immersion given by (4.2). Then M^3 has constant (mean) 1-curvature*

$$\mathfrak{C}_1 = H = -\frac{1}{r}.$$

Corollary 4.3. *Let $\mathbf{x} : M^3 \longrightarrow \mathbb{E}^4$ be an immersion given by (4.2). Then M^3 has constant 2-curvature*

$$\mathfrak{C}_2 = \frac{1}{r^2}.$$

Corollary 4.4. *Let $\mathbf{x} : M^3 \longrightarrow \mathbb{E}^4$ be an immersion given by (4.2). Then M^3 has negative constant (Gaussian) 3-curvature*

$$\mathfrak{C}_3 = K = -\frac{1}{r^3}.$$

5 Hypersphere satisfying $\Delta\mathbf{x} = \mathcal{A}\mathbf{x}$

In this section, we give the Laplace-Beltrami operator of a smooth function, then calculate it using hypersphere.

The inverse of the matrix

$$(g_{ij}) = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}$$

is as follows

$$\frac{1}{g}\begin{pmatrix} g_{22}g_{33} - g_{23}g_{32} & -(g_{12}g_{33} - g_{13}g_{32}) & g_{12}g_{23} - g_{13}g_{22} \\ -(g_{21}g_{33} - g_{31}g_{23}) & g_{11}g_{33} - g_{13}g_{31} & -(g_{11}g_{23} - g_{21}g_{13}) \\ g_{21}g_{32} - g_{22}g_{31} & -(g_{11}g_{32} - g_{12}g_{31}) & g_{11}g_{22} - g_{12}g_{21} \end{pmatrix},$$

where

$$\begin{aligned} g &= \det(g_{ij}) \\ &= g_{11}g_{22}g_{33} - g_{11}g_{23}g_{32} + g_{12}g_{31}g_{23} - g_{12}g_{21}g_{33} + g_{21}g_{13}g_{32} - g_{13}g_{22}g_{31}. \end{aligned}$$

Definition 5.1. The Laplace-Beltrami operator of a smooth function $\phi = \phi(x^1, x^2, x^3)\mid_{\mathbf{D}}$ ($\mathbf{D} \subset \mathbb{R}^3$) of class C^3 with respect to the first fundamental form of a hypersurface $\mathbf{M}$ is the operator Δ which is defined by as follows

$$\Delta\phi = \frac{1}{\sqrt{g}}\sum_{i,j=1}^{3}\frac{\partial}{\partial x^i}\left(\sqrt{g}g^{ij}\frac{\partial\phi}{\partial x^j}\right), \tag{5.1}$$

where $(g^{ij}) = (g_{kl})^{-1}$ and $g = \det(g_{ij})$.

Clearly, we can write (5.1) as follows

$$\Delta\phi = \frac{1}{\sqrt{g}}\left\{\begin{array}{c} \frac{\partial}{\partial x^1}\left(\sqrt{g}g^{11}\frac{\partial\phi}{\partial x^1}\right) - \frac{\partial}{\partial x^1}\left(\sqrt{g}g^{12}\frac{\partial\phi}{\partial x^2}\right) + \frac{\partial}{\partial x^1}\left(\sqrt{g}g^{13}\frac{\partial\phi}{\partial x^3}\right) \\ -\frac{\partial}{\partial x^2}\left(\sqrt{g}g^{21}\frac{\partial\phi}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(\sqrt{g}g^{22}\frac{\partial\phi}{\partial x^2}\right) - \frac{\partial}{\partial x^2}\left(\sqrt{g}g^{23}\frac{\partial\phi}{\partial x^3}\right) \\ +\frac{\partial}{\partial x^3}\left(\sqrt{g}g^{31}\frac{\partial\phi}{\partial x^1}\right) - \frac{\partial}{\partial x^3}\left(\sqrt{g}g^{32}\frac{\partial\phi}{\partial x^2}\right) + \frac{\partial}{\partial x^3}\left(\sqrt{g}g^{33}\frac{\partial\phi}{\partial x^3}\right) \end{array}\right\}. \tag{5.2}$$

So, we get the inverse of (4.3) as follows

$$\mathbf{I}^{-1} = \frac{1}{\det\mathbf{I}}\begin{pmatrix} CG - B^2 & AB - CF & BF - AG \\ AB - CF & CE - A^2 & AF - BE \\ BF - AG & AF - BE & EG - F^2 \end{pmatrix},$$

where $\det\mathbf{I} = (EG - F^2)C - A^2G + 2ABF - B^2E$. Hence, more clear notation of (5.2) for a smooth function $\phi = \phi(u, v, w)$ is as follows

$$\Delta\phi = \frac{1}{\sqrt{|\det\mathbf{I}|}}\left\{\begin{array}{c} \frac{\partial}{\partial u}\left(\frac{(CG-B^2)\phi_u - (AB-CF)\phi_v + (BF-AG)\phi_w}{\sqrt{|\det\mathbf{I}|}}\right) \\ -\frac{\partial}{\partial v}\left(\frac{(AB-CF)\phi_u - (CE-A^2)\phi_v + (AF-BE)\phi_w}{\sqrt{|\det\mathbf{I}|}}\right) \\ +\frac{\partial}{\partial w}\left(\frac{(BF-AG)\phi_u - (AF-BE)\phi_v + (EG-F^2)\phi_w}{\sqrt{|\det\mathbf{I}|}}\right) \end{array}\right\}. \tag{5.3}$$

We continue our calculations to find the Laplace-Beltrami operator $\Delta\mathbf{x}$ of the hypersphere $\mathbf{x}$ using (4.2) and (5.3).

