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ON RADICALS OF POLYNOMIAL SEMIRINGS

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Abstract. In this article, we introduce and investigate the polynomial extensibility of radicals. Also we generalize some results on polynomial extensibility of radicals, semisimple classes and of the Amitsur property for associative rings as given in [14] to semirings.

1 Introduction

In the literature of theory of semirings, generalized-matrix extensibility has been studied in [7]. However, generalized-polynomial extensibility has not been studied in the theory of radicals of polynomial semirings. Therefore, in this article, we have introduced and investigated the radicals of polynomial semirings. The interrelation and independence of polynomial extensibility of radical and semisimple classes and the Amitsur property are investigated for associative rings in [14]. In this paper, we generalize results known for rings in [14] to semirings. We have proved that every A- radical $\mathcal{R}$ has the Amitsur property. Throughout this article all semirings are assumed to be associative, not necessarily with unity, and by a radical we mean a radical in the sense of Kurosh Amitusar as defined in [11]. The symbol $\mapsto$ stands for a surjective homomorphism and $I \lhd A$ indicates that I is an ideal of a semiring A.

2 Preliminaries

We begin with some useful definitions and preliminaries. The preliminary notions and basic results are as given in [6]. For details, the readers are requested to see [6].

Definition 2.1. [6] A semiring is a set A together with two binary operations called addition $(+)$ and multiplication $(\cdot)$ such that $(A,+)$ is a commutative monoid with identity element 0_A; $(A,\cdot)$ is a monoid with identity element 1; multiplication distributes over addition from either side and 0 is multiplicative absorbing, that is, $a \cdot 0 = 0 \cdot a = 0$ for each $a \in A$.

Definition 2.2. [6] A semiring is commutative if $(A,\cdot)$ is a commutative semigroup.

Definition 2.3. [6] A subset I of a semiring A is said to be an ideal of A if I is an additive subsemigroup of $(A,+)$, $IA \subseteq I$ and $AI \subseteq I$.

Definition 2.4. [6] An ideal I of a semiring A is called proper if $I \neq A$ and a proper ideal I of A is called maximal if there is no ideal J of A satisfying $I \subset J \subset A$.

Definition 2.5. [6] An ideal I of a semiring A is said to be subtractive (k-ideal) if for $a \in I, a+b \in I$ for all $b \in A$ imply $b \in I$.

Definition 2.6. The additive semigroup $(A,+)$ of a semiring A is denoted by A^+ and for an abelian semigroup A^+, we may define always a semiring A^0 with zero multiplication, called a zero-semiring by the rule $xy = 0$ for all $x, y \in A$.

Definition 2.7. A radical $\mathcal{R}$ is called an $A-$radical if for any semiring $R \in \mathcal{R}$ and any additive homomorphism $f : R \to S$ such that $f(R)$ is a subsemiring of S also $f(R) \in \mathcal{R}$.

By Dorroh's Extension, every semiring A can be embedded as an ideal into a semiring A' with unity element. The semiring ring A' is refereed to be the Dorroh's extension.

3 Radical Of Polynomial Semirings

In this section, we have introduced and investigated the radicals of polynomial semirings. The interrelation and independence of polynomial extensibility of radical and semisimple classes and the Amitsur property are investigated for associative rings in [14]. We have proved that every A-radical $\mathcal{R}$ has the Amitsur property. We begin with some elementary but useful results.

Proposition 3.1. If I is a k-ideal in a semiring A, then $I[x]$ is also a k-ideal in $A[x]$.

Proof. Let $f(x) \in I[x]$ and $f(x) + g(x) \in I[x]$ for $g(x) \in A[x]$, where $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{i=0}^{n} b_i x^i$. Now $f(x)+g(x) = \sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{n} b_i x^i = \sum_{i=0}^{\infty}(a_i + b_i)x^i$. Since $a_i \in I$ and $a_i + b_i \in I \Rightarrow b_i \in I$. This shows that $\sum_{i=0}^{\infty} b_i x^i \in I[x]$. Thus $I[x]$ is a k-ideal in $A[x]$. □

Proposition 3.2. Let I be a k-ideal of a semiring A. Then $A[x]/I[x] \cong (A/I)[x]$.

Proof. Let $\pi : A \to A/I$ be a canonical map. Then π is surjective. Define $\theta : A[x] \to (A/I)[x]$ by $\theta(a_0 + a_1 x + a_2 x^2 + ... + a_n x^n) = (b_0 + b_1 x + b_2 x^2 + ... + b_n x^n)$, where $b_0 = \pi(a_0), b_1 = \pi(a_1), ..., b_n = \pi(a_n)$. It is easy to verify that $Ker\theta = I[x]$. Thus θ is an onto steady semiring homomorphism. Hence $(A/I)[x] \cong A[x]/I[x]$.

