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ON DEFERRED GENERALIZED STATISTICAL CONVERGENCE OF COMPLEX UNCERTAIN TRIPLE SEQUENCES

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Abstract Complex uncertain variables are measurable functions from an uncertain space to the set of complex numbers and are used to model complex uncertain quantities. In this manuscript, we study the deferred $\mathcal{I}_3$-statistical convergence concept of complex uncertain triple sequence. Four types of deferred $\mathcal{I}_3$-statistically convergent complex uncertain triple sequences are presented, namely deferred $\mathcal{I}_3$-statistical convergence in measure, in mean, in distribution and with respect to almost surely, and some basic properties are proved.

1 Introduction

In our daily life, we often encounter situations where there is scant or no evidence of events, not only for technical and economic problems, but also for other unexpected events. Insufficient ones data makes it difficult to apply the probability distribution of events. Experts close to the subject are consulted to give a belief degree that every event happens when a decision is made. If we insist on addressing the degree of belief using probability theory, counterintuitive results may arise. To identify such matters, Liu [19] introduced a self-duality measure, uncertain measure, which is a set function fulfilling normality, monotonicity, self-duality, and countable subadditivity axioms. In actual life, various types of uncertainty exist, such as randomness, fuzziness, and uncertainty which consists of both randomness and fuzziness. Probability measure is used to define a random event. To measure fuzzy events, Zadeh [33] initiated possibility measure but possibility measure does not have self duality. Thus, Liu and Liu [21] introduced a self-dual measure, the credibility measure. An undeniable foundation for credibility theory was given in Liu [19]. From the beginning, the uncertainty theory has been consistently studied, developed and has widespread application (see [20]). Clearly, classical measure, probability measure and credibility measure are special cases of uncertain measures. But possibility measure is not an uncertain measure. Hence the elements of uncertain measure can also be applied to classical measure, probability measure and credibility measure.

Complex uncertain variables are measurable functions from uncertainty spaces to the set of complex numbers. Convergence of sequences always plays a crucial role in different theory of mathematics. The convergence of complex uncertain sequence was first introduced by Chen et al. [4]. Studies on convergence of sequences of uncertain variables are due to You [32]. The concept of statistical convergence, which is an extension of usual idea of convergence, was introduced by Fast [11] for real and complex number sequences. Statistical convergence has several applications in different fields of mathematics like number theory, trigonometric series, summability theory, statistics and probability theory, measure theory, optimization, approximation theory, rough set theory, hopfield neural networks and fuzzy sets. The study of statistical convergence in triple sequence has been initiated by Şahiner et al. [27]. Küçükaslan and Yılmaztürk [18] defined the concept of deferred statistical convergence. Kostyrko et al. [17] extended the notion of statistical convergence to ideal convergence, and established some basic theorems. On the other hand, the new form of convergence called $\mathcal{I}$-statistical convergence has been introduced in [6]. Recently lots of interesting developments have occurred in $\mathcal{I}$-statistical convergence and related topics (see [25, 26, 31])

Since the notion of convergence of a sequence plays a very important role in the fundamental theory of mathematics, there are many convergence concepts in classical measure theory, probability theory and credibility theory, and the relationships between them are discussed. The interested reader may consult Liu [19], Zhu and Liu [34], Tripathy and Nath [30], Kişi [15], Kişi and Ünal [16], Das et al. [7] and Demirci and Gürdal [8]. Also, the readers should refer to the monographs Başar [2], Başar and Dutta [3] and Mursaleen and Başar [22] for the background on the sequence spaces and related topics. Inspired by this, in this paper, a further investigation into the mathematical properties of uncertain triple sequences will be made. Section 2 recalls some definitions and theorems in uncertainty and summability theory. In Section 3, we analyze some deferred $\mathcal{I}$-statistical convergence concepts of complex uncertain triple sequences such as deferred $\mathcal{I}$-statistical convergence in measure, deferred $\mathcal{I}$-statistical convergence in mean, deferred $\mathcal{I}$-statistical convergence in distribution and with respect to almost surely and examine the relationships among them by several theorems and examples.

2 Preliminaries

Definition 2.1. ([19]) Let $\mathcal{L}$ be a σ-algebra on a non-empty set Γ. A set function $\mathcal{M}$ on Γ is called an uncertain measure if it satisfies the following axioms:

(1^o) $\mathcal{M}\{\Gamma\} = 1$ (normality axiom);
(2^o) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any $\Lambda \in \mathcal{L}$ (duality axiom);
(3^o) We get

$$\mathcal{M}\left\{\bigcup_{j=1}^{\infty}\lambda_j\right\} \leq \sum_{j=1}^{\infty}\mathcal{M}\{\lambda_j\} \tag{2.1}$$

for all countable sequence of $\{\lambda_j\} \in \mathcal{L}$ (subadditivity axiom).

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space and each element Λ is called an event in $\mathcal{L}$. Liu [19] defined a product uncertain measure to get an uncertain measure of compound event as follows :

(4^o) For $s = 1, 2, 3, ...$, let $(\Gamma_s, \mathcal{L}_s, \mathcal{M}_s)$ be uncertainty space. The product uncertain measure $\mathcal{M}$ is a measure such that

$$\mathcal{M}\left\{\prod_{s=1}^{\infty}\Lambda_s\right\} = \bigwedge_{s=1}^{\infty}\mathcal{M}_s\{\Lambda_s\},$$

for $s \in \mathbb{N}$, where Λ_s is an arbitrarily chosen events in $\mathcal{L}_s$.

Definition 2.2. ([19]) A complex uncertain variable is a measurable function ζ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of complex numbers, that is, the set

$$\{\zeta \in \mathcal{B}\} = \{\gamma \in \Gamma : \zeta(\gamma) \in \mathcal{B}\} \tag{2.2}$$

is an event for any Borel set $\mathcal{B}$ of complex numbers.

When the range is the set of real numbers, we call it as an uncertain variable. Also, complex uncertain sequences are sequence of complex uncertain variables indexed by integers.

The notion of statistical convergence depends on the density of the subsets of the set $\mathbb{N}$ of natural numbers. The following two definitions are well known (see [12]).

