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# Passage of property $(t)$ from two operators to their tensor product

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**Abstract** An operator $T$ acting on a Banach space $\mathcal{X}$ obeys property $(t)$ if the isolated points of the spectrum $\sigma(T)$ of $T$ which are eigenvalues of finite multiplicity are exactly those points $\lambda$ of the spectrum for which $T-\lambda$ is an upper semi-Fredholm with index less than or equal to 0. In the present paper we examine the stability of property $(t)$ under perturbations. We show that if $T$ is an isoloid operator on a Banach space, that obeys property $(t)$, and $F$ is a bounded operator that commutes with $T$ and for which there exists a positive integer $n$ such that $F^n$ is finite rank, then $T+F$ obeys property $(t)$. Further, we establish that if $T$ is finite-isoloid, then property $(t)$ is transmitted from $T$ to $T+R$, for every Riesz operator $R$ commuting with $T$. Property $(t)$ does not transfer from operators $T$ and $S$ to their tensor product $T\otimes S$; we give necessary and/or sufficient conditions ensuring the passage of property $(t)$ from $T$ and $S$ to $T\otimes S$. Moreover, Perturbations by Riesz operators are considered.

## 1 Introduction

Throughout this paper, $\mathcal{X}$ denotes an infinite-dimensional complex Banach space, $\mathcal{B}(\mathcal{X})$ the algebra of all bounded linear operators on $\mathcal{X}$. For $T\in\mathcal{B}(\mathcal{X})$, let $T^*$, $\ker(T)$, $\Re(T)$, $\sigma(T)$, $\sigma_a(T)$ and $\sigma_s(T)$ denote the *adjoint*, the *null space*, the *range*, the *spectrum*, the *approximate point spectrum* and the *surjectivity spectrum* of $T$ respectively. Let $\alpha(T)$ and $\beta(T)$ be the *nullity* and the *deficiency* of $T$ defined by $\alpha(T)=\dim\ker(T)$ and $\beta(T)=\text{co}-\dim\Re(\text{T})$. Let $SF_+(\mathcal{X})=\{T\in\mathcal{B}(\mathcal{X}):\alpha(T)<\infty \text{ and } \Re(T) \text{ is closed}\}$ and $SF_-(\mathcal{X})=\{T\in\mathcal{B}(\mathcal{X}):\beta(T)<\infty\}$ denote the semigroup of upper semi-Fredholm and lower semi-Fredholm operators on $\mathcal{X}$ respectively. An operator $T\in\mathcal{B}(\mathcal{X})$ is said to be *semi-Fredholm* if $T$ is either upper semi-Fredholm or lower semi-Fredholm. If both $\alpha(T)$ and $\beta(T)$ are finite, then $T$ is called *Fredholm operator*. If $T$ is semi-Fredholm operator then *index* of $T$ is defined by ind $(T)=\alpha(T)-\beta(T)$.

A bounded linear operator $T$ acting on a Banach space $\mathcal{X}$ is *Weyl* if it is Fredholm of index zero and *Browder* if $T$ is Fredholm of finite ascent and descent. Let $\mathbb{C}$ denote the set of complex numbers and let $\sigma(T)$ denote the spectrum of $T$ . The *Weyl spectrum* $\sigma_w(T)$ and *Browder spectrum* $\sigma_b(T)$ of $T$ are defined by $\sigma_w(T)=\{\lambda\in\mathbb{C}: T-\lambda \text{ is not Weyl}\}$ and $\sigma_b(T)=\{\lambda\in\mathbb{C}: T-\lambda \text{ is not Browder }\}$ respectively. For $T\in\mathcal{B}(\mathcal{X})$, $SF_+^-(\mathcal{X})=\{T\in SF_+(\mathcal{X}):\text{ind(T)}\leq 0\}$. Then the *upper Weyl spectrum* of $T$ is defined by $\sigma_{SF_+^-}(T)=\{\lambda\in\mathbb{C}: T-\lambda\notin SF_+^-(\mathcal{X})\}$. Let $\Delta(T)=\sigma(T)\setminus\sigma_w(T)$ and $\Delta_a(T)=\sigma_a(T)\setminus\sigma_{SF_+^-}(T)$. Following Coburn [9], we say that *Weyl's theorem* holds for $T\in\mathcal{B}(\mathcal{X})$ if $\Delta(T)=E^0(T)$, where $E^0(T)=\{\lambda\in \text{iso}\,\sigma(\text{T}): 0<\alpha(\text{T}-\lambda)<\infty\}$. Here and elsewhere in this paper, for $K\subset\mathbb{C}$, iso$K$ is the set of isolated points of $K$.

According to Rakočević [24], an operator $T\in\mathcal{B}(\mathcal{X})$ is said to satisfy *a-Weyl's theorem* if $\sigma_a(T)\setminus\sigma_{SF_+^-}(T)=E_a^0(T)$, where

$$E_a^0(T)=\{\lambda\in\text{iso}\,\sigma_\text{a}(\text{T}): 0<\alpha(\text{T}-\lambda)<\infty\}.$$

It is known from [24] that an operator satisfying $a$-Weyl's theorem satisfies Weyl's theorem, but the converse does not hold in general.

The property $(t)$, which has been recently introduced in [30], is related to the classical Weyl's theorem for bounded linear operators on Banach spaces, in particular this property is related to a strong variant of Weyl's theorem, the so-called property $(w)$ introduced by Rakočević in [23] and studied extensively in [5, 6, 26, 28]. In this paper we study the stability of property $(t)$ under perturbations by finite rank operators, by nilpotent operators and, more generally, by Riesz operators commuting with $T$. Moreover, we give necessary and/or sufficient conditions ensuring the passage of property $(t)$ from $T$ and $S$ to $T \otimes S$.

## 2 Property $(t)$ for bounded linear operator

**Definition 2.1.** ([30]) Let $T \in \mathcal{B}(\mathcal{X})$. We say that $T$ obeys property $(t)$ if $\Delta_+(T) = \sigma(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$.

**Remark 2.2.** If $T \in \mathcal{B}(\mathcal{X})$ has the SVEP, then it is known from [19, Page 35] that $\sigma(T) = \sigma_s(T)$. Moreover, it is known that from [6, Theorem 2.6] that if $T^*$ has the SVEP, then $\sigma(T) = \sigma_a(T)$ and $\sigma_{SF_+^-}(T) = \sigma_w(T)$ and hence $E_a^0(T) = E^0(T)$, $\Delta_a(T) = \Delta(T)$ and $\Delta_+(T) = \Delta(T)$.

**Proposition 2.3.** *[2] Let $T \in \mathcal{B}(\mathcal{X})$. Then $T$ satisfies Weyl's theorem if and only if $T$ satisfies Browder's theorem and $\pi^0(T) = E^0(T)$.*

**Proposition 2.4.** *Let $T \in \mathcal{B}(\mathcal{X})$. Then $T$ obeys property $(t)$ if and only if the following conditions hold:*

*(i) $T$ satisfies $a$-Browder's theorem;*

*(ii) $\sigma(T) = \sigma_a(T)$;*

*(iii) $\pi_a^0(T) = E^0(T)$.*

*Proof.* The proof follows immediately from Theorem 2.6, Proposition 2.7 and Theorem 2.10 of [30]. ■

The following result is a consequence of Proposition 2.3 and [30, Theorem 2.6, Theorem 2.10].

**Proposition 2.5.** *Let $T \in \mathcal{B}(\mathcal{X})$. Then $T$ obeys property $(t)$ if and only if the following conditions hold:*

*(i) $T$ satisfies Browder's theorem;*

*(ii) $\sigma_w(T) = \sigma_{aw}(T)$;*

*(iii) $\pi^0(T) = E^0(T)$.*

Let $H_{nc}(\sigma(T))$ denote the set of all analytic functions, defined on an open neighborhood of $\sigma(T)$, such that $f$ is non-constant on each of the components of its domain. Define, by the classical calculus, $f(T)$ for every $f \in H_{nc}(\sigma(T))$.

