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On Uniformly Starlike Functions With Negative Coefficients Given by Polylogarithms

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Abstract In this work, a new subclass $TS_\lambda^k(\gamma, \xi)$ of uniformly starlike functions with negative coefficients involving polylogarithm function is introduced. Also coefficient estimates, extreme points, closure and inclusion theorems, radii of starlikeness and convexity and partial sums for this new subclass are obtained.

1 Introduction

Let A denote the class of all analytic and univalent functions in the open disc $E = \{z : z \in C \text{ and } |z| < 1\}$ having the form

$$u(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{1.1}$$

and let

$$T = \{u(z) : u \in A \text{ and } u(z) = z - \sum_{n=2}^{\infty} b_n z^n,\ (b_n \geq 0)\}. \tag{1.2}$$

After Goodman [12, 13], two new subclasses was investigated by Ronning [17, 18] as follow
(i)$u \in A$ is in $S(\gamma, \xi)$ if the following inequality holds

$$\Re\left\{\frac{zu'(z)}{u(z)} - \gamma\right\} > \xi\left|\frac{zu'(z)}{u(z)} - 1\right|,\ z \in E,\ \ -1 < \gamma \leq 1 \text{ and } \xi \geq 0, \tag{1.3}$$

$S(\gamma, \xi)$ is class of uniformly $\xi-$starlike functions.
(ii)$u \in A$ is in $UCV(\gamma, \xi)$ if the following inequality holds

$$\Re\left\{1 + \frac{zu''(z)}{u'(z)} - \gamma\right\} > \xi\left|\frac{zu''(z)}{u'(z)}\right|,\ z \in E,\ \ -1 < \gamma \leq 1 \text{ and } \xi \geq 0, \tag{1.4}$$

$UCV(\gamma, \xi)$ is the class of uniformly $\xi-$convex functions.

By using (1.3) and (1.4), one can see that

$$u \in UCV(\gamma, \xi) \Leftrightarrow zu' \in S(\gamma, \xi). \tag{1.5}$$

The generalized polylogarithm function $G(k, z)$ is considered as

$$G(k, z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k},\ (k \in C, z \in E). \tag{1.6}$$

The function $G(-1, z) = \frac{z}{(1-z)^2}$ is Koebe function. We now present a function $(G(k, z))^{-1}$ and get the linear operator $\mathfrak{D}_\lambda^k u(z)$ as following:

$$G(k, z) * (G(k, z))^{-1} = \frac{z}{(1 - z)^{\lambda+1}},\ \lambda > -1 \text{ and } \ z \in E \tag{1.7}$$

and

$$\mathfrak{D}_{\lambda}^{k}u(z) = (G(k,z))^{-1} * u(z). \tag{1.8}$$

For $\lambda > -1$,

$$\frac{z}{(1-z)^{\lambda+1}} = \sum_{n=0}^{\infty} \frac{(\lambda+1)_n}{n!} z^{n+1}, \ (z \in E). \tag{1.9}$$

By using (1.7) and (1.9) in (1.8),it is obtained that

$$\sum_{n=1}^{\infty} \frac{z^n}{n^k} * (G(k,z))^{-1} = \sum_{n=1}^{\infty} \frac{(n+\lambda-1)!}{\lambda!(n-1)!} z^n.$$

Thus $(G(k,z))^{-1}$ has of the form

$$(G(k,z))^{-1} = \sum_{n=1}^{\infty} n^k \frac{(n+\lambda-1)!}{\lambda!(n-1)!} z^n, \ (z \in E).$$

For $k, \lambda \in N_0$, let us remember polylogarithm functions $\mathfrak{D}_{\lambda}^{k}u(z)$ introduced and studied by Al-Shaqsi and Darus [10]:

$$\begin{aligned}\mathfrak{D}_{\lambda}^{k}u(z) &= z + \sum_{n=2}^{\infty} n^k \frac{(n+\lambda-1)!}{\lambda!(n-1)!} b_n z^n, \ (z \in E)\\ &= z + \sum_{n=2}^{\infty} n^k c(\lambda,n) b_n z^n, \ \text{ where } \ c(\lambda,n) = \begin{pmatrix} n+\lambda-1 \\ \lambda \end{pmatrix}.\end{aligned} \tag{1.10}$$

One can see [15] and [16] to learn more information regarding polylogarithms in the theory of univalent functions .

