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A new class of Kählerian manifolds

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Abstract In this paper, we introduce a new class of Kählerian manifolds and study their essential examples as well as their fundamental properties. Next, we investigate a particular type belonging to this class and we establish some basic results for Riemannian curvature tensor. Concrete examples are given.

1 Introduction

The theory of structures on manifolds is a very interesting topic of modern differential geometry and the differential geometric aspects of submanifolds of manifolds with certain structures are vast and very fruitful fields for Riemannian geometry.

Kähler structures were introduced and studied by Erich Kähler in 1933 [6]. But it was not until late the 1940s when the importance of Kähler manifolds in both Riemannian and algebraic geometry was finally realized. Throughout the 1950s they were studied by such giants as Borel, Chern, Hodge, Kobayashi, Kodaira and other authors.

In the present paper we study exact Kähler manifolds (M, J, η, g) as a class of Kähler manifolds with exact Kähler form Ω (i.e. $\Omega = \mathrm{d}\eta$).

The paper is organized as follows. In Section 2, we review basic definitions and results that are needed to state and prove our results. In Section 3, we introduce the basic notion of an exact Kähler manifold and we give a concrete example. In Section 4, we discuss transformations of exact Kähler metrics into exact Kähler ones. We introduce Biconformal transformations of the structure (J, η, g). In Section 5, we state and proof the construction of an exact Kählerian manifold starting from a Sasakian manifold with examples. In Section 6, we establish an interesting class of exact Kähler manifolds such as the η-Kähler manifolds and we give a correspondence between this class and Sasakian manifolds. In the last section, we investigate two particular types of η-Kähler manifolds and study some of their basic properties.

2 Review of needed notions

Let (M^n, g) be a Riemannian manifold. The Lie algebra of all C^∞ vector fields on M will be denoted by $\mathfrak{X}(M)$. We denote by R and S the Riemannian tensor and the Ricci tensor respectively defined for all $X, Y, Z \in \mathfrak{X}(M)$ by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \tag{2.1}$$

$$\begin{aligned} S(X,Y) &= \sum_{i=1}^{n} R(e_i, X, Y, e_i) \\ &= \sum_{i=1}^{n} g\big(R(e_i, X)Y, e_i\big), \end{aligned} \tag{2.2}$$

where $\{e_i\}_{i=1,n}$ is a local orthonormal basis.

An odd-dimensional Riemannian manifold (M^{2n+1}, g) is said to be an almost contact metric manifold if there exist on M a $(1,1)$ tensor field φ, a vector field ξ (called the structure vector field) and a 1-form η such that

$$\eta(\xi) = 1, \quad \varphi^2(X) = -X + \eta(X)\xi, \quad and \quad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y),$$

for any vector fields X,Y on M. In particular, in an almost contact metric manifold we also have $\varphi\xi = 0$ and $\eta \circ \varphi = 0$.

Such a manifold is said to be a contact metric manifold if $d\eta = \phi$, where $\phi(X,Y) = g(X, \varphi Y)$ is called the fundamental 2-form of M. In addition, if ξ is a Killing vector field, then M is said to be a K-contact manifold. It is well-known that a contact metric manifold is a K-contact manifold if and only if $\nabla_X \xi = -\varphi X$, for any vector field X on M.

On the other hand, the almost contact metric structure of M is said to be normal if

$$[\varphi, \varphi](X,Y) + 2d\eta\,(X,Y)\xi = 0,$$

for any X, Y, where $[\varphi, \varphi]$ denotes the Nijenhuis torsion of φ, given by

$$[\varphi, \varphi](X,Y) = \varphi^2[X,Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]$$

A normal contact metric manifold is called a Sasakian manifold. It can be proved that a Sasakian manifold is K-contact, and that an almost contact metric manifold is Sasakian if and only if for any X,Y on M

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X. \tag{2.3}$$

For more background on almost contact metric manifolds and recent study of η-Einstein manifolds, we recommend the reference [4] and [5].

An almost complex manifold with a Hermitian metric is called an almost Hermitian manifold. For an almost Hermitian manifold $(\overline{M}^{2n}, J, \overline{g})$ we have

$$J^2 = J \circ J = -Id, \qquad \overline{g}(JX, JY) = \overline{g}(X,Y).$$

An almost complex stucture J is integrable, and hence the manifold is a complex manifold, if and only if its Nijenhuis tensor N_J vanishes, with

$$N_J(X,Y) = [JX, JY] - [X,Y] - J[X, JY] - J[JX, Y].$$

For an almost Hermitian manifold $(\overline{M}, J, \overline{g})$, we define the fundamental Kähler form $\overline{\Omega}$ as:

$$\overline{\Omega}(X,Y) = \overline{g}(X, JY).$$

$(\overline{M}, J, \overline{g})$ is then called almost Kähler if $\overline{\Omega}$ is closed i.e. $d\overline{\Omega} = 0$. It can be shown that this condition for $(\overline{M}, J, \overline{g})$ to be almost Kähler is equivalent to

$$\overline{g}\big((\overline{\nabla}_X J)Y, Z\big) + \overline{g}\big((\overline{\nabla}_Y J)Z, X\big) + \overline{g}\big((\overline{\nabla}_Z J)X, Y\big) = 0.$$

An almost Kähler manifold with integrable J is called a Kähler manifold, and thus is characterized by the conditions: $d\overline{\Omega} = 0$ and $N_J = 0$. One can prove that these both conditions combined are equivalent with the single condition

$$\overline{\nabla} J = 0.$$

For more background on almost complex manifolds, we recommend the reference [11].

Torse forming vector fields were introduced by K.Yano [10] satisfies

$$\nabla_X \xi = aX + \eta(X)\xi, \tag{2.4}$$

for some smooth function a and 1-form η on M where ∇ denotes the Levi-Civita connection of a Riemannian metric. The 1-form η is called the generating form and the function a is called the conformal scalar.