The idea of statistical convergence depends upon the density of subsets of the set $\mathbb{N}$ of positive integers. The density of a subset A of $\mathbb{N}$ is defined by

$$\delta(A) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\chi_A(k),$$

provided that the limit exists, where χ_A is the characteristic function of the set A. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and that $\delta(A^c) = 1 - \delta(A)$.A sequence $x = (x_k)_{k\in\mathbb{N}}$ is said to be statistically convergent to L if, for every $\varepsilon > 0$, we have

$$\delta(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}) = 0.$$

In this case, we write st-$\lim x_k = x$.

In 1932, Agnew [1] introduced the concept of deferred Cesàro mean of real (or complex) valued sequences $x = (x_k)$ defined by

$$\left(D_a^b(x)\right)_r = \frac{1}{b_r - a_r} \sum_{k=a_r+1}^{b_r} x_k, \; r = 1, 2, 3, ..., \tag{2.3}$$

where $a = (a_r)$ and $b = (b_r)$ are two sequences of non-negative integers satisfying

$$a_r < b_r \text{ and } \lim_{r\to\infty} b_r = \infty. \tag{2.4}$$

Deferred density of $K \subset \mathbb{N}$ defined by

$$\delta_a^b(K) = \lim_{r\to\infty} \frac{|\{k : a_r < k \leq b_r, \; k \in K\}|}{b_r - a_r}.$$

Definition 2.3. ([18]) A real valued sequence $x = (x_k)$ is said to be deferred statistically convergent to L provided that

$$\lim_{r\to\infty} \frac{|\{a_r < k \leq b_r : |x_k - L| \geq \varepsilon\}|}{b_r - a_r} = 0$$

for each $\varepsilon > 0$ and it is written by $DS - \lim x_k = L$.

Remark 2.4. If $a_r = 0$ and $b_r = r$, then Definition 2.3 is coincide with the definition of statistical convergence.

Also, Dağadur and Sezgek [5] introduced deferred statistical convergence of double sequences. Let $x = (x_{jk})$ be a double sequence and $\beta_r = b_r - a_r$, $\gamma_s = d_s - c_s$. Then the double sequence x is said to be deferred statistically convergent to L if for every $\varepsilon > 0$,

$$\lim_{r,s,\to\infty} \frac{|\{(j,k) : a_r < j \leq b_r, c_s < k \leq d_s; \; |x_{jk} - L| \geq \varepsilon\}|}{\beta_r \gamma_s} = 0.$$

On the contrary, Kostyrko et al. [17] introduced $\mathcal{I}$-convergence in a metric space. This definition depends on the definition of an ideal $\mathcal{I}$ in $\mathbb{N}$.

A family $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to be an ideal in $\mathbb{N}$ provided: $\emptyset \in \mathcal{I}$; $P, R \in \mathcal{I}$ implies $P \cup R \in \mathcal{I}$; $P \in \mathcal{I}$, $R \subset P$ implies $R \in \mathcal{I}$.

A non empty family $\mathcal{F} \subset 2^{\mathbb{N}}$ is said to be a filter in $\mathbb{N}$ provided: $\emptyset \notin \mathcal{F}$; for every $P, R \in \mathcal{F}$, $P \cap R \in \mathcal{F}$; $P \in \mathcal{F}$, $P \subset R$ implies $R \in \mathcal{F}$. Let $Y \neq \emptyset$. An ideal $\mathcal{I}$ is said to be non-trivial if $\mathcal{I} \neq \emptyset$ and $Y \notin \mathcal{I}$. $\mathcal{F}(\mathcal{I}) = \{\mathbb{N}\backslash P : P \in \mathcal{I}\}$ is a filter on Y if and only if the $\mathcal{I} \subset 2^Y$ is a non-trivial ideal. $\mathcal{I} \subset 2^Y$ is a non-trivial ideal, that is named admissible if and only if $\mathcal{I} \supset \{\{y\} : y \in Y\}$.

Definition 2.5. ([17]) Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The real number sequence $x = (x_n)$ is said to be $\mathcal{I}$-convergent to L if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}$ belongs to $\mathcal{I}$. If $x = (x_n)$ is $\mathcal{I}$-convergent to L then we write $\mathcal{I}$-$\lim x = L$.

More information about $\mathcal{I}$-convergent can be found from [23, 24, 28, 29]. Utilizing the $\mathcal{I}$-convergence and statistical convergence, Das et al. [6] introduced the $\mathcal{I}$-statistical convergence as follows:

Definition 2.6. If for each $\varepsilon > 0$ and $\delta > 0$

$$\left\{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : \|x_k - \mathcal{L}\| \geq \varepsilon\}| \geq \delta\right\} \in \mathcal{I},$$

a sequence (x_n) is called to be $\mathcal{I}$-statistically convergent to $\mathcal{L}$.

We now recall that the concept of statistical convergence for triple sequences was presented by Şahiner et al. [27] as follows:

A function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} = \mathbb{N}^3 \to \mathbb{R}$ (or $\mathbb{C}$) is called a real (or complex) triple sequence. A triple sequence (x_{jkl}) is said to be convergent to L in Pringsheim's sense if for every $\varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbb{N}$ such that $|x_{jkl} - L| < \varepsilon$ whenever $j, k, l \geq n_0$.

Definition 2.7. If

$$\delta_3(K) = P - \lim_{n,l,k\to\infty} \frac{|K_{nlk}|}{nlk}$$

exists, a subset K of $\mathbb{N}^3$ is said to have natural density $\delta_3(K)$ where the vertical bars indicate the number of (n, l, k) in K so that $p \leq n$, $q \leq l$, $r \leq k$. When for all $\varepsilon > 0$,

$$\delta_3\left(\left\{(n,l,k) \in \mathbb{N}^3 : |x_{nlk} - L| \geq \varepsilon\right\}\right) = 0,$$

a real triple sequence $x = (x_{nlk})$ is called to be statistically convergent to L in Pringsheim's sense.

Throughout the paper we consider that $\mathcal{I}$ is the ideals of $2^{\mathbb{N}}$; $\mathcal{I}_2$ is the ideals of $2^{\mathbb{N}^2}$ and $\mathcal{I}_3$ is the ideals of $2^{\mathbb{N}^3}$.