Note that $\mathfrak{D}_{0}^{k} \cong \mathfrak{D}^{n}$ and $\mathfrak{D}_{\lambda}^{0} \cong \mathfrak{D}^{\delta}$ which were Salagean and Ruscheweyh derivative operators in turn [21, 19]. Obviusly, the operator $\mathfrak{D}_{\lambda}^{k}$ included two known derivative operators. In geometric funcion theory, analytic functions with negative coefficient is an important topic. In this study, we refer to the studies by [1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 14, 20, 22, 23, 24].

Also state that $\mathfrak{D}_{0}^{0}u(z) = u(z)$ and $\mathfrak{D}_{0}^{1}u(z) = \mathfrak{D}_{1}^{0}u(z) = zu'(z)$.

In the next section, we will define a new subclass $S_{\lambda}^{k}(\gamma,\xi)$ and give some important properties of the functions belonging this class.

2 The Class $S_{\lambda}^{k}(\gamma,\xi)$

Definition 2.1. A function u given by (1.1) is said to be in the class $S_{\lambda}^{k}(\gamma,\xi)$ if the following criterion is hold

$$\Re\left\{\frac{z(\mathfrak{D}_{\lambda}^{k}u(z))'}{\mathfrak{D}_{\lambda}^{k}u(z)} - \gamma\right\} > \xi\left|\frac{z(\mathfrak{D}_{\lambda}^{k}u(z))'}{\mathfrak{D}_{\lambda}^{k}u(z)} - 1\right|, \ z \in E, \tag{2.1}$$

for $-1 \le \gamma < 1, \xi \ge 0, k, \lambda \in N_0$ and $\mathfrak{D}_{\lambda}^{k}u(z)$ is given by (1.10).

We also let $TS_{\lambda}^{k}(\gamma,\xi) = S_{\lambda}^{k}(\gamma,\xi) \cap T$. In the special cases of the parameters λ and k the class $S_{\lambda}^{k}(\gamma,\xi)$ can be reduced to the class investigated by Ronning [17].

Theorem 2.2. $u \in S_{\lambda}^{k}(\gamma,\xi)$ *if the following inequality holds*

$$\sum_{n=2}^{\infty} \left[n(1+\xi) - (\gamma+\xi)\right] n^k c(\lambda,n)|b_n| \le 1-\gamma, \tag{2.2}$$

where $-1 \le \gamma < 1, \xi > 0, k, \lambda \in N_0$ *and* u *is given by* (1.1).

Proof. It is enough to show that

$$\xi\left|\frac{z(\mathfrak{D}_{\lambda}^{k}u(z))'}{\mathfrak{D}_{\lambda}^{k}u(z)}-1\right|-\Re\left\{\frac{z(\mathfrak{D}_{\lambda}^{k}u(z))'}{\mathfrak{D}_{\lambda}^{k}u(z)}-1\right\}\leq 1-\gamma. \tag{2.3}$$

We get

$$\begin{aligned}&\xi\left|\frac{z(\mathfrak{D}_{\lambda}^{k}u(z))'}{\mathfrak{D}_{\lambda}^{k}u(z)}-1\right|-\Re\left\{\frac{z(\mathfrak{D}_{\lambda}^{k}u(z))'}{\mathfrak{D}_{\lambda}^{k}u(z)}-1\right\}\\ \leq&(1+\xi)\left|\frac{z(\mathfrak{D}_{\lambda}^{k}u(z))'}{\mathfrak{D}_{\lambda}^{k}u(z)}-1\right| \\ \leq&\frac{(1+\xi)\sum\limits_{n=2}^{\infty}(n-1)n^{k}c(\lambda,n)|b_n|}{1-\sum\limits_{n=2}^{\infty}n^{k}c(\lambda,n)|b_n|}.\end{aligned} \tag{2.4}$$