A bounded operator $T \in \mathcal{B}(\mathcal{X})$ is said to be *polaroid* (respectively, *$a$-polaroid*) if $\sigma^{\text{iso}}(T) = \emptyset$ or every isolated point of $\sigma(T)$ is a pole of the resolvent of $T$ (respectively, if iso$\sigma_a(T) = \emptyset$ or every isolated point of $\sigma_a(T)$ is a pole of the resolvent of $T$ ).

**Theorem 2.6.** *Let $T$ be a bounded linear operator on $\mathcal{X}$ satisfying the SVEP. If $T - \lambda I$ has finite descent at every $\lambda \in E_a^0(T)$, then property $(t)$ holds for $f(T^*)$, for every $f \in H_{nc}(\sigma(T))$.*

*Proof.* Let $\lambda \in E_a^0(T)$, then $\lambda$ is an isolated of $\sigma_a(T)$ and hence $a(T-\lambda) = d(T-\lambda) < \infty$. Moreover, $\alpha(T-\lambda) < \infty$, so by [1, Theorem 3.4] it follows that $\beta(T-\lambda)$ is also finite, thus $\lambda \in \pi^0(T)$. This shows that $E_a^0(T) \subseteq \pi^0(T)$. Since the other inclusion is always verified, we have $E_a^0(T) = \pi^0(T)$ and hence $T$ is $a$-polaroid. Therefore, property $(t)$ holds for $T$ by [30, Theorem 3.5]. ■

The class of operators $T \in \mathcal{B}(\mathcal{X})$ for which $K(T) = \{0\}$ was introduced and studied by M. Mbekhta in [20]. It was shown that for such operators, the spectrum is connected and the SVEP holds.

**Theorem 2.7.** *Let $T \in \mathcal{B}(\mathcal{X})$. If there exists $\lambda$ such that $K(T-\lambda) = \{0\}$, then $f(T) \in ga\mathfrak{B}$, for every $f \in H_{nc}(\sigma(T))$. Moreover, if in addition* $\ker(T-\lambda) = 0$, *then property $(t)$ holds for $f(T)$*

*Proof.* Since $T$ has the SVEP, then by [18, Theorem 1.5] , generalized a-Browder's theorem holds for $f(T)$ and hence $a$-Browder's theorem holds for $f(T)$ for every $f \in H_{nc}(\sigma(T))$. Let $\gamma \in \sigma(f(T))$, then

$$f(z) - \gamma I = P(z)g(z),$$

where $g$ is complex-valued analytic function on a neighborhood of $\sigma(T)$ without any zeros in $\sigma(T)$ while $P$ is a complex polynomial of the form $P(z) = \prod_{j=1}^{n}(z - \lambda_j I)^{k_j}$ with distinct roots $\lambda_1, \cdots, \lambda_n \in \sigma(T)$. Since $g(T)$ is invertible, then we deduce that

$$\ker(f(T) - \gamma I) = \ker(P(T)) = \bigoplus_{j=1}^{n} \ker(T - \lambda_j I)^{k_j}.$$

On the other hand, it follows from [20, Proposition 2.1] that $\sigma_p(T) \subseteq \{\lambda\}$ . If we assume that $\ker(T - \lambda I) = 0$, then $T - \lambda I$ is an injective and consequently $\sigma_p(T) = \emptyset$. Hence $\ker(f(T) - \lambda I) = 0$. Therefore, $\sigma_p(f(T)) = \emptyset$. Now, we prove that

$$\pi_a^0(f(T)) = E^0(f(T)).$$

Obviously, the condition $\sigma_p(f(T)) = \emptyset$ entails that

$$E^0(f(T)) = E_a^0(f(T)) = \emptyset.$$

On the other hand, the inclusion $\pi_a^0(f(T)) \subseteq E_a^0(f(T))$ holds for every operator $T \in \mathcal{B}(\mathcal{X})$. So also $\pi_a^0(f(T)) = \emptyset$. Hence property $(w)$ and $a$-Weyl's theorem hold for $f(T)$ and so $\sigma_{SF_+^-}(f(T)) = \sigma_w(f(T)) = \sigma(T) = \sigma_a(T)$. It then follows by [30, Theorem 2.10] that $f(T)$ obeys property $(t)$. ■

In [21] Oudghiri introduced the class $H(p)$ of operators on Banach spaces for which there exists $p := p(\lambda) \in \mathbb{N}$ such that

$$H_0(\lambda I - T) = \ker(T - \lambda I)^p \qquad \text{for all } \lambda \in \mathbb{C}.$$

Let $P(\mathcal{X})$ be the class of all operators $T \in \mathcal{B}(\mathcal{X})$ having the property $H(p)$. The class $P(\mathcal{X})$ contains the classes of subscalar, algebraically $wF(p,q,r)$ operators with $p, r > 0$ and $q \geq 1$ [29], algebraically $w$-hyponormal operators [27], algebraically quasi-class $(A,k)$ [26]. It is known that if $H_0(T - \lambda I)$ is closed for every complex number $\lambda$, then T has the SVEP ( see [1, 17]). So that, the SVEP is shared by all the operators of $P(\mathcal{X})$. Moreover, $T$ is polaroid, see [3, Lemma 3.3].

**Theorem 2.8.** *Suppose that $T \in \mathcal{B}(\mathcal{X})$ is generalized scalar. Then $T$ satisfies property $(t)$ if and only if $T$ satisfies Weyl's theorem*

*Proof.* If $T$ is generalized scalar then both $T$ and $T^*$ has SVEP. Moreover, $T$ is polaroid since every generalized scalar has the property $H(p)$. Then $T$ obeys property $(t)$ by [30, Theorem 3.4]. The equivalence then follows from [30, Theorem 2.10]. ■

**Example 2.9.** Property $(t)$, as well as Weyl's theorem, is not transmitted from $T$ to its dual $T^*$. To see this, consider the weighted right shift $T \in \mathcal{B}(\ell^2(\mathbb{N}))$, defined by

$$T(x_1, x_2, \cdots) := (0, \frac{x_1}{2}, \frac{x_2}{3}, \cdots) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N}).$$

Then

$$T^*(x_1, x_2, \cdots) := (\frac{x_2}{2}, \frac{x_3}{3}, \cdots) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N}).$$

Both $T$ and $T^*$ are quasi-nilpotent, and hence are decomposable, $T$ satisfies Weyl's theorem since $\sigma(T) = \sigma_w(T) = \{0\}$ and $E^0(T) = \pi^0(T) = \emptyset$ and hence $T$ has property $(t)$. On the other hand, we have $\sigma(T^*) = \sigma_a(T^*) = \sigma_{SF_+^-}(T^*) = E_a(T^*) = \sigma_w(T^*) = E^0(T^*) = \{0\}$ and $\pi_a^0(T^*) = \emptyset$, so $T^*$ does not satisfy Weyl's theorem (and nor $a$-Weyl's theorem). Since $T^*$ has SVEP, then $T^*$ does not satisfy property $(t)$.

## 3 Property $(t)$ under perturbations

Recall that $T \in \mathcal{B}(\mathcal{X})$ is said to be a *Riesz operator* if $T - \lambda \in \mathfrak{F}(\mathcal{X})$ for all $\lambda \in \mathbb{C} \setminus \{0\}$. Evidently, quasi-nilpotent operators and compact operators are Riesz operators. The proof of the following result may be found in Rakočević [25]:

**Lemma 3.1.** *Let $T \in \mathcal{L}(\mathcal{X})$ and $R$ be a Riesz operator commuting with $T$. Then*

*(i)* $T \in \mathfrak{B}_+(\mathcal{X}) \Leftrightarrow T + R \in \mathfrak{B}_+(\mathcal{X})$.

*(ii)* $T \in \mathfrak{B}_-(\mathcal{X}) \Leftrightarrow T + R \in \mathfrak{B}_-(\mathcal{X})$.