Definition 2.8. A real triple sequence (x_{nlk}) is called to be $\mathcal{I}_3$-convergent to L if for every $\varepsilon > 0$,

$$\left\{(n,l,k) \in \mathbb{N}^3 : |x_{nlk} - L| \geq \varepsilon\right\} \in \mathcal{I}_3$$

and is written $\mathcal{I}_3\text{-}\lim x_{nlk} = L$.

Recently, some types of convergence of triple sequences have also been studied in [8, 9, 10, 13, 14].

3 Results

In this section, we introduce the notion of complex uncertain triple sequences and study deferred $\mathcal{I}_3$-statistical convergence therein. Deferred $\mathcal{I}_3$-statistical convergence with respect to all four aspects in uncertain space, i.e., deferred $\mathcal{I}_3$-statistical convergence in mean, measure, distribution and almost surely, are initiated and interrelationships among them are established.

Throughout the paper, let $\zeta = \{\zeta_{rst}\}$ be a complex uncertain triple sequence and $\beta_r = b_r - a_r$, $\gamma_s = d_s - c_s$, $\eta_t = f_t - e_t$ and $\{a_r\}_{r\in\mathbb{N}}$, $\{b_r\}_{r\in\mathbb{N}}$, $\{c_s\}_{s\in\mathbb{N}}$, $\{d_s\}_{s\in\mathbb{N}}$, $\{e_t\}_{t\in\mathbb{N}}$ and $\{f_t\}_{t\in\mathbb{N}}$ be the sequences of nonnegative integers satisfying the conditions $a_r < b_r$, $c_s < d_s$, $e_t < f_t$ and

$$\lim_r b_r = \infty, \ \lim_s d_s = \infty, \ \lim_t f_t = \infty.$$

Definition 3.1. The complex uncertain triple sequence $\{\zeta_{rst}\}$ is called to be deferred $\mathcal{I}_3$-statistically convergent almost surely (a.s.) to ζ if for all $\varepsilon, \varrho > 0$ there exists an event Λ with $\mathcal{M}(\Lambda) = 1$ such that

$$\Big\{(r,s,t) \in \mathbb{N}^3 : \frac{1}{\beta_r\gamma_s\eta_t} |\{(j,k,l) : a_r < j \leq b_r, c_s < k \leq d_s, e_t < l \leq f_t;$$
$$\|\zeta_{jkl}(\gamma) - \zeta(\gamma)\| \geq \varepsilon\}| \geq \varrho\} \in \mathcal{I}_3,$$

for every $\gamma \in \Lambda$. Hence we can write $\zeta_{rst} \to \zeta \ (DS(\mathcal{I}_3)$ a.s.).

Definition 3.2. The complex uncertain triple sequence $\{\zeta_{rst}\}$ is called to be deferred $\mathcal{I}_3$-statistically convergent in measure to ζ if

$$\Big\{(r,s,t) \in \mathbb{N}^3 : \frac{1}{\beta_r\gamma_s\eta_t} |\{(j,k,l) : a_r < j \leq b_r, c_s < k \leq d_s, e_t < l \leq f_t;$$
$$\mathcal{M}(\|\zeta_{jkl} - \zeta\| \geq \varepsilon) \geq \varrho\}| \geq \delta\} \in \mathcal{I}_3,$$

for every $\varepsilon, \delta, \varrho > 0$.

Definition 3.3. The complex uncertain triple sequence $\{\zeta_{rst}\}$ is called deferred $\mathcal{I}_3$-statistically convergent in mean to ζ if

$$\left\{(r,s,t)\in\mathbb{N}^3 : \frac{1}{\beta_r\gamma_s\eta_t}\left|\{(j,k,l) : a_r < j \le b_r, c_s < k \le d_s, e_t < l \le f_t;\right.\right.$$
$$E\left(\|\zeta_{jkl}-\zeta\|\right)\ge\varepsilon\}|\ge\varrho\}\in\mathcal{I}_3,$$

for every $\varepsilon,\varrho>0$.

Definition 3.4. Let Φ,Φ_{jkl} be the complex uncertainty distribution of complex uncertain variables ζ,ζ_{jkl} respectively, where $j,k,l\in\mathbb{N}$. Then the complex uncertain triple sequence $\{\zeta_{rst}\}$ is called to be deferred $\mathcal{I}_3$-statistically convergent in distribution to ζ if for all $\varepsilon,\varrho>0$,

$$\left\{(r,s,t)\in\mathbb{N}^3 : \frac{1}{\beta_r\gamma_s\eta_t}\left|\{(j,k,l) : a_r < j \le b_r, c_s < k \le d_s, e_t < l \le f_t;\right.\right.$$
$$\|\Phi_{jkl}(z)-\Phi(z)\|\ge\varepsilon\}|\ge\varrho\}\in\mathcal{I}_3,$$

for all complex z at which $\Phi(z)$ is continuous.

Definition 3.5. A complex uncertain triple sequence $\{\zeta_{rst}\}$ is called to be deferred $\mathcal{I}_3$-statistically convergent to ζ if for all $\varepsilon,\varrho>0$,

$$\left\{(r,s,t)\in\mathbb{N}^3 : \frac{1}{\beta_r\gamma_s\eta_t}\left|\{(j,k,l) : a_r < j \le b_r, c_s < k \le d_s, e_t < l \le f_t;\right.\right.$$
$$\|\zeta_{jkl}(\gamma)-\zeta(\gamma)\|\ge\varepsilon\}|\ge\varrho\}\in\mathcal{I}_3,$$

for every $\gamma\in\Lambda$.

Now, relationships between deferred $\mathcal{I}_3$-statistical convergence a.s., deferred $\mathcal{I}_3$-statistical convergence in mean, deferred $\mathcal{I}_3$-statistical convergence in measure, deferred $\mathcal{I}_3$-statistical convergence in distribution will be studied.