Further, a complex analogue of a torse forming vector field is called K-torse forming vector field and it was introduced by S.Yamaguchi [12] and S.Yamaguchi and W. N. Yu [13],

$$\nabla_X \xi = aX + bJX + \eta_1(X)\xi + \eta_2(X)J\xi, \tag{2.5}$$

where a and b are functions, η_1 and η_2 are 1-forms on M. The functions a and b (resp. 1-forms η_1 and η_2) appearing in (2.5) will be called the associated functions (resp. forms) of ξ. Moreover if the associated functions a and b satisfy $a^2 + b^2 \neq 0$ in M, then we call such a vector field a proper K-torse-forming vector field. For the existence of torse-forming vector field on Riemannian manifold see for example [12].

Definition 2.1. A special K-torse forming vector field on a Hermitian manifold (M, g) is a K-torse forming vector field ξ satisfying the equation (2.5) with associated forms

$$\eta_1(X) = \alpha g(\xi, X) + \mu g(J\xi, X) \quad and \quad \eta_2(X) = \lambda g(\xi, X) + \beta g(J\xi, X)$$

for all X vector field on M and α, β, λ and μ are smooth functions on M, that is

$$\nabla_X \xi = aX + bJX + \alpha g(X, \xi)\xi + \mu g(X, J\xi)\xi + \lambda g(X, \xi)J\xi + \beta g(X, J\xi)J\xi. \tag{2.6}$$

Putting $\eta(X) = g(X, \xi)$ and $\tilde{\eta} = -\eta \circ J$ i.e. $\forall X \in \mathfrak{X}(M)$

$$\tilde{\eta}(X) = -\eta(JX) = -g(JX, \xi) = g(X, J\xi),$$

which implies $\tilde{\xi} = J\xi$ then, (2.6) becomes

$$\nabla_X \xi = aX + bJX + \alpha\eta(X)\xi + \lambda\eta(X)\tilde{\xi} + \mu\tilde{\eta}(X)\xi + \beta\tilde{\eta}(X)\tilde{\xi}. \tag{2.7}$$

For the existence of special torse-forming vector field on Kählerian manifold see example 6.2.

3 Exact Kähler manifold

Let (M, J, g) be a $(2n + 2)$-dimensional noncompact Kählerian manifold and ξ a global vector field on M. We denote by η and $\tilde{\eta}$ the 1-forms corresponding to ξ and $\tilde{\xi} = J\xi$, respectively, i.e $\forall X \in \mathfrak{X}(M)$,

$$\eta(X) = g(\xi, X), \qquad \tilde{\eta}(X) = g(\tilde{\xi}, X) = -\eta(JX),$$

with $\eta(\xi) = g(\xi, \xi) = \mathrm{e}^{2\rho}$ where ρ is a function on M.

Denote by $\mathcal{D}$ the J-invariant distribution of dim $\mathcal{D} = 2n$ defined by

$$\mathcal{D} = \{X \in T_pM : \eta(X) = \tilde{\eta}(X) = 0\}.$$

In the presence of the distribution $\mathcal{D}$, the structure of any tangent space is $T_pM = \mathcal{D}(p) \oplus \mathcal{D}^\perp(p)$, where $\mathcal{D}^\perp(p)$ is the 2-dimensional J-invariant orthogonal complement to the space $\mathcal{D}(p)$. This means that the structural group of the manifolds under consideration is the subgroup $U(n-1) \times U(1)$ of $U(n)$.

Definition 3.1. (M, J, η, g) is said to be an **exact Kähler manifold** if

$$\mathrm{d}\eta = \Omega \quad and \quad \mathrm{d}\tilde{\eta} = 0. \tag{3.1}$$

where d denotes the exterior derivative and Ω is the fundamental Kähler form.

To shown the existence of such manifolds, we give an interesting class of examples:

Example 3.2. In [7], Olszak gave an important example of a 1-parameter family of Kählerian manifolds. We will show that this family is a family of exact Kählerian manifolds.

Let $(x^\alpha, y^\alpha, z, t)$ denote the Cartesian coordinates in $\mathbb{R}^{2m+2}$, $m \geq 1$. Latin indices take on values from 1 to $2m + 2$, greek indices will run from 1 to m and $\alpha' = \alpha + m$ for all $\alpha \in \{1, ..., m\}$. Assume that $M = N \times (A, B) \subset \mathbb{R}^{2m+2}$ where N is an open connected subset of $\mathbb{R}^{2m+1}$ and (A, B) is an open interval and $B > A > 0$. Suppose that $h : (A, B) \to \mathbb{R}$ is a smooth function

which is non-zero at any $t \in (A, B)$.
Let (e_i) be the frame of vector fields on M defined by:

$$e_\alpha = \frac{1}{t}\frac{\partial}{\partial x^\alpha},\ e_{\alpha'} = \frac{1}{t}(\frac{\partial}{\partial y^\alpha} + 2x^\alpha\frac{\partial}{\partial z}),\ e_{2m+1} = \frac{1}{t^2h}\frac{\partial}{\partial z},\ e_{2m+2} = th\frac{\partial}{\partial t},$$

and let (θ^i) be the dual frame of differential 1-forms,

$$\theta^\alpha = tdx^\alpha,\ \theta^{\alpha'} = tdy^\alpha,\ \theta^{2m+1} = t^2h(-2\sum_\lambda x^\lambda dy^\lambda + dz),\ \theta^{2m+2} = \frac{1}{th}dt.$$

For the non-zero Lie brackets of (e_i), we have:

$$[e_\alpha, e_{\beta'}] = 2h\delta_{\alpha\beta}e_{2m+1}, \qquad [e_\alpha, e_{2m+2}] = he_\alpha,$$

$$[e_{\alpha'}, e_{2m+2}] = he_{\alpha'}, \qquad [e_{2m+1}, e_{2m+2}] = (2h + th')e_{2m+1}.$$