□

Let $\mathcal{R}$ be a radical class, define $\mathcal{R}_x = \{A \mid A[x] \in \mathcal{R}\}$.

Theorem 3.3. For every radical class $\mathcal{R}$, the class $\mathcal{R}_x$ is a radical class. If $\mathcal{R}$ is hereditary, then so is $\mathcal{R}_x$.

Proof. (1) Let $A \in \mathcal{R}_x$ and $f : A \to B$ be a surjective homomorphism. Then $A[x] \in \mathcal{R}_x$ and $f : A[x] \to B[x]$ is surjective, i.e., $f(A[x]) = B[x] \in \mathcal{R}$. Hence $B \in \mathcal{R}_x$.
(2) Let $\{I_\lambda \mid \lambda \in \Delta\}$ be a ascending chain of ideals of a semiring A such that $A = \bigcup I_\lambda$ and $I_\lambda \in \mathcal{R}_x$ for each $\lambda \in \Delta$. Then $\{I_\lambda[x] \mid \lambda \in \Delta\}$ is an ascending chain of ideals of a semiring $A[x]$ and $I_\lambda[x] \in \mathcal{R}$, for every $\lambda \in \Delta$. Hence $A[x] = (\bigcup I_\lambda)[x] = \bigcup(I_\lambda[x]) \in \mathcal{R}$ that is $A \in \mathcal{R}_x$.
(3) Suppose that for a k-ideal I, $A/I \in \mathcal{R}_x$, therefore $I[x], (A/I)[x] \in \mathcal{R}$. Since $I[x] \in \mathcal{R}$ and $A[x]/I[x] \cong (A/I)[x] \in \mathcal{R}$. $\Rightarrow A[x] \in \mathcal{R} \Rightarrow A \in \mathcal{R}_x$. Thus $\mathcal{R}_x$ is a radical class. If A is a zero-semiring, so is $A[x]$. If $I \lhd A \in \mathcal{R}_x$, then $I[x] \lhd A[x] \in \mathcal{R}$, implies that $I[x] \in \mathcal{R}$, implies that $I \in \mathcal{R}_x$. □

A radical $\mathcal{R}$ has the Amitsur property if for every polynomial semiring $A[x]$ it holds

$$\mathcal{R}(A[x]) = (\mathcal{R}(A[x]) \cap A)[x].$$

The Amitsur property states that the radical of a polynomial semiring is a polynomial semiring.
A radical $\mathcal{R}$ has the Amitsur property if and only if $\mathcal{R}(A[x])$ is a polynomial semiring in x.

Lemma 3.4. Let R be a commutative semiring and I, J and K be ideals in R such that K is a k-ideal in R and $I \subseteq K$. Then $(I + J) \cap K = I + J \cap K$.

Proof. Clearly $I + K = K$ since $I \subseteq K$. Now $(J \cap K) + I \subseteq K + I$ and $(J \cap K) + I \subseteq J + I$, this shows that $(J \cap K) + I \subseteq (K + I) \cap (J + K) = K \cap (I + J)$. Thus $(J \cap K) + I \subseteq K \cap (I + J)$. Conversely, let $a \in K \cap (I+J)$. Then $a \in K$ and $a \in I+J$. Now $a \in I+J \Rightarrow a = b+c, b \in I$ and $c \in J \Rightarrow b \in K$ and $c \in J$. However, $b \in K$ and $a \in K \Rightarrow c \in K$ and $c \in J \Rightarrow c \in J \cap K$. Thus $a = b+c \in I + J \cap K$. Therefore, $K \cap (I + J) \subseteq I + J \cap K$. Hence $K \cap (I + J) = I + J \cap K$. □

Note that Lemma 3.4 also holds if I, J and K are k-ideals of a commutative semiring A.

Lemma 3.5. Let $\mathcal{R}$ be a radical of semirings. For any element $f \in \mathbb{N}[x]$, $f\mathcal{R}(A[x]) \subseteq \mathcal{R}(A[x])$, where $\mathbb{N}$ is a semiring of non negative integers.