The last inequality is bounded above by $(1-\gamma)$ if

$$\sum_{n=2}^{\infty}[n(1+\xi)-(\gamma+\xi)]n^{k}c(\lambda,n)|b_n|\leq 1-\gamma. \tag{2.5}$$

Thus the proof is completed. □

Theorem 2.3. *$u\in TS_{\lambda}^{k}(\gamma,\xi)$ if and only if*

$$\sum_{n=2}^{\infty}[n(1+\xi)-(\gamma+\xi)]n^{k}c(\lambda,n)b_n\leq 1-\gamma, \tag{2.6}$$

where $-1\leq\gamma<1,\ \xi\geq 0$ and $k,\lambda\in N_0$ and $u(z)$ is given by (1.2).

Proof. According to the Theorem 2.2, proving the necessity is enough. If $u\in TS_{\lambda}^{k}(\gamma,\xi)$ and z is real then

$$\frac{1-\sum\limits_{n=2}^{\infty}n^{k+1}c(\lambda,n)b_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}n^{k}c(\lambda,n)b_n z^{n-1}}-\gamma\geq\xi\left|\frac{\sum\limits_{n=2}^{\infty}(n-1)n^{k}c(\lambda,n)b_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}n^{k}c(\lambda,n)b_n z^{n-1}}\right|.$$

Getting $z\to 1$ along the real axis, it is obtained that

$$\sum_{n=2}^{\infty}[n(1+\xi)-(\gamma+\xi)]n^{k}c(\lambda,n)b_n<1-\gamma.$$

□

Corollary 2.4. *If $u\in TS_{\lambda}^{k}(\gamma,\xi)$ then*

$$b_n\leq\frac{(1-\gamma)}{[n(1+\xi)-(\gamma+\xi)]n^{k}c(\lambda,n)}\ for\ n\geq 2. \tag{2.7}$$

For the function

$$u_n(z)=\ z-\frac{(1-\gamma)}{[n(1+\xi)-(\gamma+\xi)]n^{k}c(\lambda,n)}z^{n},\ n\geq 2, \tag{2.8}$$

the result is sharp.

Theorem 2.5. *Let $u, v \in TS_\lambda^k(\gamma,\xi)$, where u is given by (1.2) and $v(z) = z - \sum_{n=2}^{\infty} b_n z^n$. Then function h given by*

$$h(z) = (1-\varrho)u(z) + \varrho v(z) = z - \sum_{n=2}^{\infty} L_n z^n \tag{2.9}$$

is in the class $TS_\lambda^k(\gamma,\xi)$. Here $L_n = (1-\varrho)b_n + \varrho b_n,\ 0 \le \varrho < 1$.

Theorem 2.6. *Let $u_1(z) = z$ and*

$$u_n(z) = z - \frac{(1-\gamma)}{[n(1+\xi)-(\gamma+\xi)]n^k c(\lambda,n)} z^n \text{ for } n = 2,3,4,\cdots . \tag{2.10}$$

Then $u \in TS_\lambda^k(\gamma,\xi)$ if and only if $u(z)$ can be given as

$$u(z) = \sum_{n=1}^{\infty} \lambda_n u_n(z) \text{ where } \lambda_n \ge 0 \text{ and} \sum_{n=1}^{\infty} \lambda_n = 1. \tag{2.11}$$

To prove the theorem, we use the analogue methods given by Silverman [22] on extreme points.

The Theorem following theorem will be proved by defining $u_j(z)(j = 1,2,\cdots,k)$, given as

$$u_j(z) = z - \sum_{n=2}^{\infty} b_{n,j} z^n \ \text{ for } b_{n,j} \ge 0,\ z \in E. \tag{2.12}$$

Theorem 2.7. *Let $u_j \in TS_\lambda^k(\gamma_j,\xi)$ $(j = 1,2,\cdots,k)$, where $u_j(z)$ is given by (1.2). Then the function h is given as*

$$h(z) = z - \frac{1}{k}\sum_{n=2}^{\infty}\Big(\sum_{j=1}^{k} b_{n,j}\Big) z^n \tag{2.13}$$

belongs to $TS_\lambda^k(\gamma,\xi)$, where $\gamma = \min_{1\le j\le k} \gamma_j$ and $-1 \le \gamma_j \le 1$.