Define an almost complex structure J on M by assuming

$$Je_\alpha = e_{\alpha'}\ ,\ Je_{\alpha'} = -e_\alpha\ ,\ Je_{2m+1} = e_{2m+2}\ ,\ Je_{2m+2} = -e_{2m+1}$$

For the Nijenhuis tensor N_j, it can be checked that

$$N_j(e_i, e_j) = [Je_i, Je_j] - J[e_i, Je_j] - J[Je_i, e_j] + J^2[e_i, e_j] = 0,$$

for any i, j. By the Newlander-Nirenberg theorem, J is a complex structure on M.
Let g be the Riemannian metric on M for which (e_i) is an orthonormal frame, so that $g = \sum \theta^i \otimes \theta^i$. It is obvious that the pair (J, g) is a Hermitian structure on M. For the fundamental form Ω, $\Omega(X, Y) = g(X, JY)$, we have

$$\begin{aligned}
\Omega &= -2\sum_\lambda \theta^\lambda \wedge \theta^{\bar\lambda} - 2\theta^{2m+1} \wedge \theta^{2m+2} \\
&= -2t^2\sum_\lambda dx^\lambda \wedge dy^\lambda - 2t\big(\sum_\lambda -2x^\lambda dy^\lambda \wedge dt + dz \wedge dt\big) \\
&= \mathrm{d}\Big(t^2\big(-2\sum_\lambda x_\lambda dy^\lambda + dz\big)\Big),
\end{aligned}$$

Putting

$$\eta = t^2\big(-2\sum_\lambda x_\lambda dy^\lambda + dz\big) = \frac{1}{h}\theta^{2m+1},$$

we get

$$\xi = \frac{1}{h}e_{2m+1}, \qquad \tilde\xi = \frac{1}{h}e_{2m+2}, \qquad \tilde\eta = \frac{1}{h}\theta^{2m+2}.$$

Hence Ω is closed, i.e., $\mathrm{d}\Omega = 0$. Thus, the pair (J, g) becomes a Kählerian structure on M. On the other hand, since Ω is exact, i.e., $\mathrm{d}\eta = \Omega$ and $\mathrm{d}\tilde\eta = 0$ then (M, J, η, g) is an exact Kählerian manifold with $\mathrm{e}^\rho = \frac{1}{h}$.

It should be noted here, that every exact Kählerian manifold is a Kählerian manifold but the converse is not true in general. The following example confirms this.

Example 3.3. Let $\{x, y, z, t\}$ denote the Cartesian coordinates on $F^4 := \mathbb{R}^2 \times \mathbb{H}^2$ where $\mathbb{H}^2 = \{\omega \in \mathbb{C} \mid Im(\omega) > 0\}$ denotes the hyperbolic plane. The complex structure J and the Kählerian metric g are given in [3], [8] and [9] by:

$$J\frac{\partial}{\partial x} = \frac{-z}{t}\frac{\partial}{\partial x} - \frac{1}{t}\frac{\partial}{\partial y}, \quad J\frac{\partial}{\partial y} = \frac{z^2 + t^2}{t}\frac{\partial}{\partial x} + \frac{z}{t}\frac{\partial}{\partial y}, \quad J\frac{\partial}{\partial z} = \frac{\partial}{\partial t}, \quad J\frac{\partial}{\partial t} = -\frac{\partial}{\partial z},$$

$$g = \frac{1}{t}(dx - zdy)^2 + tdy^2 + \frac{1}{4t^2}(dz^2 + dt^2).$$

The Kählerian manifold (F^4, J, g) possesses a fundamental 2-form Ω, given by

$$\begin{aligned}\Omega &= 2\mathrm{d}x\wedge\mathrm{d}y - \frac{1}{2t^2}\mathrm{d}z\wedge\mathrm{d}t \\ &= \mathrm{d}\big(2x\mathrm{d}y - \frac{1}{2t}\mathrm{d}z\big),\end{aligned}$$

putting $\eta = 2x\mathrm{d}y - \frac{1}{2t}\mathrm{d}z$ and knowing that $\tilde{\eta} = -\eta\circ J$ we obtain

$$\tilde{\eta} = \frac{2x}{t}\mathrm{d}x - \frac{2xz}{t}\mathrm{d}y - \frac{1}{2t}\mathrm{d}t.$$

We can easily see that $\mathrm{d}\tilde{\eta} \neq 0$. So, (F^4, J, g) is not an exact Kählerian manifold.

4 From Sasakian manifold to exact Kählerian manifold

For this construction, we use our techniques included in [2].

Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a Sasakian manifold and $(\overline{M} = \mathbb{R}\times M, J, \overline{g})$ the 1-parameter family of Kählerian structures defined by (see [1] and [2]):

$$\left\{\begin{array}{l}\overline{g} = dt^2 + f^2 g + f^2(f'^2 - 1)\eta\otimes\eta,\\ J(a\frac{\partial}{\partial t}, X) = \Big(ff'\eta(X)\frac{\partial}{\partial t}\ ,\ \varphi X - \frac{a}{ff'}\xi\Big),\end{array}\right. \tag{4.1}$$

where $f = f(t)\in C^\infty(\mathbb{R})$ and $ff' \neq 0$ anywhere.