*(iii)* $T \in \mathfrak{B}(\mathcal{X}) \Leftrightarrow T + R \in \mathfrak{B}(\mathcal{X})$.

It is known that if $K \in \mathcal{B}(\mathcal{X})$ is a finite-rank operator commuting with $T$, then

$$\lambda \in \operatorname{acc} \sigma_a(T) \Leftrightarrow \lambda \in \operatorname{acc} \sigma_a(T + K), \tag{3.1}$$

for a proof see Theorem 3.2 of [10].

The classes $\mathcal{W}_+(\mathcal{X}), \mathcal{W}_-(\mathcal{X})$ and $\mathcal{W}(\mathcal{X})$ are stable under some perturbations. The proof of following result may be found in [4].

**Lemma 3.2.** *Let $T, K \in \mathcal{B}(\mathcal{X})$ be such that $K$ is a compact operator. Then*

*(i)* $T \in \mathcal{W}_+(\mathcal{X}) \Leftrightarrow T + K \in \mathcal{W}_+(\mathcal{X})$.

*(ii)* $T \in \mathcal{W}_-(\mathcal{X}) \Leftrightarrow T + K \in \mathcal{W}_-(\mathcal{X})$.

*(iii)* $T \in \mathcal{W}(\mathcal{X}) \Leftrightarrow T + K \in \mathcal{W}(\mathcal{X})$.

Define

$$E^{0f} := \{\lambda \in \sigma^{\text{iso}}(T) : \alpha(T - \lambda) < \infty\}.$$

Evidently, $E^0(T) \subseteq E^{of}$ for every operator $T \in \mathcal{B}(\mathcal{X})$.

**Lemma 3.3.** *Let $T \in \mathcal{B}(\mathcal{X})$. If $R$ is a Riesz operator that commutes with $T$, then*

$$E^0(T + R) \cap \sigma(T) \subseteq \sigma^{iso}(T). \tag{3.2}$$

*Proof.* By [22, Lemma 2.3] we have

$$E^0(T + R) \cap \sigma(T) \subseteq E^{0f}(T + R) \cap \sigma(T) \subseteq \sigma^{\text{iso}}(T).$$

■

**Lemma 3.4.** *Suppose that $T \in \mathcal{B}(\mathcal{X})$ obeys property $(t)$ and $R$ is a Riesz operator commuting with $T$ such that $\sigma_a(T) = \sigma_a(T + R)$. Then $\pi_a^0(T + R) \subseteq E^0(T + R)$.*

*Proof.* Let $\lambda \in \pi_a^0(T + R)$ be arbitrary given. Then $\lambda \in \sigma_a^{\text{iso}}(T + R)$ and $T + R - \lambda \in \mathfrak{B}_+(\mathcal{X})$, so $\alpha(T + R - \lambda) < \infty$. Since $T + R - \lambda$ has closed range, the condition $\lambda \in \sigma_a(T + R)$ entails that $\alpha(T + R - \lambda) > 0$. Therefore, in order to show that $\lambda \in E^0(T + R)$, we need only to prove that $\lambda$ is an isolated point of $\sigma(T + R)$.
We know that $\lambda \in \sigma_a^{\text{iso}}(T)$. We have from Lemma 3.1 that $(T + R) - \lambda - R = T - \lambda \in \mathfrak{B}_+(\mathcal{X})$ so that $\lambda \in \sigma_a(T) \setminus \sigma_{ub}(T) = \pi_a^0(T)$.
Now, by assumption $T$ obeys property $(t)$ so, by [30, Proposition 2.7], $\pi_a^0(T) = E^0(T)$. Moreover, $T$ satisfies Weyl's theorem and hence

$$E^0(T) = \pi^0(T) = \sigma(T) \setminus \sigma_b(T).$$

Therefore, $T - \lambda$ is Browder and hence $T + F - \lambda$ is Browder, so

$$0 < a(T + R - \lambda) = d(T + R - \lambda) < \infty$$

and hence $\lambda$ is a pole of the resolvent of $T + R$. Consequently, $\lambda$ is an isolated point of $\sigma(T + R)$. ■

**Lemma 3.5.** *Suppose that $T \in \mathcal{B}(\mathcal{X})$ obeys property $(t)$. If $R$ is a Riesz operator commuting with $T$ and $\sigma_a(T) = \sigma_a(T+R)$, then $E^0(T) \subseteq E^0(T+R)$.*

*Proof.* Suppose that $T$ obeys property $(t)$. Hence we conclude from [30] that

$$E^0(T) = \sigma(T) \setminus \sigma_{SF_+^-}(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \sigma_a(T+R) \setminus \sigma_{ub}(T+R) = \pi_a^0(T+R). \quad (3.3)$$

Let $\lambda \in E^0(T)$ be arbitrary given. Set $W := T+R$ then $W$ commutes with $R$. By [7, Lemma 2.3] we have

$$\begin{aligned}\lambda \in E^0(T) \cap \sigma_a(T+R) &= E^0(W-R) \cap \sigma_a(T)\\ &\subseteq \sigma^{\text{iso}}(W) = \sigma^{\text{iso}}(T+R).\end{aligned}$$

Moreover, we have from (3.3) that $T+R-\lambda \in \mathfrak{B}_+(\mathcal{X})$ and so has closed range. Since $\lambda \in \sigma_a(T+R)$ it follows that $\lambda$ is an eigenvalue and hence $0 < \alpha(T+R-\lambda) < \infty$. That is, $\lambda \in E^0(T+R)$. ■

Recall that $T \in \mathcal{B}(\mathcal{X})$ is said to be isoloid if $\sigma^{\text{iso}}(T) \subseteq \sigma_p(T)$. As a consequence of [30, Theorem 2.4] and [4, Lemma 2.4] we have

**Corollary 3.6.** *Suppose that $T \in \mathcal{B}(\mathcal{X})$ obeys property $(t)$ and $F$ is a finite rank operator commuting with $T$ such that $\sigma_a(T) = \sigma_a(T+F)$. Then $\pi_a^0(T+F) \subseteq E^0(T+F)$.*

We first recall two well-known results: if $R$ is a Riesz operator commuting with $T \in \mathcal{B}(\mathcal{X})$, then

$$\sigma_{SF_+^-}(T+R) = \sigma_{SF_+^-}(T) \text{ and } \sigma_{ub}(T+R) = \sigma_{ub}(T). \quad (3.4)$$

Since $\sigma(T+R) = \sigma(T)$ and $\sigma_b(T) = \sigma_b(T+R)$, we then have $\pi^0(T) = \pi^0(T+R)$ and $\pi_a^0(T) = \pi_a^0(T+R)$.

**Theorem 3.7.** *Suppose that $T \in \mathcal{B}(\mathcal{X})$ is an isoloid operator for which property $(t)$ holds and $F$ be a bounded operator commuting with $T$ such that $F^n$ is a finite rank operator for some $n \in \mathbb{N}$. Then*

*(i) $E^0(T) = E^0(T+F)$.*

*(ii) $T+F$ has property $(t)$.*

*Proof.* (i) Observe first that $F$ is a Riesz operator, so, it follows from Lemma 3.5 that $E^0(T) \subseteq E^0(T+F)$. Hence it suffices to show that $E^0(T+F) \subseteq E^0(T)$. Let $\lambda \in E^0(T+F)$. Then $\lambda \in \sigma^{\text{iso}}(T+F)$. Since $\alpha(T+F-\lambda) > 0$ and $\sigma(T) = \sigma(T+F)$. Therefore, by Lemma 3.3, $\lambda \in E^0(T+F) \cap \sigma(T) \subseteq \sigma^{\text{iso}}(T)$. Since $T$ is isoloid then $\alpha(T-\lambda) > 0$. We show now $\alpha(T-\lambda) < \infty$. Let $Z = (T+F-\lambda)^n|_{\ker(T-\lambda)}$. Clearly, if $x \in \ker(T-\lambda)$, then $Zx = (-1)^n F^n x$ thus $Z$ is a finite rank operator. Moreover, since $\lambda \in E^0(T+F)$ we have $\alpha(T+F-\lambda) < \infty$ and hence $\alpha(Z) \leq \alpha(T+F-\lambda)^n < \infty$. Then it follows that $\ker(T-\lambda)$ is finite dimensional. Therefore, $\lambda \in E^0(T)$.
(ii) As $T$ obeys property $(t)$ and $F$ is a Riesz operator, we have

$$E^0(T+F) = E^0(T) = \sigma(T) \setminus \sigma_{SF_+^-}(T) = \sigma(T+F) \setminus sfpm(T+F),$$

hence $T+F$ obeys property $(t)$. ■

As an immediate consequence we have:

**Corollary 3.8.** *Let $T \in \mathcal{B}(\mathcal{X})$ be an isoloid operator. If property $(t)$ holds for $T$ then property $(t)$ holds also for $T+F$, for every finite rank operator $F$ commuting with $T$.*

**Theorem 3.9.** *Suppose that $T \in \mathcal{B}(\mathcal{X})$ obeys property $(t)$ and $\sigma^{iso}(T) = \emptyset$. If $F$ is a finite rank operator commuting with $T$, then $T+F$ obeys property $(t)$.*

*Proof.* The condition $\sigma^{\text{iso}}(T) = \emptyset$ entails that $T$ is an isoloid. Hence the result follows by Corollary 3.8. ■

we shall consider nilpotent perturbations of operators satisfying property $(t)$. It easy to check that if $N$ is a nilpotent operator commuting with $T$ , then

$$\sigma(T)=\sigma(T+N)\ \sigma_a(T)=\sigma_a(T+N) \text{ and } \sigma_{SF_+^-}(T)=\sigma_{SF_+^-}(T+N). \tag{3.5}$$

Hence it follows from Equation ( 3.5)

$$E^0(T)=E^0(T+N), \text{ and } E_a^0(T)=E_a^0(T+N). \tag{3.6}$$

**Theorem 3.10.** *Suppose that $T\in\mathcal{B}(\mathcal{X})$ and let $N\in\mathcal{B}(\mathcal{X})$ be a nilpotent operator which commutes with $T$. Then $T$ obeys property $(t)$ if and only if $T+N$ obeys property $(t)$.*

*Proof.* Suppose that $T$ obeys property $(t)$. Then

$$\begin{aligned} E^0(T+N)=E^0(T)&=\sigma(T)\setminus\sigma_{SF_+^-}(T)\\ &=\sigma(T+N)\setminus\sigma_{SF_+^-}(T+N), \end{aligned}$$

hence $T+N$ obeys property $(t)$. The converse follows by symmetry. ■

**Example 3.11.** In general property $(t)$ is not transmitted from an operator to a commuting quasinilpotent perturbation as the following example shows.
If we consider on the Hilbert space $\ell^2(\mathbb{N})$ the operators $T=0$ and $Q$ defined by

$$Q(x_1,x_2,\cdots)=\left(\frac{x_2}{2},\frac{x_3}{3},\cdots\right) \quad \text{for all } x_n\in\ell^2(\mathbb{N}).$$

Then $Q$ is quasinilpotent operator commuting with $T$. Moreover, we have $\sigma(T)=\{0\},\sigma_{SF_+^-}(T)=\emptyset$, $E(T)=\{0\}$. Hence $T$ obeys property $(t)$. But property $(t)$ fails for $T+Q=Q$. Indeed, $\sigma_{SF_+^-}(T+Q)=\{0\},E^0(T+Q)=E^0(T)=\{0\}$ and $\sigma(T+Q)=\{0\}$.

A bounded linear operator $T\in\mathcal{B}(\mathcal{X})$ is said to be *finite-isoloid* if every isolated point of $\sigma(T)$ is an eigenvalue of $T$ having finite multiplicity.

**Theorem 3.12.** *Suppose that $T\in\mathcal{B}(\mathcal{X})$ is a finite-isoloid operator which obeys property $(t)$. If $R$ is a Riesz operator which commutes with $T$, then $E^0(T)=E^0(T+R)$.*

*Proof.* Suppose that $T$ obeys property $(t)$. Then it follows from [30, Theorem 2.10] that $T$ satisfies Weyl's theorem and $\sigma_w(T)=\sigma_{SF_+^-}(T)$. Since $R$ is a Riesz operator commuting with $T$ then by [22, Theorem 2.7] that $T+R$ satisfies Weyl's theorem. Hence

$$\begin{aligned} E^0(T+R)=\sigma(T+R)\setminus\sigma_w(T+R)&=\sigma(T)\setminus\sigma_w(T)\\ &=\sigma(T)\setminus\sigma_{SF_+^-}(T)=E^0(T). \end{aligned}$$

■

**Corollary 3.13.** *Suppose that $T\in\mathcal{B}(\mathcal{X})$ is a finite-isoloid operator which obeys property $(t)$. If $R$ is a Riesz operator which commutes with $T$, then $T+R$ obeys property $(t)$.*

*Proof.* As $T$ obeys property $(t)$, we have $E^0(T)=\sigma(T)\setminus\sigma_{SF_+^-}(T)$. As we known that the equalities $\sigma(T)=\sigma(T+R)$ and $\sigma_{SF_+^-}(T)=\sigma_{SF_+^-}(T+R)$ hold for every Riesz operator commuting with $T$. So, it follows from Theorem 3.12 that

$$E^0(T+R)=E^0(T)=\sigma(T)\setminus\sigma_{SF_+^-}(T)=\sigma(T+R)\setminus\sigma_{SF_+^-}(T+R).$$

That is, $T+R$ obeys property $(t)$. ■

**Corollary 3.14.** *Suppose that $T\in\mathcal{B}(\mathcal{X})$ is a finite-isoloid operator which obeys property $(t)$.*

*(i) If $Q$ is a quasi-nilpotent which commutes with $T$, then $T+Q$ obeys property $(t)$.*

*(ii) If $K$ is a compact operator which commutes with $T$ and $\sigma_a(T) = \sigma_a(T + K)$, then $T + K$ obeys property $(t)$.*

*Proof.* (i) This follows immediately from the fact that $\sigma_a(T) = \sigma_a(T + Q)$ and Theorem 3.12.
(ii) It is clear since every compact operator is a Riesz operator. ■

**Theorem 3.15.** *Let $T$ be an operator on $\mathcal{X}$ that obeys property $(t)$ and such that $\sigma_p(T) \cap \sigma^{iso}(T) \subseteq E^0(T)$. If $Q$ is a quasi-nilpotent operator that commutes with $T$, then $T + Q$ obeys property $(t)$*

*Proof.* As $T$ obeys property $(t)$, we have by [30, Theorem 2.10] that $T$ satisfies Weyl's theorem and $\sigma_w(T) = \sigma_{SF_+^-}(T)$. Hence by [22, Proposition 2.9], we have $T+Q$ satisfies Weyl's theorem. Since $\sigma_{SF_+^-}(T+Q) = \sigma_{SF_+^-}(T)$ and $\sigma_w(T) = \sigma_w(T+Q)$ we have $\sigma_{SF_+^-}(T+Q) = \sigma_w(T+Q)$ and so $T + Q$ obeys property $(t)$. ■

**Definition 3.16.** A bounded linear operator $T$ is said to be *algebraic* if there exists a non-trivial polynomial $h$ such that $h(T) = 0$.

From the spectral mapping theorem it easily follows that the spectrum of an algebraic operator is a finite set. A nilpotent operator is a trivial example of an algebraic operator. Also finite rank operators $K$ are algebraic; more generally, if $K^n$ is a finite rank operator for some $n \in \mathbb{N}$ then $K$ is algebraic. Clearly, if $T$ is algebraic then its dual $T^*$ is algebraic, as well as $T'$ in the case of Hilbert space operators.