Proof. Since $u_j(z) \in TS_\lambda^k(\gamma_j,\xi)$ $(j = 1,2,3,\cdots,k)$ if we apply Theorem 2.3 to (2.12), it is observed that

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n(1+\xi)-(\gamma+\xi)]n^k c(\lambda,n)\Big(\frac{1}{k}\sum_{j=1}^{k} b_{n,j}\Big)\\
=&\frac{1}{k}\sum_{j=1}^{k}\Big(\sum_{n=2}^{\infty} n(1+\xi)-(\gamma+\xi)n^k c(\lambda,n)\Big)b_{n,j}\\
\le&\frac{1}{k}\sum_{j=1}^{\infty}(1-\gamma_j)\\
\le&(1-\gamma).
\end{aligned}$$

Taking into account Theorem 2.3 it is obtained that $h \in TS_\lambda^k(\gamma,\xi)$.

□

Theorem 2.8. *Let $u \in TS_\lambda^k(\gamma,\xi)$, where the function u given by (1.2). Then u is close to convex of order δ $(0 \le \delta < 1)$ in $|z| < r_1$ where*

$$r_1 = \inf_{n\ge 2}\left\{\frac{(1-\delta)[n(1+\xi)-(\gamma+\xi)]n^k c(\lambda,n)}{n(1-\gamma)}\right\}^{\frac{1}{(n-1)}}. \tag{2.14}$$

The result is sharp, with the extremal function $u_n(z)$ given by (2.8)

Proof. Firstly, the correctness of the inequality $|u'(z)-1| \leq 1-\delta$ for $|z| < r_1$, where r_1 is defined by (2.14), must be shown. Actually, we have $|u'(z)-1| \leq \sum\limits_{n=2}^{\infty} nb_n|z|^{n-1}$. Thus

$$|u'(z)-1| \leq 1-\delta \; if \; \sum_{n=2}^{\infty}(\frac{n}{1-\delta})b_n|z|^{n-1} \leq 1. \tag{2.15}$$

Using the situation $f \in TS_{\lambda}^{k}(\gamma,\xi)$ if and only if

$$\sum_{n=2}^{\infty}[n(1+\xi)-(\gamma+\xi)]n^k c(\lambda,n)|b_n| \leq 1-\gamma.$$

Then (2.15) holds true if

$$\frac{(\frac{n}{1-\delta})|z|^{n-1} \leq [n(1+\xi)-(\gamma+\xi)]n^k c(\lambda,n)}{1-\gamma}.$$

That is, if

$$r_1 = \inf_{n\geq 2}\left\{\frac{(1-\delta)[n(1+\xi)-(\gamma+\xi)]n^k c(\lambda,n)}{n(1-\gamma)}\right\}^{\frac{1}{n-1}}.$$

Thus the proof is completed. □

Theorem 2.9. *Let $u \in TS_{\lambda}^{k}(\gamma,\xi)$, where the function u is defined by (1.2). Then u is starlike of order δ $(0 \leq \delta < 1)$ in $|z| < r_2$, where*

$$|z| \leq \left\{\frac{(1-\delta)[n(1+\xi)-(\gamma+\xi)]n^k c(\lambda,n)}{(n-\delta)(1-\gamma)}\right\}^{\frac{1}{n-1}}.$$

If the external function u is given by (2.8), then the result is sharp .

Proof. Given $u \in A$ and u is starlike of order δ, we have

$$\left|\frac{zu'(z)}{u(z)}-1\right| \leq 1-\delta. \tag{2.16}$$

For the left hand side of (2.16) we have

$$\left|\frac{zu'(z)}{u(z)}-1\right| \leq \frac{\sum\limits_{n=2}^{\infty}(n-1)b_n|z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}b_n|z|^{n-1}}.$$

The last inequality is less than $1-\delta$ if

$$\sum_{n=2}^{\infty}\frac{n-\delta}{1-\delta}b_n|z|^{n-1} < 1.$$

Using the fact that $f \in TS_{\lambda}^{k}(\gamma,\xi)$ if and only if

$$\sum_{n=2}^{\infty}[n(1+\xi)-(\gamma+\xi)]n^k c(\lambda,n)|b_n| \leq 1-\gamma.$$