The fundamental 2-form $\overline{\Omega}$ of $(J, \overline{g})$ is

$$\overline{\Omega}\Big(\big(a\frac{\partial}{\partial t}, X\big), \big(b\frac{\partial}{\partial t}, Y\big)\Big) = \overline{g}\Big(\big(a\frac{\partial}{\partial t}, X\big), J\big(b\frac{\partial}{\partial t}, Y\big)\Big),$$

we can check that is very simply as follows:

$$\begin{aligned}\overline{\Omega} &= f(2f'\ \mathrm{d}t\wedge\eta + f\phi)\\ &= \mathrm{d}\big(f^2\eta\big),\end{aligned}$$

where ϕ denotes the fundamental 2-form of (φ, ξ, η, g). Putting

$$\overline{\eta} = f^2\eta,$$

we get

$$\tilde{\overline{\eta}} = -\overline{\eta}\circ J = -f^2\eta\circ J.$$

Since the vanishing of the differential 2-form $\mathrm{d}\tilde{\overline{\eta}}$ is a necessary condition for this construction, from formula

$$2\mathrm{d}\tilde{\overline{\eta}}(\overline{X}, \overline{Y}) = \overline{X}\big(\tilde{\overline{\eta}}(\overline{Y})\big) - \overline{Y}\big(\tilde{\overline{\eta}}(\overline{X})\big) - \tilde{\overline{\eta}}([\overline{X}, \overline{Y}]),$$

for all vector fields $\overline{X}$ and $\overline{Y}$ on $\overline{M}$, one easily obtains

$$\mathrm{d}\tilde{\overline{\eta}}(X, Y) = 0, \qquad \mathrm{d}\tilde{\overline{\eta}}(X, \frac{\partial}{\partial t}) = 0,$$

for all vector fields X and Y on M^{2n+1}, which proves that $\mathrm{d}\tilde{\overline{\eta}} = 0$ and so $(\overline{M} = \mathbb{R}\times M, J, \overline{\eta}, \overline{g})$ is a 1-parameter family of exact Kählerian manifolds.

Example 4.1. For this example, we rely on the example of Blair [4]. We know that $\mathbb{R}^{2n+1}$ with coordinates (x^i, y^i, z), $i = 1..n$, admits the Sasakian structure

$$\begin{aligned} g &= \frac{1}{4}\begin{pmatrix}\delta_{ij} + y^i y^j & 0 & -y^i\\ 0 & \delta_{ij} & 0\\ -y^j & 0 & 1\end{pmatrix}, \qquad \varphi = \begin{pmatrix}0 & \delta_{ij} & 0\\ -\delta_{ij} & 0 & 0\\ 0 & y^j & 0\end{pmatrix},\\ \xi &= 2\left(\frac{\partial}{\partial z}\right), \qquad \eta = \frac{1}{2}(dz - y^i dx^i).\end{aligned}$$

So, using this structure and formulas (4.1), we can define a family of exact Kählerian structures $(J, \overline{\eta}, \overline{g})$ on $\mathbb{R}^{2n+2}$ as follows

$$\overline{g} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{4}f^2(\delta_{ij} + f'^2 y^i y^j) & 0 & -\frac{1}{4}f^2 f'^2 y^i \\ 0 & 0 & \frac{\delta_{ij}}{4}f^2 & 0 \\ 0 & -\frac{1}{4}f^2 f'^2 y^j & 0 & \frac{1}{4}f^2 f'^2 \end{pmatrix}$$

$$J = \begin{pmatrix} 0 & -\frac{1}{2}y^j f f' & 0 & \frac{1}{2}ff' \\ 0 & 0 & \delta_{ij} & 0 \\ 0 & -\delta_{ij} & 0 & 0 \\ -\frac{2}{ff'} & 0 & y^j & 0 \end{pmatrix}.$$

We can easily check that

$$\overline{\Omega} = \mathrm{d}\overline{\eta} = \mathrm{d}\big(f^2\eta\big) \quad and \quad \mathrm{d}\tilde{\overline{\eta}} = \mathrm{d}\Big(\frac{f}{f'}\mathrm{d}t\Big) = 0.$$

Remark 4.2. Since any exact Kahlerian manifold has an exact 2-form (i.e. $\Omega = \mathrm{d}\eta$) then, by using the Proposition 7 in [1], we can confirm the inverse, i.e. we can construct a Sasakian manifold from an exact Kahlerian manifold.

5 Biconformal Transformations of exact Kähler metrics

It is well known that any conformal transformation

$$\overline{g} = fg, \qquad f > 0 \quad and \quad \mathrm{d}f \neq 0$$

of the metric g in a Kähler manifold (M, J, g) gives rise to a Hermitian manifold $(M, J, \overline{g})$ which is no longer Kählerian.
The aim of our considerations in this section is to find the class of J-invariant distributions which admit biconformal changes of the given exact Kähler metric so that the new metrics continue being exact Kählerian.
Let (M, J, η, g) be an exact Kählerian manifold. Putting

$$g' = 2fg + \eta \otimes \eta + \tilde{\eta} \otimes \tilde{\eta}, \qquad \tilde{\eta} = -\eta \circ J, \tag{5.1}$$

with $f > 0 \quad and \quad \mathrm{d}f \neq 0$. One can prove that (M, g', J) is a Hermitian manifold. Its fundamental Kähler form is denoted by Ω'. Then, for all $X, Y \in \mathfrak{X}(M)$ we have

$$\begin{aligned} \Omega'(X,Y) &= g'(X, JY) \\ &= 2f\Omega(X,Y) - 2(\eta \wedge \tilde{\eta})(X,Y) \end{aligned}$$

i.e.