**Theorem 3.17.** *Suppose that $T \in \mathcal{B}(\mathcal{X})$ and $K \in \mathcal{B}(\mathcal{X})$ is an algebraic operator which commutes with $T$.*

*(i) If $T^*$ is hereditarily polaroid and has SVEP, then $T + K$ obeys property $(t)$.*

*(ii) If $T$ is hereditarily polaroid and has SVEP, then $T^* + K^*$ obeys property $(t)$.*

*Proof.* (i) Obviously, $K^*$ is algebraic and commutes with $T^*$. Moreover, by [7, Theorem 2.15], we have $T^* + K^*$ is polaroid, or equivalently, $T + K$ is polaroid. Since $T^*$ has SVEP then by [6, Theorem 2.14], we have $T^* + K^*$ has SVEP . Therefore, $T + K$ obeys property $(t)$ by [30, Theorem 3.4 (i)].
(ii) It follows from the proof of Theorem 2.15 of [7] that $T + K$ is polaroid and hence by duality $T^* + K^*$ is polaroid. Since $T$ has SVEP then it follows from [6, Theorem 2.14] that $T + K$ has SVEP. Therefore, $T^* + K^*$ obeys property $(t)$ by [30, Theorem 3.4 (ii)]. ■

**Theorem 3.18.** *Suppose that $T \in \mathcal{B}(\mathcal{X})$ and $K \in \mathcal{B}(\mathcal{X})$ is an algebraic operator which commutes with $T$.*

*(i) If $T^*$ is hereditarily polaroid and has SVEP, then $f(T + K)$ obeys property $(t)$ for all $f \in H_{nc}(\sigma(T))$.*

*(ii) If $T$ is hereditarily polaroid and has SVEP, then $f(T^* + K^*)$ obeys property $(t)$ for all $f \in H_{nc}(\sigma(T))$.*

*Proof.* (i) We conclude from [7, Theorem 2.15] that $T+K$ is polaroid and hence by [8, Lemma 3.11], we have $f(T + K)$ is polaroid and from [6, Theorem 2.14] that $T^* + K^*$ has SVEP. The SVEP of $T^* + K^*$ entails the SVEP for $f(T^* + K^*)$ by [1, Theorem 2.40]. So, $f(T+K)$ obeys property $(t)$ by [30, Theorem 3.6 (i)].
(ii) The proof of part (ii) is analogous. ■

## 4 Property $(t)$ and tensor product

The problem of transferring Weyl's theorem, property $(w)$ and property $(b)$ from operators $A$ and $B$ to their tensor product $A \otimes B$ was considered in [16], [12] and [31]. The main objective of this section is to study the transfer of property $(t)$ from a bounded linear operator $A$ acting on a Banach space $\mathcal{X}$ and a bounded linear operator $B$ acting on a Banach space $\mathcal{Y}$ to their tensor product $A \otimes B$.

**Example 4.1.** Let $U \in \mathcal{B}(\ell^2)$ denote the forward unilateral shift, and let $A, B \in \mathcal{B}(\ell^2 \otimes \ell^2)$ be the operators

$$A = (1 - UU^*) \oplus \left(\frac{1}{2}U - 1\right), \; B = -(1 - UU^*)\left(\frac{1}{2}U^* - 1\right).$$

Then $A$ and $B^*$ have SVEP, so $A, B \in a\mathfrak{B}$. Furthermore, $1 \in \sigma(A \otimes B) \setminus \sigma_w(A \otimes B)$. However, since

$$\sigma(A \otimes B) = \left\{\{0,1\} \cup \{\frac{1}{2}\mathbb{D} - 1\}\right\} . \left\{\{0,-1\} \cup \{\frac{1}{2}\mathbb{D} + 1\}\right\},$$

where $\mathbb{D}$ is the closed unit disc in the complex plane $\mathbb{C}$, $1 \in \operatorname{acc} \sigma(A \otimes B) \Longrightarrow 1 \in \sigma_b(A \otimes B)$. Then $A \otimes B \notin \mathfrak{B}$, and hence $A \otimes B$ does not obey property $(t)$.

The following example shows that property $(t)$ does not transfer from $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$ to $A \otimes B$.

**Example 4.2.** Let $Q \in \mathcal{B}(\ell^2)$ be an injective quasi-nilpotent, and let

$$A = B = (I + Q) \oplus \alpha \oplus \beta \in Ł(\ell^2) \oplus \mathbb{C} \oplus \mathbb{C},$$

where $\alpha\beta = 1 \neq \alpha$. Then

$$\sigma(A) = \sigma(B) = \{1, \alpha, \beta\}, \sigma_{aw}(A) = \sigma_{aw}(B) = \{1\}, \sigma(A \otimes B) = \{1, \alpha, \beta, \alpha^2, \beta^2\}.$$

The operators $A, B$ have SVEP, hence $a$-Browder's theorem transfers from $A$ and $B$ to $A \otimes B$, which implies that

$$\sigma_{aw}(A \otimes B) = \{1, \alpha, \beta\}, \; 1 \notin \sigma(A \otimes B) \setminus \sigma_{aw}(A \otimes B) \text{ and } 1 = \alpha\beta \in E^0(A \otimes B).$$

Note that the operators $A$ and $B$ are not isoloid.

**Example 4.3.** Choose $A = (I + Q) \oplus \alpha \oplus \beta \in \mathcal{B}(\ell^2) \oplus \mathbb{C} \oplus \mathbb{C}$ as in the previous example, and let $B = \frac{1}{4}U \oplus 1 \oplus \beta \in B(\ell^2) \oplus \mathbb{C} \oplus \mathbb{C}$, where $U$ is the forward unilateral shift and $\alpha = \frac{\sqrt{3}}{2} < \beta = \frac{2}{\sqrt{3}}$. Let $\mathbb{D}$ be the closed unit disc in $\mathbb{C}$ and $\partial\mathbb{D}$ denote the boundary of the closed unit disc $\mathbb{D}$ in $\mathbb{C}$. Then $A$ and $B$ have SVEP, and it follows that

$$\sigma(A) = \{1, \alpha, \beta\}, \; \sigma(B) = \frac{1}{4}\mathbb{D} \cup \{1, \beta\}, \; \sigma(A \otimes B) = \frac{1}{2\sqrt{3}}\mathbb{D} \cup \{1, \alpha, \beta, \beta^2\},$$

and

$$\sigma_{aw}(A) = \{1\}, \; \sigma_{aw}(B) = \frac{1}{4}\partial\mathbb{D}, \; \sigma_{aw}(A \otimes B) = \frac{1}{2\sqrt{3}}\partial\mathbb{D} \cup \{1, \alpha, \beta\}.$$

Evidently, $1 \notin \sigma(A \otimes B) \setminus \sigma_{aw}(A \otimes B)$ and $1 \in E^0(A \otimes B)$. Here the operator $B$ is isoloid but $A$ is not isoloid.

The following theorem gives a necessary and sufficient condition for the transference of property $(t)$ from isoloid $A$ and $B$ to $A \otimes B$. Let $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$. Then $\sigma^{\text{iso}}(A \otimes B) \subseteq \sigma^{\text{iso}}(A).\sigma^{\text{iso}}(B) \cup \{0\}$. If 0 is in the point spectrum of either of $A$ and $B$, then $\alpha(A \otimes B) = \infty$; in particular, $0 \notin E^0(A \otimes B)$. It is easily seen, see the argument of the proof of [16, Proposition 2], that $E^0(A \otimes B) \subseteq E^0(A)E^0(B)$.