(2.16) is true if

$$\frac{n-\delta}{1-\delta}|z|^{n-1} \leq \frac{[n(1+\xi)-(\gamma+\xi)]n^k c(\lambda,n)}{1-\gamma}$$

or, equivalently,

$$|z| \leq \left\{\frac{(1-\delta)[n(1+\xi)-(\gamma+\xi)]n^k c(\lambda,n)}{(n-\delta)(1-\gamma)}\right\}^{\frac{1}{n-1}}$$

which ensures the starlikeness of the family. □

Using the fact that u is convex if and only if zu' is starlike, we get the Corollary 2.10:

Corollary 2.10. *Let $u \in TS_\lambda^k(\gamma,\xi)$, where the function u is defined by (1.2). Then u is convex of order δ $(0 \le \delta < 1)$ in $|z| < r_3$ where*

$$r_3 = \inf_{n\ge 2}\left\{\frac{(1-\delta)[n(1-\xi)-(\gamma+\xi)]n^k c(\lambda,n)}{(n-\delta)(1-\gamma)}\right\}^{\frac{1}{n-1}}.$$

If the external function u is given by (2.8), then the result is sharp.

In the next theorems, partial sums of functions in the class $TS_\lambda^k(\gamma,\xi)$ will be given. Also, it will be obtained that sharp lower bounds for the ratios of real part of $u(z)$ to $u_k(z)$ and $u'(z)$ to $u'_k(z)$ by using the technics earlier works of Silverman [23] and Silvia [24].

Theorem 2.11. *Let $u \in TS_\lambda^k(\gamma,\xi)$ be given by (1.1) and define the partial sums $u_1(z)$ and $u_k(z)$, by*

$$u_1(z) = z \text{ and } u_k(z) = z + \sum_{n=2}^{k} b_n z^n, (k \in N/1) \tag{2.17}$$

and assume that

$$\sum_{n=2}^{\infty} q_n |b_n| \le 1 \tag{2.18}$$

where

$$q_n = \frac{[n(\gamma+\xi)-(\gamma+\xi)]n^k c(\lambda,n)}{(1-\gamma)}.$$

Then $u \in TS_\lambda^k(\gamma,\xi)$. Furthermore

$$\Re\left\{\frac{u(z)}{u_k(z)}\right\} > 1 - \frac{1}{q_{k+1}},\ z \in E, k \in N \tag{2.19}$$

and

$$\Re\left\{\frac{u_k(z)}{u(z)}\right\} > \frac{q_{k+1}}{1+q_{k+1}}. \tag{2.20}$$

Proof. It is easy to confirm that

$$q_n + 1 > q_n > 1, \tag{2.21}$$

for q_n given in (2.18) Thus we get

$$\sum_{n=2}^{k} |b_n| + q_{k+1}\sum_{n=k+1}^{\infty} |b_n| \le \sum_{n=2}^{\infty} q_n |b_n| \le 1. \tag{2.22}$$

By using assumption (2.18) and taking

$$\begin{aligned} v_1(z) &= q_{k+1}\left\{\frac{u(z)}{u_k(z)} - \left(1 - \frac{1}{q_{k+1}}\right)\right\} \\ &= 1 + \frac{q_{k+1}\sum\limits_{n=k+1}^{\infty} b_n z^{n-1}}{1 + \sum\limits_{n=2}^{\infty} b_n z^{n-1}} \end{aligned} \tag{2.23}$$

also applying (2.22), it is obtained that

$$\left|\frac{v_1(z)-1}{v_2(z)+1}\right| \le \frac{q_{k+1}\sum\limits_{n=k+1}^{\infty}|b_n|}{2 - 2\sum\limits_{n=2}^{\infty}|b_n| - q_{k+1}\sum\limits_{n=k+1}^{\infty}|b_n|} \le 1, z \in E \tag{2.24}$$

which easily yields the assertion (2.19) of Theorem 2.11. Due to the fact that prove the function

$$u(z) = z + \frac{z^{k+1}}{q_{k+1}} \tag{2.25}$$

gives sharp result, assume that for $z = re^{\frac{i\pi}{k}}$ that for

$$\frac{u(z)}{u_k(z)} = 1 + \frac{z^k}{q_{k+1}} \to 1 - \frac{1}{q_{k+1}} \; as \; z \to 1^-.$$