$$\Omega' = 2f\Omega - 2\eta \wedge \tilde{\eta}.$$

Then,

$$\begin{aligned} \mathrm{d}\Omega' &= 2\mathrm{d}f \wedge \Omega - 2\mathrm{d}\eta \wedge \tilde{\eta} \\ &= 2(\mathrm{d}f - \tilde{\eta}) \wedge \Omega, \end{aligned}$$

which shows that the 2-form Ω' is closed (i.e $\mathrm{d}\Omega' = 0$) if $\tilde{\eta} = \mathrm{d}f$. Then, we get $\eta = \mathrm{d}f \circ J$, in this case, we notice that $\mathrm{d}\tilde{\eta} = 0$ and $\mathrm{d}\eta = \Omega$ which give $\tilde{\xi} = \mathrm{grad}f$ and $\xi = -J\mathrm{grad}f$.
On the other hand, replacing η and $\tilde{\eta}$ in formula (5.1), we get

$$g' = 2fg + \big(\mathrm{d}f \circ J \otimes \mathrm{d}f \circ J + \mathrm{d}f \otimes \mathrm{d}f\big),$$

and then,

$$\begin{aligned}\Omega' &= 2f\Omega + 2\mathrm{d}f \wedge \mathrm{d}f \circ J \\ &= \mathrm{d}(2f\ \mathrm{d}f \circ J),\end{aligned}$$

putting $\eta' = 2f\ \mathrm{d}f \circ J$ we get

$$\tilde{\eta'} = -\eta' \circ J = 2f\mathrm{d}f = \mathrm{d}f^2,$$

and we have

$$\mathrm{d}\tilde{\eta'} = 0.$$

Then, we get the following proposition:

Proposition 5.1. *Any biconformal transformation*

$$g' = 2fg + \mathrm{d}f \otimes \mathrm{d}f + \mathrm{d}f \circ J \otimes \mathrm{d}f \circ J,$$

with $f > 0$ and $\mathrm{d}f \neq 0$ of the metric g in an exact Kählerian manifold (M, J, η, g) where $\eta = \mathrm{d}f \circ J$, gives rise to an exact Kählerian manifold (M, J, η', g') where $\eta' = 2f\eta = 2f\ \mathrm{d}f \circ J$.

6 η-Kählerian manifolds

While the exact Kählerian manifold is an area of possible future research, we focus on an interesting case. First of all, we prepare

Lemma 6.1. *Let (M, J, η, g) be an exact Kählerian manifold. We have*

1) $\nabla_{JX}\xi + J\nabla_X\xi = 2X$,

2) $\nabla_\xi\xi = -2\tilde{\xi} + \mathrm{e}^{2\rho}\mathrm{grad}\rho$,

3) $\nabla_{\tilde{\xi}}\xi = -\mathrm{e}^{2\rho}J\mathrm{grad}\rho$,

4) $\nabla_\xi\tilde{\xi} + \nabla_{\tilde{\xi}}\xi = 2\xi$, *where ∇ denotes the Levi-Civita connection of a Riemannian metric.*

Proof. For all $X, Y \in \mathfrak{X}(M)$ we have

$$\Omega(X,Y) = \mathrm{d}\eta(X,Y) \Leftrightarrow 2g(X, JY) = g(\nabla_X\xi, Y) - g(\nabla_Y\xi, X), \tag{6.1}$$

For the first formula, just replace Y by JX.

For the second, taking $Y = \xi$ and knowing that

$$\begin{aligned}g(\nabla_X\xi, \xi) &= Xg(\xi,\xi) - g(\xi, \nabla_X\xi) \\ &= \frac{1}{2}X(\mathrm{e}^{2\rho}) \\ &= g(\mathrm{e}^{2\rho}\mathrm{grad}\rho, X),\end{aligned}$$

we get

$$2g(X, J\xi) = g(\mathrm{e}^{2\rho}\mathrm{grad}\rho, X) - g(\nabla_\xi\xi, X),$$

for all $X \in \mathfrak{X}(M)$, then

$$\nabla_\xi\xi = -2\tilde{\xi} + \mathrm{e}^{2\rho}\mathrm{grad}\rho.$$

For the third one, we have

$$\mathrm{d}\tilde{\eta}(X,Y) = 0 \Leftrightarrow g(\nabla_X\tilde{\xi}, Y) - g(\nabla_Y\tilde{\xi}, X) = 0, \tag{6.2}$$

for $Y = \tilde{\xi}$ and knowing that $g(\nabla_X\tilde{\xi}, \tilde{\xi}) = g(\mathrm{e}^{2\rho}\mathrm{grad}\rho, X)$, we get

$$\nabla_{\tilde{\xi}}\tilde{\xi} = \mathrm{e}^{2\rho}\mathrm{grad}\rho \Leftrightarrow \nabla_{\tilde{\xi}}\xi = -\mathrm{e}^{2\rho}J\mathrm{grad}\rho.$$

Finally,

$$\nabla_\xi\tilde{\xi} + \nabla_{\tilde{\xi}}\xi = J\nabla_\xi\xi + \nabla_{\tilde{\xi}}\xi = 2\xi.$$

□

Regarding the lemma 6.1, one can ask if it is possible that ξ is a K-torse forming vector field. For the answer, assuming (see (2.7))

$$\nabla_X \xi = aX + bJX + \alpha\eta(X)\xi - \lambda\eta(X)\tilde{\xi} + \mu\tilde{\eta}(X)\xi - \beta\tilde{\eta}(X)\tilde{\xi}, \tag{6.3}$$

where $a, b, \alpha, \beta, \lambda, \mu$ are functions on M.

First, we have for all $X \in \mathfrak{X}(M)$

$$g(\nabla_X \xi, \xi) = \mathrm{e}^{2\rho} X(\rho) \Leftrightarrow (a + \mathrm{e}^{2\rho}\alpha)\xi - (b - \mathrm{e}^{2\rho}\mu)\tilde{\xi} = \mathrm{e}^{2\rho}\mathrm{grad}\rho,$$

which gives

$$\alpha = \mathrm{e}^{-2\rho}\Big(\xi(\rho) - a\Big) \quad \textit{and} \quad \mu = \mathrm{e}^{-2\rho}\Big(\tilde{\xi}(\rho) + b\Big), \tag{6.4}$$

knowing that $\nabla_\xi \xi = -2\tilde{\xi} + \mathrm{e}^{2\rho}\mathrm{grad}\rho$ (see lemma 6.1), we get