**Theorem 4.4.** *If $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$ are isoloid operators which satisfy property $(t)$ and $0 \notin \sigma^{iso}(A \otimes B)$, then the following conditions are equivalent:*

*(i) $A \otimes B$ satisfies property $(t)$.*

*(ii) The $a$-Weyl spectrum equality $\sigma_{aw}(A \otimes B) = \sigma(A)\sigma_{aw}(B) \cup \sigma_{aw}(A)\sigma(B)$ is satisfied.*

*(iii) $A \otimes B$ satisfies $a$-Browder's theorem.*

*Proof.* Since property $(t)$ implies $a$-Browder's theorem, the equivalence (ii)⇔(iii) and (i)⇒(iii) follows from [11, Theorem 3]. We prove (iii)⇒(i). The hypothesis $A$ and $B$ satisfy property $(t)$ implies

$$\sigma(A) \setminus \sigma_{aw}(A) = E^0(A), \qquad \sigma(B) \setminus \sigma_{aw}(B) = E^0(B).$$

Observe that (iii) implies $a$-Browder's theorem transfers from $A$ and $B$ to $A \otimes B$ : hence $\sigma_{aw}(A \otimes B) = \sigma_a(A)\sigma_{aw}(B) \cup \sigma_{aw}(A)\sigma_a(B)$. Let $\lambda \in E^0(A \otimes B)$; then $\lambda \neq 0$ and there exist $\mu \in \sigma^{\text{iso}}(A)$ and $\nu \in \sigma^{\text{iso}}(B)$ such that $\lambda = \mu\nu$. By hypothesis, $A$ and $B$ are isoloid; hence $\mu$ is an eigenvalue of $A$ and $\nu$ is an eigenvalue of $B$. Since $A \otimes B - (\mu I \otimes \nu I) = (A - \mu) \otimes B + \mu(I \otimes (B - \nu))$, if either of $\alpha(A - \mu)$ or $\alpha(B - \nu)$ is infinite then so is $\alpha(A \otimes B - (\mu I \otimes \nu I))$. Hence $\mu \in E^0(A) = \sigma(A) \setminus \sigma_{aw}(A)$ and $\nu \in E^0(B) = \sigma(B) \setminus \sigma_{aw}(B)$, consequently, $\lambda \in \sigma(A \otimes B) \setminus \sigma_{aw}(A \otimes B)$; hence $E^0(A \otimes B) \subseteq \sigma(A \otimes B) \setminus \sigma_{aw}(A \otimes B)$. Conversely, if $\lambda \in \sigma(A \otimes B) \setminus \sigma_{aw}(A \otimes B)$, then $\lambda \neq 0$, and there exist $\mu \in \sigma(A) \setminus \sigma_{aw}(A) = E^0(A)$ and $\nu \in \sigma(B) \setminus \sigma_{aw}(B) = E^0(B)$ such that $\lambda = \mu\nu$. But then $\lambda \in E^0(A \otimes B)$. Therefore, $\sigma(A \otimes B) \setminus \sigma_{aw}(A \otimes B) \subseteq E^0(A \otimes B)$. ■

Let

$$\sigma_s(T) = \{\lambda \in \sigma(T) : T - \lambda \text{ is not onto}\},$$

$$\sigma_{sb} = \{\lambda \in \sigma_s(T) : T - \lambda \text{ is not lower semi-Fredholm or } d(T - \lambda) = \infty\}$$

and

$$\sigma_{sw}(T) = \{\lambda \in \sigma_s(T) : T - \lambda \text{ is not lower semi-Fredholm or } \text{ind}(T - \lambda) < 0$$

denote, respectively, the surjectivity spectrum, the Browder essential surjectivity spectrum and the Weyl essential surjectivity spectrum of $T \in \mathcal{B}(\mathcal{X})$. Then $T$ satisfies $s$-Browder's theorem ($T \in s\mathfrak{B}$) if $\sigma_{sb}(T) = \sigma_{sw}(T)$. Apparently, $T$ satisfies s-Browder's theorem if and only if $T^*$ satisfies a-Bt. A necessary and sufficient condition for T to satisfy a-Browder's theorem is that $T$ has SVEP at every $\lambda \in \sigma_a(T) \setminus \sigma_{aw}(T)$; by duality, $T$ satisfies s-Browder's theorem if and only if $T^*$ has SVEP at every $\lambda \in \sigma_s(T) \setminus \sigma_{sw}(T)$.

$T \in \mathcal{B}(\mathcal{X})$ is polaroid implies $T^*$ polaroid. It is well known that if $T$ or $T^*$ has SVEP and $T$ is polaroid, then $T$ and $T^*$ satisfy Weyl's theorem. Note as well known is the fact, [30, Theorem 3.4], that if $T$ is polaroid and $T^*$ (resp., $T$) has SVEP, then $T$ (resp., $T^*$) satisfies property $(gt)$. The following theorem is the tensor product analogue of this result.

**Theorem 4.5.** *Suppose that the operators $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$ are polaroid.*

*(i) If $A^*$ and $B^*$ have SVEP, then $A \otimes B$ satisfies property $(t)$.*

*(ii) If $A$ and $B$ have SVEP, then $A^* \otimes B^*$ satisfies property $(t)$.*

*Proof.* (i) The hypotheses $A^*$ and $B^*$ have SVEP implies

$$\sigma(A) = \sigma_a(A), \qquad \sigma(B) = \sigma_a(B), \qquad \sigma_{aw}(A) = \sigma_w(A), \qquad \sigma_{aw}(B) = \sigma_w(B)$$

and

$$A^*, B^* \qquad \text{and } A^* \otimes B^* \text{ satisfy } s\text{-Browder's theorem.}$$

Thus $s$-Browder's theorem and Browder's theorem ($s\mathfrak{B} \implies \mathfrak{B}$) transfer from $A^*$ and $B^*$ to $A^* \otimes B^*$. Hence

$$\begin{aligned}\sigma_{aw}(A \otimes B) &= \sigma_{sw}(A^* \otimes B^*) = \sigma_s(A^*)\sigma_{sw}(B^*) \cup \sigma_{sw}(A^*)\sigma_s(B^*) \\ &= \sigma_a(A)\sigma_{aw}(B) \cup \sigma_{aw}(A)\sigma_a(B) = \sigma(A)\sigma_w(B) \cup \sigma_w(A)\sigma(B),\end{aligned}$$

and

$$\begin{aligned}\sigma_w(A \otimes B) &= \sigma_w(A^* \otimes B^*) = \sigma_w(A^*)\sigma(B^*) \cup \sigma_w(B^*)\sigma(A^*) \\ &= \sigma(A)\sigma_w(B) \cup \sigma(B)\sigma_w(A).\end{aligned}$$

Consequently,

$$\sigma_{aw}(A \otimes B) = \sigma_w(A \otimes B).$$

Evidently, $A \otimes B$ is polaroid [12, Lemma 2]; combining this with $A \otimes B$ satisfies Browder's theorem, it follows that $A \otimes B$ satisfies Weyl's theorem, i.e., $\sigma(A \otimes B) \setminus \sigma_w(A \otimes B) = E^0(A \otimes B)$. But then

$$\sigma(A \otimes B) \setminus \sigma_{aw}(A \otimes B) = \sigma(A \otimes B) \setminus \sigma_w(A \otimes B) = E^0(A \otimes B),$$

i.e., $A \otimes B$ satisfies property $(t)$.
(ii) In this case $\sigma(A) = \sigma_a(A^*), \sigma(B) = \sigma_a(B^*)$, $\sigma_w(A^*) = \sigma_{aw}(A^*), \sigma_w(B^*) = \sigma_{aw}(B^*)$, $\sigma(A^* \otimes B^*) = \sigma_a(A^* \otimes B^*)$, polaroid property transfer from $A$ and $B$ to $A^* \otimes B^*$, and both $s$-Browder's theorem and Browder's theorem transfer from $A$ and $B$ to $A \otimes B$. Hence

$$\begin{aligned}\sigma_{aw}(A^* \otimes B^*) &= \sigma_{sw}(A \otimes B) = \sigma_s(A)\sigma_{sw}(S) \cup \sigma_{sw}(A)\sigma_s(B)\\ &= \sigma_a(A^*)\sigma_{aw}(B^*) \cup \sigma_{aw}(A^*)\sigma_a(B^*)\\ &= \sigma(A)\sigma_w(B) \cup \sigma_w(A)\sigma(B)\\ &= \sigma_w(A \otimes B) = \sigma_w(A^* \otimes B^*).\end{aligned}$$