Likewise, taking

$$v_2(z) = (1 + q_{k+1})\Big\{\frac{u_k(z)}{u(z)} - \frac{q_{k+1}}{1 + q_{k+1}}\Big\} = 1 - \frac{(1 + q_{k+1}) \sum\limits_{n=k+1}^{\infty} b_n z^{n-1}}{1 + \sum\limits_{n=2}^{\infty} b_n z^{n-1}} \tag{2.26}$$

and by using (2.22), it can be deduced that

$$\Big|\frac{v_2(z) - 1}{v_2(z) + 1}\Big| \leq \frac{(1 + q_{k+1}) \sum\limits_{n=k+1}^{\infty} |b_n|}{2 - 2\sum\limits_{n=2}^{k} |b_n| - (1 - q_{k+1}) \sum\limits_{n=k+1}^{\infty} |b_n|} \tag{2.27}$$

This leads instantly to the assertion (2.20) of Theorem 2.11. The bound in (2.20) is sharp for each $k \in N$ with external function u given by (2.25). Thus proof is completed. □

Theorem 2.12. *If $u \in S^k_\lambda(\gamma, \xi)$ satisfies the condition (2.2) then*

$$\Re\Big\{\frac{u'(z)}{u'_k(z)}\Big\} \geq 1 - \frac{k+1}{q_{k+1}}. \tag{2.28}$$

Proof. Choosing

$$\begin{aligned} v(z) &= q_{k+1}\Big\{\frac{u'(z)}{u'_k(z)}\Big\} - \big(1 - \frac{k+1}{q_{k+1}}\big). \\ &= \frac{1 + \frac{q_{k+1}}{k+1} \sum\limits_{n=k+1}^{\infty} nb_n z^{n-1} + \sum\limits_{n=2}^{\infty} nb_n z^{n-1}}{1 + \sum\limits_{n=2}^{k} nb_n z^{n-1}} \\ &= 1 + \frac{\frac{q_{k+1}}{k+1} \sum\limits_{n=k+1}^{\infty} nb_n z^{n-1}}{1 + \sum\limits_{n=2}^{k} nb_n z^{n-1}} \end{aligned}$$

$$\Big|\frac{v(z) - 1}{v(z) + 1}\Big| \leq \frac{\frac{q_{k+1}}{k+1} \sum\limits_{n=k+1}^{\infty} |b_n|}{2 - 2\sum\limits_{n=2}^{k} n|b_n| - \frac{q_{k+1}}{k+1} \sum\limits_{n=k+1}^{\infty} n|b_n|}. \tag{2.29}$$

Now

$$\Big|\frac{v(z) - 1}{v(z) + 1}\Big| \leq 1.$$

$$\text{If } \sum_{n=2}^{k} n|b_n| + \frac{q_{k+1}}{k+1} \sum_{n=k+1}^{\infty} n|b_n| \leq 1. \tag{2.30}$$

Due to the fact that the left hand side of (2.30) is bounded above by $\sum_{t=2}^{k} q_t|a_t|$ if

$$\sum_{t=2}^{k}(q_n - n)|b_n| + \sum_{n=k+1}^{\infty} q_n - \frac{q_{k+1}}{k+1}n|b_n| \geq 0, \tag{2.31}$$

and the proof is completed.

For the extremal function $u(z) = z + \frac{z^{k+1}}{q_{k+1}}$, the result is sharp. □

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References

[1] A. Akgül, Some families of meromorphic functions with positive coefficients given by an integral operator, *Acta Universitatis Apulensis,* **50** , 81-95 ,(2017).

[2] A. Akgül, A new subclass of meromorphic functions with positive and second coefficients defined by the Rafid-Operator,*Commun. Fac. Sci. Univ.Ank.Series A1,* **66 (2)**, 1-13 , (2017).