Theorem 3.6. *If the complex uncertain triple sequence $\{\zeta_{rst}\}$ deferred $\mathcal{I}_3$-statistically converges in the mean to ζ, then $\{\zeta_{rst}\}$ deferred $\mathcal{I}_3$-statistically converges to ζ in measure.*

Proof. Let the complex uncertain triple sequence $\{\zeta_{rst}\}$ be deferred $\mathcal{I}$-statistically convergent in mean to ζ. For any taken $\varepsilon,\varrho,\delta>0$ with the Markov's inequality, we have

$$\left\{(r,s,t)\in\mathbb{N}^3 : \frac{1}{\beta_r\gamma_s\eta_t}\left|\{(j,k,l) : a_r < j \le b_r, c_s < k \le d_s, e_t < l \le f_t;\right.\right.$$
$$\mathcal{M}\left(\|\zeta_{jkl}-\zeta\|\ge\varepsilon\right)\ge\varrho\}|\ge\delta\}$$
$$\subseteq\left\{(r,s,t)\in\mathbb{N}^3 : \frac{1}{\beta_r\gamma_s\eta_t}\left|\{(j,k,l) : a_r < j \le b_r, c_s < k \le d_s, e_t < l \le f_t;\right.\right.$$
$$\left.\left.\left(\frac{E\left(\|\zeta_{jkl}-\zeta\|\right)}{\varepsilon}\right)\ge\varrho\right\}\right|\ge\delta\right\}\in\mathcal{I}_3.$$

Hence $\{\zeta_{rst}\}$ deferred $\mathcal{I}_3$-statistically converges in measure to ζ. □

The converse of Theorem 3.6 does not hold, in general, i.e. deferred $\mathcal{I}_3$-statistically convergence in measure does not imply deferred $\mathcal{I}_3$-statistically convergence in mean. This can be demonstrated by Example 3.7, given below.

Example 3.7. Take into account the uncertaintly space $(\Gamma,L,\mathcal{M})$. It becomes $\Gamma=\{\gamma_1,\gamma_2,...\}$ with

$$\mathcal{M}\{\Lambda\}=\begin{cases}\sup\limits_{\gamma_{r+s+t}\in\Lambda}\frac{1}{r+s+t+1}, & \text{if } \sup\limits_{\gamma_{r+s+t}\in\Lambda}\frac{1}{r+s+t+1}<0.5,\\ 1-\sup\limits_{\gamma_{r+s+t}\in\Lambda^C}\frac{1}{r+s+t+1}, & \text{if } \sup\limits_{\gamma_{r+s+t}\in\Lambda^C}\frac{1}{r+s+t+1}<0.5,\\ 0.5, & \text{otherwise},\end{cases}\tag{3.1}$$

also

$$\zeta_{rst}(\gamma)=\begin{cases}(r+s+t+1)\,i, & \text{if } \gamma=\gamma_{r+s+t},\\ 0, & \text{otherwise,}\end{cases}$$

describes the complex uncertain variables for $r,s,t\in\mathbb{N}$ and $\zeta(\gamma)\equiv 0$, $\forall\gamma\in\Gamma$. Take $b_r=r, d_s=s, f_t=t$ and $a_r=c_s=e_t=0$. For some small numbers $\varepsilon,\varrho,\delta>0$ and $r,s,t\geq 2$, we have

$$\Big\{(r,s,t)\in\mathbb{N}^3:\frac{1}{\beta_r\gamma_s\eta_t}\left|\{(j,k,l):a_r<j\leq b_r, c_s<k\leq d_s, e_t<l\leq f_t;\right.$$
$$\left.\mathcal{M}\left(\|\zeta_{jkl}-\zeta\|\geq\varepsilon\right)\geq\varrho\}\right|\geq\delta\}$$

$$=\left\{(r,s,t):\frac{1}{rst}\left|\{(j,k,l):j\leq r,k\leq s,l\leq t;\ \mathcal{M}\left(\gamma:\|\zeta_{jkl}(\gamma)-\zeta(\gamma)\|\geq\varepsilon\right)\geq\varrho\}\right|\geq\delta\right\}$$
$$=\left\{(r,s,t)\in\mathbb{N}^3:\frac{1}{rst}\left|\{(j,k,l):j\leq r,k\leq s,l\leq t;\ \mathcal{M}\{\gamma=\gamma_{j+k+l}\}\geq\varrho\}\right|\geq\delta\right\}$$
$$=\left\{(r,s,t)\in\mathbb{N}^3:\frac{1}{rst}\left|\left\{(j,k,l):j\leq r,k\leq s,l\leq t;\ \frac{1}{j+k+l+1}\geq\varrho\right\}\right|\geq\delta\right\}\in\mathcal{I}_3.$$

Thus, the complex triple sequence $\{\zeta_{rst}\}$ deferred $\mathcal{I}_3$-statistically converges in measure to ζ.

Φ_{rst} is the uncertainty distribution function for $r,s,t\geq 2$ and of the complex uncertain variable $\|\xi_{rst}-\xi\|=\|\xi_{rst}\|$. This is,

$$\Phi_{rst}(x)=\begin{cases}0, & \text{if } x<0,\\ \left(1-\frac{1}{r+s+t+1}\right), & \text{if } 0\leq x<(r+s+t+1),\\ 1, & \text{if } x\geq r+s+t+1.\end{cases}\tag{3.2}$$

And,

$$\Big\{(r,s,t)\in\mathbb{N}^3:\frac{1}{\beta_r\gamma_s\eta_t}\left|\{(j,k,l):a_r<j\leq b_r, c_s<k\leq d_s, e_t<l\leq f_t;\right.$$
$$\left.E\left(\|\zeta_{jkl}-\zeta\|-1\right)\geq\varepsilon\}\right|\geq\varrho\}$$

$$=\Big\{(r,s,t):\frac{1}{rst}\Big|\{(j,k,l):j\leq r,k\leq s,l\leq t;$$
$$\left.\left.\left(\left[\int_0^{j+k+l+1}1-\left(1-\frac{1}{j+k+l+1}\right)dx\right]-1\right)\geq\varepsilon\right\}\right|\geq\varrho\Big\}.$$

Hence, for each $r,s,t\geq 2$, and for every $\varepsilon,\varrho>0$, we have

$$\Big\{(r,s,t)\in\mathbb{N}^3:\frac{1}{\beta_r\gamma_s\eta_t}\left|\{(j,k,l):a_r<j\leq b_r, c_s<k\leq d_s, e_t<l\leq f_t;\right.$$
$$\left.E\left(\|\zeta_{jkl}-\zeta\|\right)\geq\varepsilon\}\right|\geq\varrho\}$$
$$=\left\{(r,s,t)\in\mathbb{N}^3:\frac{1}{rst}\left|\{(j,k,l):j\leq r,k\leq s,l\leq t\}\right|\geq\varrho\right\}\in\mathcal{F}(\mathcal{I}_3)$$

which is impossible. So, the complex uncertain triple sequence $\{\zeta_{rst}\}$ doesn't deferred $\mathcal{I}_3$-statistically converges in mean to ζ.