Proof. Clearly $f\mathcal{R}(A[x])$ is an ideal of $A[x]$. Since $\mathcal{R}(A[x])$ is an ideal in $A[x]$, we have $I = \mathcal{R}(A[x]) + f\mathcal{R}(A[x])$ is an ideal in $A[x]$. Let $\phi : I \to I/\mathcal{R}(A[x])$ be the natural homomorphism. Define the surjective homomorphism $\psi : \mathcal{R}(A[x]) \to I/\mathcal{R}(A[x])$ by $\psi(a) = \phi(f \cdot a)$. Thus $I/\mathcal{R}(A[x]) \in \mathcal{R}$ and, by extension closure, $I \in \mathcal{R}$ and $I \subseteq \mathcal{R}(A[x])$. This shows that $f\mathcal{R}(A[x]) \subseteq \mathcal{R}(A[x])$. □

Proposition 3.6. If $\mathcal{R}$ is a radical, then for any semiring A, $(A \cap \mathcal{R}(A[x]))[x] \subseteq \mathcal{R}(A[x])$.

Proof. Since $(A \cap \mathcal{R}(A[x]))[x] = (A \cap \mathcal{R}(A[x]))\mathbb{N}[x]$ and $(A \cap \mathcal{R}(A[x]))[x] \subseteq \mathcal{R}(A[x])$. By above Lemma 3.5, $(A \cap \mathcal{R}(A[x]))[x] \subseteq \mathcal{R}(A[x])$. □

Lemma 3.7. Let θ be an isomorphism between R and R^θ. Then θ induces one to one correspondence between the $\mathcal{R}$-ideals of R and R^θ.

Proof. Proof is straightforward. □

Lemma 3.8. If the kernel θ of the homomorphic mapping of R and R^θ is in the upper $\mathcal{R}$-radical $\mathcal{R}(R)$, then $\mathcal{R}(R^\theta) = \mathcal{R}(R)^\theta$.

Proof. Proof is similar to rings. □

In Lemma 3.8, upper radical $\mathcal{R}(R)$ is a maximal k-ideal of R.

Proposition 3.9. For a radical $\mathcal{R}$ to have the Amitsur property a necessary and sufficient condition is, $\mathcal{R}(A[x]) \cap A = 0 \Rightarrow \mathcal{R}(A[x]) = 0$ for all semirings A.

Proof. Assume that $\mathcal{R}$ has the Amitsur property, that is, $\mathcal{R}(A[x]) = (\mathcal{R}(A[x]) \cap A)[x]$. Then, if $\mathcal{R}(A[x]) \cap A = 0$, we have that $\mathcal{R}(A[x]) = (\mathcal{R}(A[x]) \cap A)[x] = 0[x] = 0$.

Conversely, assume that $\mathcal{R}(A[x]) = 0$ if $(\mathcal{R}(A[x]) \cap A) = 0$. Claim that $(A \cap \mathcal{R}(A[x]))[x] = \mathcal{R}(A[x])$ for any semiring A. Set $J = A \cap \mathcal{R}(A[x])$. It should be noted that J is a $k-$ideal as $\mathcal{R}(A[x])$ is a maximal $k-$ideal in $A[x]$ and $A/J \cong (A + J[x])/J[x]$ as $a \to \overline{a}/J[x]$, where $\overline{a} = a + J[x]$ is an onto steady homomorphism with kernel J. Consider the homomorphism $A[x] \to A[x]/J[x]$. The kernel of this homomorphism is $J[x] \subseteq \mathcal{R}(A[x])$. Hence by Lemma 3.8, for any radical $\mathcal{R}$, $\mathcal{R}(A[x]/J[x]) = \mathcal{R}(A[x])/J[x]$. But by Proposition 3.2, $\mathcal{R}((A/J)[x]) \cong \mathcal{R}(A[x]/J[x]) = \mathcal{R}(A[x])/J[x]$.

Consider, $\mathcal{R}((A/J)[x]) \cap (A/J) \cong (\mathcal{R}(A[x])/J[x]) \cap ((A + J[x])/J[x]) = (\mathcal{R}(A[x]) \cap (A + J[x]))/J[x]$ since by Lemma 3.4 $\Rightarrow ((\mathcal{R}(A[x]) \cap A) + J[x])/J[x] = J + J[x]/J[x] = 0$. Thus $\mathcal{R}((A/J)[x]) \cap (A/J) = 0 \Rightarrow \mathcal{R}((A/J)[x]) = 0 \Rightarrow \mathcal{R}(A[x])/J[x] = 0 \Rightarrow \mathcal{R}(A[x]) \subseteq J[x]$. This shows that $\mathcal{R}(A[x]) = J[x] = \mathcal{R}(A[x]) \cap A$. Hence the Amitsur property. □

Let $Z(A')$ denote the center of the Dorroh's extension A' of a semiring A. We say that a radical $\mathcal{R}$ is closed under linear substitution if $f(x) \in \mathcal{R}(A[x]) \Rightarrow f(ax + b) \in \mathcal{R}(A[x])$ for all semirings A and $a, b \in Z(A')$.