$$\lambda = \mathrm{e}^{-2\rho}\Big(2 + b - \tilde{\xi}(\rho)\Big), \tag{6.5}$$

replacing formula (2) from lemma 6.1 in (6.3), we get

$$\beta = \mathrm{e}^{-2\rho}\Big(\xi(\rho) + a\Big), \tag{6.6}$$

replacing (6.4), (6.5) and (6.6) in (6.3) we obtain

$$\begin{aligned} \nabla_X \xi &= aX + bJX + \mathrm{e}^{-2\rho}\Big(\xi(\rho) - a\Big)\eta(X)\xi + \mathrm{e}^{-2\rho}\Big(\tilde{\xi}(\rho) - b - 2\Big)\eta(X)\tilde{\xi} \\ &+ \mathrm{e}^{-2\rho}\Big(\tilde{\xi}(\rho) + b\Big)\tilde{\eta}(X)\xi - \mathrm{e}^{-2\rho}\Big(\xi(\rho) + a\Big)\tilde{\eta}(X)\tilde{\xi}. \end{aligned} \tag{6.7}$$

On the other hand, using (6.7) in the equation

$$2\mathrm{d}\tilde{\eta}(X, Y) = 0 \Leftrightarrow g(\nabla_X \xi, JY) = g(\nabla_Y \xi, JX),$$

we get $a = 0$, so

$$\begin{aligned} \nabla_X \xi &= bJX + \mathrm{e}^{-2\rho}\xi(\rho)\eta(X)\xi + \mathrm{e}^{-2\rho}\Big(\tilde{\xi}(\rho) - b - 2\Big)\eta(X)\tilde{\xi} \\ &+ \mathrm{e}^{-2\rho}\Big(\tilde{\xi}(\rho) + b\Big)\tilde{\eta}(X)\xi - \mathrm{e}^{-2\rho}\xi(\rho)\tilde{\eta}(X)\tilde{\xi}. \end{aligned} \tag{6.8}$$

Using (6.8) in (6.1) we have

$$g(X, JY) = g\Big(-bX + (1 + b)\big(\eta(X)\xi + \tilde{\eta}(X)\tilde{\xi}\big), JY\Big),$$

which gives

$$(1 + b)\big(-X + \eta(X)\xi + \tilde{\eta}(X)\tilde{\xi}\big) = 0,$$

which implies that $b = -1$ finally, we find

$$\begin{aligned} \nabla_X \xi &= -JX + \mathrm{e}^{-2\rho}\xi(\rho)\eta(X)\xi + \mathrm{e}^{-2\rho}\Big(\tilde{\xi}(\rho) - 1\Big)\eta(X)\tilde{\xi} \\ &+ \mathrm{e}^{-2\rho}\Big(\tilde{\xi}(\rho) - 1\Big)\tilde{\eta}(X)\xi - \mathrm{e}^{-2\rho}\xi(\rho)\tilde{\eta}(X)\tilde{\xi}. \end{aligned}$$

This equality can be written by the form

$$\nabla_X \xi = -JX - \mathrm{e}^{-2\rho}\left(1 - \tilde{\xi}(\rho) + \xi(\rho)J\right)\left(\eta(X)\tilde{\xi} + \tilde{\eta}(X)\xi\right). \tag{6.9}$$

In the remainder of this paper, we study a type of exact Kählerian manifold which satisfies (6.9). Briefly, we denote such a manifold by **η-Kählerian manifold**.

Example 6.2. Using the above example 3.2. For the Levi-Civita connection corresponding to g, we have

$$\nabla_{e_\alpha} e_\beta = \nabla_{e_{\alpha'}} e_{\beta'} = -h\delta_{\alpha\beta} e_{2m+2},$$

$$\nabla_{e_\alpha} e_{\beta'} = -\nabla_{e_{\alpha'}} e_\beta = h\delta_{\alpha\beta} e_{2m+1},$$

$$\nabla_{e_\alpha} e_{2m+1} = \nabla_{e_{2m+1}} e_\alpha = -\nabla_{e_{\alpha'}} e_{2m+2} = -he_{\alpha'},$$

$$\nabla_{e_\alpha} e_{2m+2} = \nabla_{e_{\alpha'}} e_{2m+1} = \nabla_{e_{2m+1}} e_{\alpha'} = he_\alpha,$$

$$\nabla_{e_{2m+1}} e_{2m+1} = -(2h + th')e_{2m+2},$$

$$\nabla_{e_{2m+1}} e_{2m+2} = (2h + th')e_{2m+1}.$$

We can easily check that for all $X \in \mathfrak{X}(M)$

$$\nabla_X \xi = -JX - \mathrm{e}^{-2\rho}\left(1 - \tilde{\xi}(\rho) + \xi(\rho)J\right)\left(\eta(X)\tilde{\xi} + \tilde{\eta}(X)\xi\right),$$

i.e. for all $i \in \{2n+2\}$,

$$\nabla_{e_i} \xi = -Je_i - h\left(h + th'\right)\left(\eta(e_i)\tilde{\xi} + \tilde{\eta}(e_i)\xi\right),$$

in other words, as long as $\xi = \frac{1}{h} e_{2m+1}$ we can write

$$\begin{aligned}\nabla_{e_i} e_{2m+1} &= -hJe_i - he_i\left(\frac{1}{h}\right)e_{2m+1} \\ &\quad - \left(h + th'\right)\left(\theta^{e_{2m+1}}(e_i)e_{2m+2} + \theta^{e_{2m+2}}(e_i)e_{2m+1}\right).\end{aligned}$$

The following paragraph establishes a one-to-one correspondence between η-Kählerian manifolds and Sasakian manifolds.
First, using the remark 4.2, we can easily conclude that for any η-Kählerian manifold we can build a Sasakian manifold.
For the inverse, we will base on section 4, where we have shown that with a single Sasakian structure we can build a family of η-Kählerian structures.

Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a Sasakian manifold and $(\overline{M} = \mathbb{R} \times M, J, \overline{g})$ the 1-parameter family of exact Kählerian structures defined above with $f^2 = 2t$ and $t > 0$ which gives:

$$\begin{cases} \overline{g} = dt^2 + 2tg + (1-2t)\eta \otimes \eta \\ JX = \varphi X + \eta(X)\frac{\partial}{\partial t} \\ J\frac{\partial}{\partial t} = -\xi, \end{cases} \tag{6.10}$$

then we have

$$\overline{\eta} = 2t\eta, \qquad \tilde{\overline{\eta}} = -2t\eta \circ J.$$

We can check that is very simply as follows:

$$\overline{\xi} = 2t\xi, \qquad \tilde{\overline{\xi}} = 2t\frac{\partial}{\partial t}, \qquad \rho = \ln 2t.$$

Using formula (6.9), for all X vector field on M, we can get

$$\begin{cases} \overline{\nabla}_{\frac{\partial}{\partial t}} \overline{\xi} = 2\xi, \\ \overline{\nabla}_X \overline{\xi} = -\varphi X. \end{cases} \tag{6.11}$$

On the other hand, for all $\overline{X}, \overline{Y}$ and $\overline{Z}$ vectors fields on $\overline{M}$, using Koszul's formula for the metric $\overline{g}$,

$$\begin{aligned} 2\overline{g}(\overline{\nabla}_{\overline{X}}\overline{Y}, \overline{Z}) &= \overline{X}\overline{g}(\overline{Y}, \overline{Z}) + \overline{Y}\overline{g}(\overline{Z}, \overline{X}) - \overline{Z}\overline{g}(\overline{X}, \overline{Y}) \\ &- \overline{g}(\overline{X}, [\overline{Y}, \overline{Z}]) + \overline{g}(\overline{Y}, [\overline{Z}, \overline{X}]) + \overline{g}(\overline{Z}, [\overline{X}, \overline{Y}], \end{aligned}$$

we obtain:

$$\begin{aligned}\overline{\nabla}_{\frac{\partial}{\partial t}} X &= \overline{\nabla}_X \frac{\partial}{\partial t} = \frac{1}{2t}(X - \eta(X)\xi)\\ \overline{\nabla}_X Y &= \nabla_X Y + \Big(1 - \frac{1}{2t}\Big)\big(\eta(X)\varphi Y + \eta(Y)\varphi X\big) - g(\varphi X, \varphi Y)\frac{\partial}{\partial t}.\end{aligned}$$

This implies that

$$\begin{cases} \overline{\nabla}_{\frac{\partial}{\partial t}}\overline{\xi} = 2\xi + 2t\overline{\nabla}_{\frac{\partial}{\partial t}}\xi = 2\xi \\ \overline{\nabla}_X\overline{\xi} = 2t\overline{\nabla}_X\xi = -\varphi X. \end{cases} \tag{6.12}$$

Comparing this result with (6.11), we can state the following theorem:

Theorem 6.3. *Let $(M, \varphi, \xi, \eta, g)$ be a Sasakian manifold. The product $\tilde{M} = \mathbb{R} \times M$ provided with the almost Hermitian structure $(J, \overline{g})$ given in (6.10) is an η-Kählerian manifold.*

Lemma 6.4. *For all η-Kählerian manifold, we have*

$$\text{grad } \mathrm{e}^{2\rho} = 2\Big(\tilde{\xi}(\rho)\tilde{\xi} + \xi(\rho)\xi\Big), \tag{6.13}$$

with $|\xi| = \mathrm{e}^{\rho}$.

Proof. In formula (6.9), replacing X with ξ we get

$$\nabla_\xi \xi = -2\tilde{\xi} + \xi(\rho)\xi + \tilde{\xi}(\rho)\tilde{\xi}. \tag{6.14}$$

Comparing this result with the formula (1) in lemma 6.4 we obtain

$$\text{grad}\rho = \mathrm{e}^{-2\rho}\Big(\tilde{\xi}(\rho)\tilde{\xi} + \xi(\rho)\xi\Big). \tag{6.15}$$

This completes the proof of the lemma. □

The famous equation (6.9) gives important informations about the curvature properties of η-Kähler manifold. We start with the first proposition

Proposition 6.5. *Let (M, J, η, g) be an η-Kählerian manifold. Then we have:*

$$\begin{aligned} R(X,Y)\xi &= -\big(\nabla_X T_\rho\big)\big(\eta(Y)\tilde{\xi} + \tilde{\eta}(Y)\xi\big) + \big(\nabla_Y T_\rho\big)\big(\eta(X)\tilde{\xi} + \tilde{\eta}(X)\xi\big)\\ &- T_\rho\big((X \wedge Y)\xi + (JX \wedge JY)\xi + 2g(X, JY)J\xi\big), \end{aligned} \tag{6.16}$$

where $T_\rho = \mathrm{e}^{-2\rho}\left(1 - \tilde{\xi}(\rho) + \xi(\rho)J\right)$.

$$\begin{aligned} R(X,\xi)Y &= \big(\nabla_Y T_\rho\big)\big(\eta(X)\tilde{\xi} + \tilde{\eta}(X)\xi\big) + \big(\eta(X)\eta(Y) \qquad (6.17)\\ &\quad -\tilde{\eta}(X)\tilde{\eta}(Y)\big)\text{grad}\left(\mathrm{e}^{-2\rho}\xi(\rho)\right)\\ &\quad -\big(\eta(X)\tilde{\eta}(Y) + \tilde{\eta}(X)\eta(Y)\big)\text{grad}\left(\mathrm{e}^{-2\rho}\big(1 - \tilde{\xi}(\rho)\big)\right)\\ &- T_\rho\big((X \wedge \xi)Y + (JX \wedge J\xi)Y + 2\tilde{\eta}(X)JY\big), \end{aligned} \tag{6.18}$$

$$\begin{aligned} S(X,\xi) &= 2\mathrm{e}^{-2\rho}\big(\tilde{\xi}(\rho)\eta(X) + \xi(\rho)\tilde{\eta}(X)\big)\\ &\quad + \left(\xi\big(\mathrm{e}^{-2\rho}\xi(\rho)\big) + \tilde{\xi}\big(\mathrm{e}^{-2\rho}\tilde{\xi}(\rho)\big)\right)\eta(X)\\ &\quad -2(n+2)\mathrm{e}^{-2\rho}\left(\big(1 - \tilde{\xi}(\rho)\big)\eta(X) + \xi(\rho)\tilde{\eta}(X)\right). \end{aligned} \tag{6.19}$$

where R denote the curvature tensor and S is the Ricci curvature defined in (2.1) and (2.2) respectively and $(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y$ for all vectors fields X and Y on M.