The Laplace-Beltrami operator of the hypersphere (4.2) is given by

$$\Delta\mathbf{x} = \frac{1}{\sqrt{|\det\mathbf{I}|}}\left(\frac{\partial}{\partial u}\mathcal{U} - \frac{\partial}{\partial v}\mathcal{V} + \frac{\partial}{\partial w}\mathcal{W}\right), \tag{5.4}$$

where

$$\begin{aligned}\mathcal{U} &= \frac{\left(CG-B^2\right)\mathbf{x}_u-(AB-CF)\,\mathbf{x}_v+(BF-AG)\,\mathbf{x}_w}{\sqrt{|\det \mathbf{I}|}},\\ \mathcal{V} &= \frac{(AB-CF)\,\mathbf{x}_u-\left(CE-A^2\right)\mathbf{x}_v+(AF-BE)\,\mathbf{x}_w}{\sqrt{|\det \mathbf{I}|}},\\ \mathcal{W} &= \frac{(BF-AG)\,\mathbf{x}_u-(AF-BE)\,\mathbf{x}_v+\left(EG-F^2\right)\mathbf{x}_w}{\sqrt{|\det \mathbf{I}|}}.\end{aligned}$$

Here, $A=B=F=0$. Hence, we briefly can write $\mathcal{U}$,$\mathcal{W}$,$\mathcal{W}$, as follows

$$\mathcal{U}=\frac{CG}{\sqrt{|\det \mathbf{I}|}}\mathbf{x}_u,\ \mathcal{V}=-\frac{CE}{\sqrt{|\det \mathbf{I}|}}\mathbf{x}_v,\ \mathcal{W}=\frac{EG}{\sqrt{|\det \mathbf{I}|}}\mathbf{x}_w.$$

Finally, substituting $\frac{\partial}{\partial u}\mathcal{U}$, $\frac{\partial}{\partial v}\mathcal{V}$, $\frac{\partial}{\partial w}(\mathcal{W})$ into (5.4) , we get

$$\Delta\mathbf{x}=\begin{pmatrix}\Delta\mathbf{x}_1\\ \Delta\mathbf{x}_2\\ \Delta\mathbf{x}_3\\ \Delta\mathbf{x}_4\end{pmatrix}=\begin{pmatrix}-\frac{3}{r}\cos u\cos v\cos w\\ -\frac{3}{r}\sin u\cos v\cos w\\ -\frac{3}{r}\sin v\cos w\\ -\frac{3}{r}\sin w\end{pmatrix}.$$

Therefore, we have following results:

Corollary 5.2. *Let* $\mathbf{x}: M^3 \longrightarrow \mathbb{E}^4$ *be an immersion given by* (4.2)*. Then* $\mathbf{x}$ *has*

$$\Delta\mathbf{x}=3\mathfrak{C}_1 e,$$

where $\mathfrak{C}_1$ *and* e *are the mean 1-curvature and the Gauss map, respectively.*

Corollary 5.3. *Let* $\mathbf{x}: M^3 \longrightarrow \mathbb{E}^4$ *be an immersion given by* (4.2)*. Then* $\mathbf{x}$ *has*

$$\Delta\mathbf{x}=-3\left(\mathfrak{C}_2\right)^{1/2}e,$$

where $\mathfrak{C}_2$ *and* e *are the 2-curvature and the Gauss map, respectively.*

Corollary 5.4. *Let* $\mathbf{x}: M^3 \longrightarrow \mathbb{E}^4$ *be an immersion given by* (4.2)*. Then* $\mathbf{x}$ *has*

$$\Delta\mathbf{x}=3\left(\mathfrak{C}_3\right)^{1/3}e,$$

where $\mathfrak{C}_3$ *and* e *are the 3-curvature and the Gauss map, respectively.*

Corollary 5.5. *Let* $\mathbf{x}: M^3 \longrightarrow \mathbb{E}^4$ *be an immersion given by* (4.2)*. Then* $\mathbf{x}$ *has*

$$\Delta\mathbf{x}=\mathcal{A}\mathbf{x},$$

where

$$\mathcal{A}=-3\left(\mathfrak{C}_1\right)^2 I_4=-3\mathfrak{C}_2 I_4=-3\left(\mathfrak{C}_1\right)^{-1}\mathfrak{C}_3 I_4=-3\left(\left(\mathfrak{C}_2\right)^{-1}\mathfrak{C}_3\right)^2 I_4,$$

and $\mathcal{A}\in Mat\,(4,4)$, $I_4=diag(1,1,1,1)$.

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