Proposition 3.10. If a radical $\mathcal{R}$ has the Amitsur-property, then $\mathcal{R}$ is closed under linear substitution. If a radical $\mathcal{R}$ is closed under linear substitution then $\mathcal{R}$ satisfies condition $f(x) \in \mathcal{R}(A[x]) \Rightarrow f(0) \in \mathcal{R}(A[x])$, for all semirings A.

Proof. Suppose that $\mathcal{R}(A[x]) = (\mathcal{R}(A[x]) \cap A)[x]$. Let $f(x) = \sum_{i=0}^{n} c_i x^i \in \mathcal{R}(A[x]) = (\mathcal{R}(A[x]) \cap A)[x]$. Then $f(ax + b) = \sum_{i=0}^{n} c_i(ax + b)^i = g(x)$. Since each $c_i \in (\mathcal{R}(A[x]) \cap A)$, and $a, b \in Z(A')$, $(a = (a, 0)\ b = (b, 0))$, all the coefficient of $g(x)$ are in $\mathcal{R}(A[x]) \cap A$. Therefore $g(x) \in (\mathcal{R}(A[x]) \cap A) = \mathcal{R}(A[x])$. Thus $f(x) \in \mathcal{R}(A[x]) \Rightarrow f(ax + b) \in \mathcal{R}(A[x])$. If $f(x) \in \mathcal{R}(A[x])$, then $f(ax + b) \in \mathcal{R}(A[x])$. Substitute $a = 0$ and $b = 0$, then $f(0) \in \mathcal{R}(A[x])$. □

Definition 3.11. The semisimple class $S_\mathcal{R}$ of a radical class $\mathcal{R}$ is polynomially extensible if $A \in S_\mathcal{R} \Rightarrow A[x] \in S_\mathcal{R}$.

Proposition 3.12. Let $\mathcal{R}$ be a radical class. Then the radical $\mathcal{R}$ is polynomial extensible if and only if $\mathcal{R} = \mathcal{R}_x$.

Proof. Since $\theta : A[x] \to A$, defined by $\theta(f(x)) = f(0)$ is an onto homomorphism. i.e. $\theta(A[x]) = A$, and $\mathcal{R}$ is a radical class. Therefore $A[x] \in \mathcal{R} \Rightarrow A \in \mathcal{R}$. Shows that for any semiring $A \in \mathcal{R}_x, A \in \mathcal{R}$ i.e. $\mathcal{R}_x \subseteq \mathcal{R}$, for every radical class $\mathcal{R}$. If $\mathcal{R}$ is polynomially extensible, then $\mathcal{R}_x = \mathcal{R}$ and conversely. □

Proposition 3.13. Let $\mathcal{R}$ be a hereditary radical class. Then $\mathcal{R}(A[x]) \cap A \subseteq \mathcal{R}(A)$, for all semirings A.

Proof. Let A' be the Dorroh's extension of a semiring A and let $\sigma(A) = \mathcal{R}(A[x]) \cap A$. Then $\sigma(A') = \mathcal{R}(A'[x]) \cap A' \subset \mathcal{R}(A'[x]) = A'[x]$ and $x \in A'[x]$, we have $\sigma(A')[x] \subseteq \mathcal{R}(A'[x])$. But $\sigma(A') \lhd A'$, so that $\sigma(A')[x] \lhd \mathcal{R}(A'[x])$ and thus $\sigma(A')[x] \in \mathcal{R}$ by the hereditary of $\mathcal{R}$. This shows that $\sigma(A') \in \mathcal{R}_x$. Hence $\sigma(A') \subseteq \mathcal{R}_x(A')$.

Moreover, $\mathcal{R}_x(A')[x] \lhd A'[x]$ and $\mathcal{R}_x(A')[x] \in \mathcal{R}$, so $\mathcal{R}_x(A') = \mathcal{R}_x(A')[x] \cap A' \subseteq \mathcal{R}(A'[x]) \cap A' = \sigma(A')$. The hereditary of $\mathcal{R}$ gives, $\mathcal{R}_x(A) = \mathcal{R}_x(A') \cap A = \sigma(A') \cap A = \mathcal{R}(A'[x]) \cap A' \cap A = \mathcal{R}(A'[x]) \cap A = \mathcal{R}(A'[x]) \cap A \cap A[x] = \mathcal{R}(A[x]) \cap A = \sigma(A)$. Thus $\mathcal{R}_x(A) = \sigma(A)$. But $\mathcal{R}_x \subseteq \mathcal{R}$, hence $\mathcal{R}_x(A) \subseteq \mathcal{R}(A)$.