Theorem 3.8. *Let the complex uncertain triple sequence $\{\zeta_{rst}\}$ where $\{\xi_{rst}\}$ is the real part and $\{\eta_{rst}\}$ is the imaginary part, for $r,s,t\in\mathbb{N}$. When uncertain triple sequences $\{\xi_{rst}\}$ and $\{\eta_{rst}\}$ deferred $\mathcal{I}_3$-statistically converges to ξ as measure and γ, respectively, complex uncertain triple sequence $\{\zeta_{rst}\}$ deferred $\mathcal{I}_3$-statistically converges to $\zeta=\xi+i\eta$ as measure.*

Proof. Let $\{\xi_{rst}\}$ and $\{\eta_{rst}\}$ deferred $\mathcal{I}_3$-statistically converges to ξ and η respectively in measure. Then for any small numbers $\varepsilon, \varrho, \delta > 0$,

$$\left\{(r,s,t) \in \mathbb{N}^3 : \frac{1}{\beta_r \gamma_s \eta_t} \left|\{(j,k,l) : a_r < j \leq b_r, c_s < k \leq d_s, e_t < l \leq f_t; \mathcal{M}\left(\|\xi_{jkl} - \xi\| \geq \frac{\varepsilon}{\sqrt{2}}\right) \geq \varrho\right\}\right| \geq \delta\right\} \in \mathcal{I}_3$$

and

$$\left\{(r,s,t) \in \mathbb{N}^3 : \frac{1}{\beta_r \gamma_s \eta_t} \left|\{(j,k,l) : a_r < j \leq b_r, c_s < k \leq d_s, e_t < l \leq f_t; \mathcal{M}\left(\|\eta_{jkl} - \eta\| \geq \frac{\varepsilon}{\sqrt{2}}\right) \geq \varrho\right\}\right| \geq \delta\right\} \in \mathcal{I}_3.$$

Note that $\|\zeta_{rst} - \zeta\| = \sqrt{|\xi_{rst} - \xi|^2 + |\eta_{rst} - \eta|^2}$. Therefore, we have

$$\{\|\zeta_{rst} - \zeta\| \geq \varepsilon\} \subset \left\{\|\xi_{rst} - \xi\| \geq \frac{\varepsilon}{\sqrt{2}} \cup \|\eta_{rst} - \eta\| \geq \frac{\varepsilon}{\sqrt{2}}\right\}.$$

Taking advantage of the subadditivity axiom of uncertain measure, we get

$$\left\{(r,s,t) \in \mathbb{N}^3 : \frac{1}{\beta_r \gamma_s \eta_t} |\{(j,k,l) : a_r < j \leq b_r, c_s < k \leq d_s, e_t < l \leq f_t; \mathcal{M}\left(\|\zeta_{jkl} - \zeta\| \geq \varepsilon\right) \geq \varrho\}| \geq \delta\right\}$$

$$\subseteq \left\{(r,s,t) \in \mathbb{N}^3 : \frac{1}{\beta_r \gamma_s \eta_t} \left|\{(j,k,l) : a_r < j \leq b_r, c_s < k \leq d_s, e_t < l \leq f_t; \mathcal{M}\left(\|\xi_{jkl} - \xi\| \geq \frac{\varepsilon}{\sqrt{2}}\right) \geq \varrho\right\}\right| \geq \delta\right\}$$

$$\cup \left\{(r,s,t) \in \mathbb{N}^3 : \frac{1}{\beta_r \gamma_s \eta_t} \left|\{(j,k,l) : a_r < j \leq b_r, c_s < k \leq d_s, e_t < l \leq f_t; \mathcal{M}\left(\|\eta_{jkl} - \eta\| \geq \frac{\varepsilon}{\sqrt{2}}\right) \geq \varrho\right\}\right| \geq \delta\right\} \in \mathcal{I}_3.$$

Hence, we have

$$\left\{(r,s,t) \in \mathbb{N}^3 : \frac{1}{\beta_r \gamma_s \eta_t} |\{(j,k,l) : a_r < j \leq b_r, c_s < k \leq d_s, e_t < l \leq f_t; \mathcal{M}\left(\|\zeta_{jkl} - \zeta\| \geq \varepsilon\right) \geq \varrho\}| \geq \delta\right\} \in \mathcal{I}_3.$$

That is, $\{\zeta_{rst}\}$ deferred $\mathcal{I}_3$-statistically converges to ζ in measure. □

To prove Theorem 3.10, we give the following definition :

Definition 3.9. (i) There exists a $o \in \mathbb{R}$ so that $M_o \notin \mathcal{I}_3$, then we have

$$D\mathcal{I}_3 - \limsup x = \sup\{o \in \mathbb{R} : M_o \notin \mathcal{I}_3\} \tag{3.3}$$

where

$$M_o = \left\{(r,s,t) \in \mathbb{N}^3 : \frac{1}{\beta_r \gamma_s \eta_t} |\{(j,k,l) : a_r < j \leq b_r, c_s < k \leq d_s, e_t < l \leq f_t; x_{jkl} > o\}| > \gamma\right\}.$$

When $M_o \in \mathcal{I}_3$ takes for each one $o \in \mathbb{R}$, we have $D\mathcal{I}_3 - \limsup x = -\infty$.