Thus, since $A^* \otimes B^*$ polaroid and $A \otimes B$ satisfies Browder's theorem imply $A^* \otimes B^*$ satisfies Weyl's theorem,

$$\sigma(A^* \otimes B^*) \setminus \sigma_{aw}(A^* \otimes B^*) = \sigma(A^* \otimes B^*) \setminus \sigma_w(A^* \otimes B^*) = E^0(A^* \otimes B^*),$$

i.e., $A^* \otimes B^*$ satisfies property $(t)$. ■

## 5 Perturbations and Tensor Product

Let $[A, Q] = AQ - QA$ denote the commutator of the operators $A$ and $Q$. If $Q_1 \in \mathcal{B}(\mathcal{X})$ and $Q_2 \in \mathcal{B}(\mathcal{Y})$ are quasinilpotent operators such that $[Q_1, A] = [Q_2, B] = 0$ for some operators $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$, then

$$(A + Q_1) \otimes (B + Q_2) = (A \otimes B) + Q,$$

where $Q = Q_1 \otimes B + A \otimes Q_2 + Q_1 \otimes Q_2 \in \mathcal{B}(\mathcal{X} \otimes \mathcal{Y})$ is a quasinilpotent operator. If in the above, $Q_1$ and $Q_2$ are nilpotents then $(A + Q_1) \otimes (B + Q_2)$ is the perturbation of $A \otimes B$ by a commuting nilpotent operator.

**Theorem 5.1.** *Let $Q_1 \in \mathcal{B}(\mathcal{X})$ and $Q_2 \in \mathcal{B}(\mathcal{Y})$ be quasinilpotent operators such that $[Q_1, A] = [Q_2, B] = 0$ for some operators $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$. If $A \otimes B$ is finitely isoloid, then $A \otimes B$ satisfies property $(t)$ implies $(A + Q_1) \otimes (B + Q_2)$ satisfies property $(t)$.*

*Proof.* Start by recalling that $\sigma((A+Q_1) \otimes (B+Q_2)) = \sigma(A \otimes B)$, $\sigma_a((A+Q_1) \otimes (B+Q_2)) = \sigma_a(A \otimes B)$, $\sigma_{aw}((A + Q_1) \otimes (B + Q_2)) = \sigma_{aw}(A \otimes B)$ and that the perturbation of an operator by a commuting quasinilpotent has SVEP if and only if the operator has SVEP. If $A \otimes B$ satisfies property $(t)$, then

$$\begin{aligned}E^0(A \otimes B) &= \sigma(A \otimes B) \setminus \sigma_{aw}(A \otimes B)\\ &= \sigma((A + Q_1) \otimes (B + Q_2)) \setminus \sigma_{aw}((A + Q_1) \otimes (B + Q_2)).\end{aligned}$$

We prove that $E^0(A \otimes B) = E^0((A + Q_1) \otimes (B + Q_2))$. Observe that if $\lambda \in \sigma^{iso}(A \otimes B)$, then $A^* \otimes B^*$ has SVEP at $\lambda$; equivalently, $(A^* + Q_1^*) \otimes (B^* + Q_2^*)$ has SVEP at $\lambda$. Let $\lambda \in E^0(A \otimes B)$, then $\lambda \in \sigma((A+Q_1) \otimes (B+Q_2)) \setminus \sigma_{aw}((A+Q_1) \otimes (B+Q_2))$. Since $(A+Q_1)^* \otimes (B+Q_2)^*$ has SVEP at $\lambda$, it follows that $\lambda \notin \sigma_{aw}((A+Q_1) \otimes (B+Q_2))$ and $\lambda \in \sigma^{iso}((A+Q_1) \otimes (B+Q_2))$. Thus $\lambda \in E^0((A+Q_1) \otimes (B+Q_2))$. Hence $E^0(A \otimes B) \subseteq E^0((A+Q_1) \otimes (B+Q_2))$. Conversely, if $\lambda \in E^0((A+Q_1) \otimes (B+Q_2))$, then $\lambda \in \sigma^{iso}(A \otimes B)$, and this, since $A \otimes B$ is finitely isoloid, implies that $\lambda \in E^0(A \otimes B)$. Therefore, $E^0((A+Q_1) \otimes (B+Q_2)) \subseteq E^0(A \otimes B)$. So, the proof of the theorem is achieved. ■

**Corollary 5.2.** *If $Q_1 \in \mathcal{B}(\mathcal{X})$ and $Q_2 \in \mathcal{B}(\mathcal{Y})$ are nilpotent operators such that $[Q_1, A] = [Q_2, B] = 0$ for some operators $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$, then $A \otimes B$ satisfies property $(t)$ implies $(A + Q_1) \otimes (B + Q_2)$ satisfies property $(t)$.*

The situation for perturbations by commuting Riesz operators is a bit more delicate. The equality $\sigma_a(T) = \sigma_a(T+R)$ does not always hold for operators $T, R \in \mathcal{B}(\mathcal{X})$ such that R is Riesz and $[T, R] = 0$; the tensor product $T \otimes R$ is not a Riesz operator (the Fredholm spectrum $\sigma_e(T \otimes R) = \sigma(T)\sigma_e(R) \cup \sigma_e(T)\sigma(R) = \sigma_e(T)\sigma(R) = \{0\}$ for a particular choice of $T$ only). However, $\sigma_w$ (also, $\sigma_b$) is stable under perturbation by commuting Riesz operators [32], and so $T$ satisfies Browder's theorem if and only if $T+R$ satisfies Browder's theorem. Thus, if $\sigma(T) = \sigma(T+R)$ for a certain choice of operators $T, R \in \mathcal{B}(\mathcal{X})$ (such that $R$ is Riesz and $[T, R] = 0$), then

$$\pi^0(T) = \sigma(T) \setminus \sigma_w(T) = \sigma(T+R) \setminus \sigma_w(T+R) = \pi^0(T+R),$$

where $\pi^0(T)$ is the set of $\lambda \in \sigma^{iso}(T)$ which are finite rank poles of the resolvent of $T$. If we now suppose additionally that $T$ satisfies property $(t)$, then

$$E^0(T) = \sigma(T) \setminus \sigma_{aw}(T) = \sigma(T+R) \setminus \sigma_{aw}(T+R), \tag{5.1}$$

and a necessary and sufficient condition for $T+R$ to satisfy property $(t)$ is that $E^0(T+R) = E^0(T)$. One such condition, namely $T$ is finitely isoloid.

**Theorem 5.3.** *Let $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$ be finitely isoloid operators which satisfy property $(t)$. If $R_1 \in \mathcal{B}(\mathcal{X})$ and $R_2 \in \mathcal{B}(\mathcal{Y})$ are Riesz operators such that $[A, R_1] = [B, R_2] = 0$, $\sigma_a(A+R_1) = \sigma_a(A)$ and $\sigma_a(B+R_2) = \sigma_a(B)$, then $A \otimes B$ satisfies property $(t)$ implies $(A+R_1) \otimes (B+R_2)$ satisfies property $(t)$ if and only if Browder's theorem transforms from $A+R_1$ and $B+R_2$ to their tensor product.*

*Proof.* The hypotheses imply (by Corollary 3.13) that both $A+R_1$ and $B+R_2$ satisfy property $(t)$. Suppose that $A \otimes B$ satisfies property $(t)$. Then $\sigma(A \otimes B) \setminus \sigma_{aw}(A \otimes B) = E^0(A \otimes B)$. Evidently $A \otimes B$ satisfies $a$-Browder's theorem, and so the hypothesis $A$ and $B$ satisfy property $(t)$ implies that $a$-Browder's theorem transfers from $A$ and $B$ to $A \otimes B$. Furthermore, since , $\sigma_a(A+R_1) = \sigma_a(A)$, $\sigma_a(B+R_2) = \sigma_a(B)$, and $\sigma_{aw}$ is stable under perturbations by commuting Riesz operators,

$$\begin{aligned}\sigma_{aw}(A \otimes B) &= \sigma_a(A)\sigma_{aw}(B) \cup \sigma_{aw}(A)\sigma_a(B) \\ &= \sigma_a(A+R_1)\sigma_{aw}(B+R_2) \cup \sigma_{aw}(A+R_1)\sigma_a(B+R_2) \\ &= \sigma(A+R_1)\sigma_{aw}(B+R_2) \cup \sigma_{aw}(A+R_1)\sigma(B+R_2)\end{aligned}$$