Proof. The relation (6.16) follows from (2.1) with $Z = \xi$ and formula (6.9). For the second relation, we have for all vectors fields X, Y, Z on M

$$g\big(R(X,\xi)Y, Z\big) = -g\big(R(Y,Z)\xi, X\big),$$

and using (6.16). Finally, knowing that

$$S(X,Y) = \sum_{i=1}^{2n+2} g\big(R(X,e_i)e_i, Y\big),$$

then

$$S(X,\xi) = -\sum_{i=1}^{2n+2} g\big(R(X,e_i)\xi, e_i\big),$$

using (6.16), we obtain (6.19). This completes the proof of the proposition. □

Notice that, from equation (6.16) we can get

$$R(\xi,\tilde{\xi},\xi,\tilde{\xi}) = \mathrm{e}^{2\rho}\left(4 - \xi\big(\xi(\rho)\big) - \tilde{\xi}\big(\tilde{\xi}(\rho)\big) + 2\xi(\rho)^2 + 2\tilde{\xi}(\rho)^2 - 6\tilde{\xi}(\rho)\right).$$

If ξ is a global unit vector field (i.e. $\rho = 0$), then the holomorphic sectional curvature of the plane section generated by $\{\xi, \tilde{\xi}\}$ is -4.

7 Some kinds of η-Kählerian manifolds

In this section we will highlight two important kinds of η-Kählerian manifolds.

7.1 η-U-Kählerian manifold

Definition 7.1. An η-U-Kählerian manifold is an η-Kählerian manifold where η is an unit 1-form i.e. $\eta(\xi) = 1$.

Let (M, J, η, g) be an η-U-Kählerian manifold. Since $\eta(\xi) = 1$ i.e. $\rho = 0$, from the formula (6.9) we get

$$\nabla_X \xi = -JX - \eta(X)\tilde{\xi} - \tilde{\eta}(X)\xi,$$

and the proposition 6.5 becomes:

Proposition 7.2. *Let (M, J, η, g) be an η-U-Kählerian manifold. Then we have:*

$$R(X,Y)\xi = -(X \wedge Y)\xi - (JX \wedge JY)\xi - 2g(X, JY)J\xi, \tag{7.1}$$

$$R(X,\xi)Y = -(X \wedge \xi)Y - (JX \wedge J\xi)Y - 2\tilde{\eta}(X)JY, \tag{7.2}$$

$$S(X,\xi) = -2(n+2)\eta(X) \tag{7.3}$$

Example 7.3. Using the above example 3.2. Taking $\rho = 0$ i.e. $h = 1$ we get an η-U-Kählerian manifold.

7.2 η-K-Kählerian manifold

Definition 7.4. An η-K-Kählerian manifold is an η-Kählerian manifold where ξ is a Killing vector field with respect to g.

Let (M, J, η, g) be an η-Kählerian manifold. We have for all vectors fields X and Y on M

$$\begin{aligned} \mathrm{d}\eta(X,Y) = g(X, JY) \quad &\Leftrightarrow \quad g(\nabla_X\xi, Y) - g(\nabla_Y\xi, X) = 2g(X, JY) \\ &\Leftrightarrow \quad g(\nabla_X\xi, Y) = 2g(X, JY) + g(\nabla_Y\xi, X). \end{aligned}$$

On the other hand,

$$(\mathcal{L}_\xi g)(X,Y) = g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) = 2g(X, JY) + 2g(\nabla_Y \xi, X),$$

where $(\mathcal{L}_\xi g)$ denotes the Lie derivative of g. If ξ is a Killing vector field with respect to g since $\mathcal{L}_\xi g = 0$ then we get

$$\nabla_X \xi = -JX. \tag{7.4}$$

Using formula (6.9) we obtain

$$\big(1 - \tilde{\xi}(\rho) + \xi(\rho)J\big)\big(\eta(X)\tilde{\xi} + \tilde{\eta}(X)\xi\big) = 0,$$

which gives the following ODE system:

$$\begin{cases} \xi(\rho) = 0 \\ \tilde{\xi}(\rho) = 1. \end{cases} \tag{7.5}$$

This is the case which the proposition 6.5 becomes:

Proposition 7.5. *Let (M, J, η, g) be an η-K-Kählerian manifold. Then we have:*

$$R(X,Y)\xi = R(X,\xi)Y = 0 \tag{7.6}$$

$$S(X,\xi) = 0 \tag{7.7}$$

Example 7.6. Again, let's go back to our example 3.2, we can see that

$$\begin{cases} \xi(\rho) = 0 \\ \tilde{\xi}(\rho) = 1. \end{cases} \Leftrightarrow \begin{cases} \frac{1}{h} e_{2m+1}\left(\frac{1}{h}\right) = 0 \\ \frac{1}{h} e_{2m+2}\left(\frac{1}{h}\right) = 1. \end{cases} \Leftrightarrow th' + h = 0$$

the solution of this ODE is of the form:

$$h = \frac{c}{t} \qquad with \;\; c \in \mathbb{R}^*.$$

So, for $h = \frac{c}{t}$, (M, J, η, g) is an η-K-Kählerian manifold.

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