(ii) There exists a $o \in \mathbb{R}$ so that $M^o \notin \mathcal{I}_3$ then we have

$$D\mathcal{I}_3 - \liminf x = \inf \{o \in \mathbb{R} : M^o \notin \mathcal{I}_3\} \tag{3.4}$$

where

$$M^o = \left\{(r,s,t) \in \mathbb{N}^3 : \frac{1}{\beta_r \gamma_s \eta_t} |\{(j,k,l) : a_r < j \leq b_r, c_s < k \leq d_s, e_t < l \leq f_t; \right.$$
$$\left. x_{jkl} < o\}| > \gamma\right\}.$$

When $M^o \in \mathcal{I}_3$ takes for each one $o \in \mathbb{R}$, we have $D\mathcal{I}_3 - \liminf x = +\infty$.

Theorem 3.10. *Let consider the complex uncertain triple sequence $\{\zeta_{rst}\}$ with the real part $\{\xi_{rst}\}$ and the imaginary part $\{\eta_{rst}\}$ for all $r, s, t \in \mathbb{N}$. When uncertain triple sequences $\{\xi_{rst}\}$ and $\{\eta_{rst}\}$ deferred $\mathcal{I}_3$-statistically converges to ξ as measure and η, respectively, complex uncertain triple sequence $\{\zeta_{rst}\}$ deferred $\mathcal{I}_3$-statistically converges to $\zeta = \xi + i\eta$ as distribution.*

Proof. The complex uncertainty distribution Φ should have a definite point of continuity $z = u + iv$. Otherwise, we have

$$\{\xi_{rst} \leq u, \eta_{rst} \leq v\} = \{\xi_{rst} \leq u, \eta_{rst} \leq v, \xi \leq \widehat{x}, \eta \leq \widehat{y}\}$$
$$\cup \{\xi_{rst} \leq u, \eta_{rst} \leq v, \xi > \widehat{x}, \eta > \widehat{y}\}$$

$$\cup \{\xi_{rst} \leq u, \eta_{rst} \leq v, \xi \leq \widehat{x}\ , \eta > \widehat{y}\} \cup \{\xi_{rst} \leq u, \eta_{rst} \leq v, \xi > \widehat{x}, \eta \leq \widehat{y}\}$$
$$\subset \{\xi \leq u, \eta \leq v\} \cup \{|\xi_{rst} - \xi| \geq \widehat{x} - u\} \cup \{|\eta_{rst} - \eta| \geq \widehat{y} - v\}$$

for any $\widehat{x} > u, \widehat{y} > v$. From here with the axiom of subadditivity,

$$\Phi_{rst}(z) = \Phi_{rst}(u + iv)$$
$$\leq \Phi(\widehat{x} + i\widehat{y}) + \mathcal{M}\{|\xi_{rst} - \xi| \geq \widehat{x} - u\} + \mathcal{M}\{|\eta_{rst} - \eta| \geq \widehat{y} - v\}.$$

Since $\{\xi_{rst}\}$ and $\{\eta_{rst}\}$ deferred $\mathcal{I}_3$-statistically converge to ξ as measure and η, respectively, hence, for every small numbers $\varepsilon, \varrho > 0$, we get

$$\left\{(r,s,t) \in \mathbb{N}^3 : \frac{1}{\beta_r \gamma_s \eta_t} |\{(j,k,l) : a_r < j \leq b_r, c_s < k \leq d_s, e_t < l \leq f_t; \right.$$
$$\left. \mathcal{M}(\|\xi_{jkl} - \xi\| \geq \widehat{x} - u) \geq \varepsilon\}| \geq \varrho\right\} \in \mathcal{I}_3$$

and

$$\left\{(r,s,t) \in \mathbb{N}^3 : \frac{1}{\beta_r \gamma_s \eta_t} |\{(j,k,l) : a_r < j \leq b_r, c_s < k \leq d_s, e_t < l \leq f_t; \right.$$
$$\left. \mathcal{M}(\|\eta_{jkl} - \eta\| \geq \widehat{y} - v) \geq \varepsilon\}| \geq \varrho\right\} \in \mathcal{I}_3.$$

Thus, we obtain

$$D\mathcal{I}_3 - \limsup_{r,s,t \to \infty} \Phi_{r,s,t}(z) \leq \Phi(\widehat{x} + i\widehat{y})$$

for any $\widehat{x} > u, \widehat{y} > v$. Letting $\widehat{x} + i\widehat{y} \to u + iv$, we get

$$D\mathcal{I}_3 - \limsup_{r,s,t \to \infty} \Phi_{r,s,t}(z) \leq \Phi(z). \tag{3.5}$$

Furthermore, we have

$$\{\xi \leq a, \eta \leq b\} = \{\xi_{rst} \leq u, \eta_{rst} \leq v, \xi \leq a, \eta \leq b\}$$
$$\cup \{\xi_{rst} \leq u, \eta_{rst} > v, \xi \leq a, \eta \leq b\}$$
$$\cup \{\xi_{rst} > u, \eta_{rst} \leq v, \xi \leq a, \eta \leq b\}$$
$$\cup \{\xi_{rst} > u, \eta_{rst} > v, \xi \leq a, \eta \leq b\}$$

$$\subset \{\xi_{rst} \le u, \eta_{rst} \le v\} \cup \{|\xi_{rst} - \xi| \ge u - a\} \cup \{|\eta_{rst} - \eta| \ge v - b\}$$

for any $a < u, b < v$. This means,

$$\Phi(a + ib) \le \Phi_{rst}(u + iv) + \mathcal{M}\{\|\xi_{rst} - \xi\| \ge u - a\} + \mathcal{M}\{\|\eta_{rst} - \eta\| \ge v - b\}.$$