Suppose now that $a$-Browder's theorem transfers from $A+R_1$ and $B+R_2$ to $(A+R_1)\otimes(B+R_2)$. Then

$$\sigma_{aw}(A \otimes B) = \sigma_{aw}((A+R_1) \otimes (B+R_2))$$

and

$$E^0(A \otimes B) = \sigma((A+R_1) \otimes (B+R_2)) \setminus \sigma_{aw}((A+R_1) \otimes (B+R_2)).$$

Let $\lambda \in E^0(A \otimes B)$. Then $\lambda \neq 0$, and hence there exist $\mu \in \sigma(A+R_1) \setminus \sigma_{aw}(A+R_1)$ and $\nu \in \sigma(B+R_2) \setminus \sigma_{aw}(B+R_2)$ such that $\lambda = \mu\nu$. As observed above, both $A+R_1$ and $B+R_2$ satisfy property $(t)$; hence $\mu \in E_a^0(A+R_1)$ and $\nu \in E_a^0(B+R_2)$. This, since $\lambda \in \sigma(A \otimes B) = \sigma((A+R_1)\otimes(B+R_2))$, implies $\lambda \in E^0((A+R_1)\otimes(B+R_2))$. Conversely, if $\lambda \in E^0((A+R_1)\otimes(B+R_2))$, then $\lambda \neq 0$ and there exist $\mu \in E^0(A+R_1) \subseteq \sigma_a^{iso}(A)$ and $\nu \in E^0(B+R_2) \subseteq \sigma_a^{iso}(B)$ such that $\lambda = \mu\nu$. Recall that $E^0((A+R_1)\otimes(B+R_2)) \subseteq E^0(A+R_1)E^0(B+R_2)$. Since $A$ and $B$ are finite isoloid, $\mu \in E^0(A)$ and $\nu \in E^0(B)$. Hence, since $\sigma((A+R_1)\otimes(B+R_2)) = \sigma(A \otimes B)$, $\lambda = \mu\nu \in E^0(A \otimes B)$. To complete the proof, we observe that if the implication of the statement of the theorem holds, then (necessarily) $(A+R_1) \otimes (B+R_2)$ satisfies Browder's theorem. This, since $A+R_1$ and $B+R_2$ satisfy Browder's theorem, implies Browder's theorem transfers from $A+R_1$ and $B+R_2$ to $(A+R_1) \otimes (B+R_2)$. ■

## References

[1] P. Aiena, *Fredholm and local spectral theory with applications to multipliers*,Kluwer Acad. Publishers, Dordrecht, 2004.

[2] P. Aiena, Classes of operators satisfying $a$-Weyl's theorem, *Studia Math.* **169**, 105–122 (2005).

[3] P. Aiena, C. Carpintero, Weyl's theorem, $a$-Weyl's theorem and single-valued extension property, *Extracta Math.* **20**, 25–41 (2005).

[4] P. Aiena, M.T. Biondi, Property $(w)$ and perturbations, *J. Math. Anal. Appl.* **336**, 683–692 (2007).

[5] P. Aiena and P. Peña, Variations on Weyl's theorem, *J. Math. Anal. Appl.* **324**, 566–579 (2006).

[6] P. Aiena , J. R. Guillen and P. Peña, Property $(w)$ for perturbations of polaroid operators, *Linear Alg. Appl.* **428**, 1791–1802 (2008).

[7] P. Aiena, E. Aponte, J. R. Guillen and P. Peña, Property $(R)$ under perturbations, *Mediterr. J. Math.* **10** (1), 367–382 (2013).

[8] P. Aiena, E. Aponte, Polaroid type operators under perturbations, *Studia Math.* **214** (2), 121–136 (2013).

[9] L. A. Coburn, Weyl's theorem for nonnormal operators, *Michigan Math. J.* **13**, 285–288 (1966).

[10] D.S. Djordjević, Operators obeying $a$-Weyl's theorem, *Publ. Math. Debrecen* **55** (3), 283–298 (1999).

[11] B.P. Duggal, S.V. Djordjeviȅ and C.S. Kubrusly, On the $a$-Browder and $a$-Weyl spectra of tensor products. *Rend. Circ. Mat. Palermo* **59**, 473–481 (2010).

[12] B.P. Duggal, Tensor product and property $(w)$, *Rend. Circ. Mat. Palermo* **60** (1), 23–30 (2011).

[13] J. K. Finch, The single valued extension property on a Banach space, *Pacific J. Math.* **58**, 61–69 (1975).

[14] R. Harte and W.Y. Lee, Another note on Weyl's theorem, *Trans. Amer. Math. Soc.* **349**, 2115–2124 (1997).

[15] H. Heuser, *Functional analysis*, Marcel Dekker, New York (1982).

[16] C.S. Kubrusly and B.P. Duggal,On Weyl and Browder spectra of tensor product, *Glasgow Math. J.* **50**, 289–302 (2008).

[17] K. B. Laursen, Operators with finite ascent, *Pacific J. Math.* **152**, 323–336 (1992).

[18] M. Lahrouz and M. Zohry, Weyl type theorems and the approximate point spectrum, *Irish Math. Soc. Bulletin* **55**, 41–51 (2005).

[19] K. B. Laursen, M. M. Neumann, *An introduction to local spectral theory*, Oxford. Clarendon (2000).

[20] M. Mbekhta, Sur la th'eorie spectrale locale et limite de nilpotents, *Proc. Amer. Math. Soc.* **3**, 621–631 (1990).

[21] M. Oudghiri, Weyl's and Browder's theorem for operators satysfying the SVEP, *Studia Math.* **163**, 85–101 (2004).

[22] M. Oudghiri, Weyl's Theorem and perturbations, *Integral Equations Operator Theory* **53**, 535–545 (2005).

[23] V. Rakočević, On a class of operators, *Math. Vesnik* **37** , 423–426 (1985).

[24] V. Rakočević, Operators obeying $a$-Weyl's theorem, *Rev. Roumaine Math. Pures Appl.* **10**, 915–919 (1986).

[25] V. Rakočević, Semi-Browder operators and perturbations, *Studia Math.* **122**, 131–137 (1997).

[26] M.H.M.Rashid, Property $(w)$ and quasi-class $(A, k)$ operators, *Revista De Le Unión Math. Argentina* **52**, 133–142 (2011).

[27] M.H.M.Rashid and M.S.M.Noorani, Weyl's type theorems for algebraically $w$-hyponormal operators, *Arab. J. Sci. Eng.* **35**,103–116 (2010).

[28] M.H.M.Rashid, Property $(gw)$ and perturbations, *Bull. Belg. Math. Soc. Simon Stevin* **18**, 635–654 (2011).

[29] M.H.M.Rashid, Weyl's theorem for algebraically $wF(p, r, q)$ operators with $p, r > 0$ and $q \geq 1$, *Ukrainian Math. J.* **63** (8), 1256–1267 (2011).

[30] M.H.M. Rashid, Properties $(t)$ and $(gt)$ For Bounded Linear Operators, *Mediterr. J. Math.* **11**, 729–744 (2014).

[31] M.H.M. Rashid, Tensor product and property $(b)$, *Periodica Math. Hungarica* **75**, 376–386 (2017).

[32] M. Schechter and R. Whitley, Best Fredholm perturbation theorems, *Stud. Math.* **90**, 175–190 (1988).

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