[3] A. Akgül, A subclass of meromorphic functions defined by a certain integral operator on Hilbert space, *Creat. Math. Inform.,* **26 (2)**, 115 - 125 , (2017).

[4] A. Akgül, A new subclass of meromorphic functions defined by Hilbert space operator, *Honam Mathematical J.,* **38(3)**, 495-506 , (2016).

[5] A. Akgül, A certain subclass of meromorphic functions with positive coefficients associated with an integral operator, *Honam Mathematical J. ,* **39 (3)**, 331-347, (2017).

[6] N. Alessa, B. Venkateswarlu, P. Thirupathi Reddy, K. Loganathan and K. Tamilvanan, A New subclass of analytic functions related to Mittag-Leffler type Poisson distribution series, *Journal of Function Spaces ,* **2021**, Article ID 6618163, 7 pages, (2021).

[7] M. K. Aouf, p-valent regular functions with negative coefficients of order α, *Bull. Inst. Math. Acad. Sinica,* **17(3)** , 255– 267, (1988).

[8] M. K. Aouf, Certain multivalent functions with negative coefficients defined by using a differential operator, *J. Math. Appl.,* **30**, 5 – 21, (2008).

[9] M. K. Aouf, Generalization of certain subclasses of multivalent functions with negative coefficients defined by using a differential operator, *Math. Comput. Modelling,* **50** , 1367 – 1378, (2009).

[10] K. Al-Shaqsi, Darus M. An operator defined by convolution involving the polylogarithms functions, *J. Math. Stat. ,* **4(1)**, 46-50, (2008).

[11] S. Altınkaya and S. Yalçin, Poisson Distribution Series for Certain Subclasses of Starlike Functions with Negative Coefficients, *Annals of Oradea Univ. Math. Fascicola,* **24(2)**, 5-8, 2017 .

[12] A. W. Goodman, On uniformly convex functions, *Annales Polonici Math.,* **56(1)**, 87-92, (1991).

[13] A. W. Goodman, On uniformly starlike functions. Journal of *Math. Anal. and Appl.,* **155 (2)**, 364-370, (1991).

[14] A. O. Mostafa and M.K. Aouf, Neighborhoods of certain p-valent analytic functions with complex order,*Comput. Math. Appl.,* **58**, 1183–1189, (2009).

[15] S. Ponnusamy and S. Sabapathy , Polylogarithms in the theory of univalent functions, *Results in Math.,* **30 (1-2)**, 136-150, (1996).

[16] S. Ponnusamy, Inclusion theorems for convolution product of second order polylogarithms and functions with the derivative in a halfplane, *Rocky Mountain J. Math.,* **28(1)**, 695 – 733, (1998).

[17] F. Ronning, Uniformly convex functions and a corresponding class of starlike functions, *Proc. of the Amer. Math. Soc.,* **118(1)**, 189-196 , (1993).

[18] F. Ronning, Integral representations of bounded starlike functions, *Annales Polonici Math. ,* **60(3)**, 289 – 297, (1995).

[19] S. Ruscheweyh, New criteria for univalent functions, *Proc. of the Amer. Math. Soc.,* **49** (1975), 109-115.

[20] F. M. Sakar and M.Aydoğan, Subclass of m-quasiconformal harmonic functions in association with Janowski starlike functions, *Applied Math. and Computation ,* **319**, 461-468 , (2018).

[21] G. S. Salagean, *Subclasses of univalent functions* , Complex Analysis–Fifth Romanian-Finnish Seminar. Springer, Berlin, Heidelberg, (1983).

[22] H. Silverman, Univalent functions with negative coefficients, *Proc. of the Amer. Math. Soc.,* **51(1)**, 109–116 , (1975).

[23] H. Silverman, Partial sums of starlike and convex functions, *J. of Math. Anal. and Appl.,* **209 (1)**, 221 – 227 , (1997).

[24] E. M. Silvia, On partial sums of convex functions of order, *Houston J. Math. ,* **11 (3)**, 397 – 404 , (1985) .

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