Therefore, one can see that

$$\Big\{(r,s,t) \in \mathbb{N}^3 : \frac{1}{\beta_r\gamma_s\eta_t} |\{(j,k,l) : a_r < j \le b_r, c_s < k \le d_s, e_t < l \le f_t;$$
$$\mathcal{M}(\|\zeta_{jkl} - \zeta\| \ge u - a) \ge \varepsilon\}| \ge \varrho\} \in \mathcal{I}_3$$

and

$$\Big\{(r,s,t) \in \mathbb{N}^3 : \frac{1}{\beta_r\gamma_s\eta_t} |\{(j,k,l) : a_r < j \le b_r, c_s < k \le d_s, e_t < l \le f_t;$$
$$\mathcal{M}(\|\eta_{jkl} - \eta\| \ge v - b) \ge \varepsilon\}| \ge \varrho\} \in \mathcal{I}_3.$$

we gain

$$\Phi(a + ib) \le D\mathcal{I}_3 - \liminf_{r,s,t\to\infty} \Phi_{rst}(u + iv)$$

for any $a < u, b < v$. Taking $a + ib \to u + iv$, we get

$$\Phi(z) \le D\mathcal{I}_3 - \liminf_{r,s,t\to\infty} \Phi_{rst}(z) \tag{3.6}$$

From (3.5) and (3.6) we find $\Phi_{rst}(z) \to \Phi(z)$ as $r, s, t \to \infty$. This is the complex uncertain triple sequence $\{\zeta_{rst}\}$ and it is deferred $\mathcal{I}_3$-statistically convergent in distribution to $\zeta = \xi + i\eta$. □

Deferred $\mathcal{I}_3$-statistically convergence in distribution doesn't allude to the deferred $\mathcal{I}_3$-statistically convergence in measure. This indicates that the contrary of the Theorem 3 does not necessarily hold.

To see this, Example 3.11 may be useful.

Example 3.11. For $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$, there are eight events. Define

$$\mathcal{M}\{\gamma_1\} = 0.6, \quad \mathcal{M}\{\gamma_2\} = 0.3, \quad \mathcal{M}\{\gamma_3\} = 0.2, \quad \mathcal{M}\{\gamma_1, \gamma_2\} = 0.8,$$
$$\mathcal{M}\{\gamma_1, \gamma_3\} = 0.7, \quad \mathcal{M}\{\gamma_2, \gamma_3\} = 0.4, \quad \mathcal{M}\{\emptyset\} = 0, \quad \mathcal{M}\{\Gamma\} = 1.$$

Obviously, the set function $\mathcal{M}$ is an uncertain measure. We describe the complex uncertain variables as

$$\zeta_{rst}(\gamma) = \begin{cases} i, & \text{if } \gamma = \gamma_1, \\ -i, & \text{if } \gamma = \gamma_2, \\ 2i, & \text{if } \gamma = \gamma_3. \end{cases} \tag{3.7}$$

We describe $-\zeta_{rst} = \zeta$ for $r, s, t \in \mathbb{N}$. Take $b_r = r, d_s = s, f_t = t$ and $a_r = c_s = e_t = 0$. Therefore, by Examples 1.1 and 1.5 in [19] we can say $\{\zeta_{rst}\}$ deferred $\mathcal{I}_3$-statistically converges in distribution to ζ. But, the complex uncertain triple sequence $\{\zeta_{rst}\}$ doesn't deferred $\mathcal{I}_3$-statistically converge in measure to ζ.

Deferred $\mathcal{I}_3$-statistically convergence almost surely doesn't indicate deferred $\mathcal{I}_3$-statistically convergence in measure.

Example 3.12. With Borel algebra and Lebesque measure, take the uncertainty space $(\Gamma, L, \mathcal{M})$ as $[0, 1]$. There are integers y_1, y_2 and y_3 such that $r = 2^{y_1} + p$, $s = 2^{y_2} + p$ and $t = 2^{y_3} + p \in \mathbb{N}$, for any positive integer r, s, t where p is an integer between 0 and $\min\{2^{y_1}, 2^{y_2}, 2^{y_3}\} - 1$. If so, we determine a complex uncertain variable by

$$\zeta_{rst}(\gamma) = \begin{cases} i, & \text{if } \frac{p}{2^{y_1+y_2+y_3}} \le \gamma \le \frac{p+1}{2^{y_1+y_2+y_3}}, \\ 0, & \text{otherwise}, \end{cases} \tag{3.8}$$

for $r,s,t\in\mathbb{N}$ and $\zeta\equiv 0$. Take $b_r=r, d_s=s, f_t=t$ and $a_r=c_s=e_t=0$. For some small numbers $\varepsilon,\varrho,\delta>0$ and $r,s,t\geq 2$, we have

$$\Big\{(r,s,t)\in\mathbb{N}^3:\frac{1}{\beta_r\gamma_s\eta_t}\left|\{(j,k,l):a_r<j\leq b_r,c_s<k\leq d_s,e_t<l\leq f_t;\right.$$
$$\mathcal{M}\left(\|\xi_{jkl}-\xi\|\geq\varepsilon\right)\geq\varrho\}|\geq\delta\}$$
$$=\Big\{(r,s,t)\in\mathbb{N}^3:\frac{1}{\beta_r\gamma_s\eta_t}\left|\{(j,k,l):a_r<j\leq b_r,c_s<k\leq d_s,e_t<l\leq f_t;\right.$$
$$\mathcal{M}\left(\gamma:\|\zeta_{jkl}(\gamma)-\zeta(\gamma)\|\geq\varepsilon\right)\geq\varrho\}|\geq\delta\}$$
$$=\left\{(r,s,t)\in\mathbb{N}^3:\frac{1}{rst}\left|\{(j,k,l):j\leq r,k\leq s,l\leq t;\ \mathcal{M}\{\gamma_{jkl}\}\geq\varrho\}\right|\geq\delta\right\}\in\mathcal{I}_3.$$

Thus, the triple sequence $\{\zeta_{rst}\}$ deferred $\mathcal{I}_3$-statistically converges to ζ as measure. Also, for all $\varepsilon,\varrho>0$, we have

$$\left\{(r,s,t)\in\mathbb{N}^3:\frac{1}{rst}\left|\{(j,k,l):j\leq r,k\leq s,l\leq t;\ E\left(\|\zeta_{jkl}-\zeta\|\right)\geq\varepsilon\}\right|\geq\varrho\right\}\in\mathcal{I}_3$$

Hence, the sequence $\{\zeta_{rst}\}$ also deferred $\mathcal{I}_3$-statistically converges in mean to ζ. But, there exists an infinite number of intervals of the form $\left[\frac{p}{2^{y_1+y_2+y_3}},\frac{p+1}{2^{y_1+y_2+y_3}}\right]$ containing γ, for any $\gamma\in[0,1]$. Thus, $\zeta_{rst}(\gamma)$ doesn't deferred $\mathcal{I}_3$-statistically converge to 0. Another way, the triple uncertain sequence $\{\zeta_{rst}\}$ doesn't deferred $\mathcal{I}_3$-statistically converge a.s. to ζ. This completes the proof.

Deferred $\mathcal{I}_3$-statistically convergence a.s. doesn't allude to the deferred $\mathcal{I}_3$-statistically convergence in mean.

Example 3.13. Take into account the uncertainty space $(\Gamma,L,\mathcal{M})$ to be $\Gamma=\{\gamma_1,\gamma_2,\gamma_3,...\}$ with

$$\mathcal{M}\{\Lambda\}=\sum_{\gamma_d,\gamma_e,\gamma_f\in\Lambda}\frac{1}{2^{d+e+f}}. \tag{3.9}$$

Take $b_r=r,d_s=s,f_t=t$ and $a_r=c_s=e_t=0$. For $r,s,t\in\mathbb{N}$ and $\zeta(\gamma)\equiv 0$, $\forall\gamma\in\Gamma$,

$$\zeta_{rst}(\gamma)=\begin{cases} i2^{r+s+t}, & \text{if } \gamma=\gamma_{r+s+t},\\ 0, & \text{otherwise,}\end{cases} \tag{3.10}$$

is the complex uncertain variables. Thereafter, the sequence $\{\zeta_{rst}\}$ deferred $\mathcal{I}_3$-statistically converges a.s. to ζ. But, the uncertainty distributions of $\|\zeta_{rst}\|$ are

$$\Phi_{rst}(x)=\begin{cases} 0, & \text{if } x<0,\\ 1-\frac{1}{2^{r+s+t}}, & \text{if } 0\leq x<2^{r+s+t}\\ 1 & \text{if } x\geq 2^{r+s+t},\end{cases} \tag{3.11}$$

for $r,s,t\in\mathbb{N}$, respectively. Then, we have

$$\Big\{(r,s,t)\in\mathbb{N}^3:\frac{1}{\beta_r\gamma_s\eta_t}\left|\{(j,k,l):a_r<j\leq b_r,c_s<k\leq d_s,e_t<l\leq f_t;\right.$$
$$E\left(\|\zeta_{jkl}-\zeta\|\right)\geq 1\}|\geq\varrho\}$$
$$=\left\{(r,s,t)\in\mathbb{N}^3:\frac{1}{rst}\left|\{(j,k,l):j\leq r,k\leq s,l\leq t;\ E\left(\|\zeta_{jkl}-\zeta\|\right)\geq 1\}\right|\geq\varrho\right\}\in\mathcal{I}_3.$$

Therefore, the complex uncertain triple sequence $\{\zeta_{rst}\}$ doesn't deferred $\mathcal{I}_3$-statistically converge in mean to ζ.

From Example 3.13, we can acquire that deferred $\mathcal{I}_3$-statistically convergence in mean doesn't allude to the deferred $\mathcal{I}_3$-statistically converge a.s.

Example 3.14. Take into account the uncertainty space $(\Gamma, L, \mathcal{M})$ to be $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ with

$$\mathcal{M}\{\Lambda\} = \begin{cases} 0, & \text{if } \Lambda = \emptyset \\ 0.6, & \text{if } \gamma_1 \in \Lambda \\ 0.4, & \text{if } \gamma_1 \notin \Lambda \\ 1, & \text{if } \Lambda = \Gamma. \end{cases} \tag{3.12}$$

Take $b_r = r, d_s = s, f_t = t$ and $a_r = c_s = e_t = 0$. For $r, s, t \in \mathbb{N}$ and $\zeta(\gamma) \equiv 0, \forall \gamma \in \Gamma$,

$$\zeta_{rst}(\gamma) = \begin{cases} i, & \text{if } \gamma = \gamma_1 \\ 2i, & \text{if } \gamma = \gamma_2 \\ 3i, & \text{if } \gamma = \gamma_3 \\ 4i, & \text{if } \gamma = \gamma_4 \\ 0, & \text{otherwise} \end{cases} \tag{3.13}$$

is the complex uncertain variables. We find that the sequence $\{\zeta_{rst}\}$ is deferred $\mathcal{I}_3$-statistically convergent to ζ with respect to almost surely, but the sequence $\{\zeta_{rst}\}$ is not deferred $\mathcal{I}_3$-statistically convergent in the measure.

If the complex uncertainty distribution of the complex uncertain variables ζ_{rst} and ζ are $\Phi_{rst}(z)$ and $\Phi(z)$ respectively in the uncertainty space taken in Example 3.14, then for $r, s, t \in \mathbb{N}$

$$\Phi_{rst}(z) = \begin{cases} 0, & \text{if } a < 0, \ -\infty < b < \infty \\ 0, & \text{if } a \geq 0, \ b < 1 \\ 0.6, & \text{if } a \geq 0, \ 1 \leq b < 4 \\ 1 & \text{if } a \geq 0, \ b \geq 4 \end{cases} \tag{3.14}$$

and

$$\Phi(z) = \begin{cases} 0, & \text{if } a < 0, \ -\infty < b < \infty \\ 0, & \text{if } a \geq 0, \ b < 0 \\ 1 & \text{if } a \geq 0, \ b \geq 0. \end{cases} \tag{3.15}$$

From the examples given above, it is obvious that the complex uncertain triple sequence $\{\zeta_{rst}\}$ isn't deferred $\mathcal{I}_3$-statistically converge in distribution to ζ.

4 Conclusions

Here, the study of deferred ideal statistical convergence in mean, in measure, in distribution, with respect to almost surely of a complex uncertain triple sequence has been made and interrelationships among them were established. These concepts can be generalized and applied for further studies. For example, this study can be extended by introducing deferred $\mathcal{I}_3$-statistically Cauchy triple sequence of complex uncertain variables. In further studies, the deferred ideal invariant convergence by using triple sequences can be defined and examined for complex uncertain sequence.

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