□

We say that the radical $\mathcal{R}$ has the intersection property relative to the class $\mathcal{M}$ of semirings, if $\mathcal{R}(A) = \cap\{I \lhd A \mid \ 0 \neq A/I \in \mathcal{M}, I \ is \ k - ideal\}$ for all semirings A.

In this case the class $\mathcal{M}$ is regular and $\mathcal{R} = \mathcal{U}\mathcal{M} = \{A \mid 0 \neq A/I \notin \mathcal{M}\}$.

Theorem 3.14. Let $\mathcal{R}$ be a radical which has the intersection property relative to a class $\mathcal{M}$ of additively cancellative and semisubtractive semirings. If both $\mathcal{M}$ and $\mathcal{R}$ are polynomial extensible, then $\mathcal{R}$ has the Amitsur property.

Proof. In view of Proposition 3.9, we have to prove that $\mathcal{R}(A[x]) \cap A = 0$ implies that $\mathcal{R}(A[x]) = 0$. Since $\mathcal{R}$ is polynomial extensible, we have $\mathcal{R}(A) = \mathcal{R}_x(A) = \mathcal{R}(A[x]) \cap A = 0$. Thus by the intersection property there exist k-ideals I_λ, $\lambda \in \Delta$ of A such that $A/I_\lambda \in \ \mathcal{M}$ and $\cap I_\lambda = 0$ for all λ. Since also $\mathcal{M}$ is polynomial extensible, we have $A[x]/I_\lambda[x] \cong (A/I_\lambda)[x] \in \mathcal{M}, \lambda \in \Delta$. From $\cap I_\lambda = 0$, it follows that $\cap I_\lambda[x] = 0$, so $A[x]$ is a subdirect sum the semirings $A[x]/I_\lambda[x] \in \mathcal{M}$. Thus $A[x] \in S_\mathcal{R}$. □

Given two radicals $\mathcal{R}_1$ and $\mathcal{R}_2$, the radical $\mathcal{R}_1$ is polynomially extensible to $\mathcal{R}_2$ whenever

$$A[x] \in \mathcal{R}_2 \ for \ all \ A \in \mathcal{R}_1. \qquad (A)$$

Obiviously $\mathcal{R}_1 \subseteq \mathcal{R}_2$.

Corollary 3.15. If a radical $\mathcal{R}$ and its semisimple class $S_\mathcal{R}$ are polynomially extensible, then $\mathcal{R}$ has the Amitsur property.

Proposition 3.16. If $\mathcal{R}_1$ and $\mathcal{R}$ are radicals satisfying condition (A) then,

$$A \in S_{\mathcal{R}_1} \ for \ all \ A[x] \in S_\mathcal{R}. \qquad (B)$$

In particular, $A \in S_{\mathcal{R}_x}$ for all $A[x] \in S_\mathcal{R}$. If $\mathcal{R}_1 \subseteq \mathcal{R}$ and $\mathcal{R}$ has the Amitsur property, then condition (B) implies (A).

Proof. Suppose that $A[x] \in S_\mathcal{R}$ but $A \notin S_{\mathcal{R}_1}$. Then $0 \neq \mathcal{R}_1(A) \lhd A$, and so by (A) we have $\mathcal{R}_1(A)[x] \in \mathcal{R}$. Hence by $\mathcal{R}_1(A)[x] \lhd A[x]$. We conclude that $0 \neq \mathcal{R}_1(A)[x] \subseteq \mathcal{R}(A[x])$, contradicting $A[x] \in S_\mathcal{R}$. This proves the validity of (B).
Now assume that $\mathcal{R}_1 \subseteq \mathcal{R}$ and $\mathcal{R}$ has the Amitsur property. For any $A \in \mathcal{R}_1$, we have

$$A[x]/\mathcal{R}(A[x]) = A[x]/(\mathcal{R}(A[x]) \cap A)[x] \cong (A/(\mathcal{R}(A[x]) \cap A)[x] .... \qquad (C)$$

and by $A[x]/\mathcal{R}(A[x]) \in S_\mathcal{R}$ also $(A/(\mathcal{R}(A[x]) \cap A)[x] \in S_\mathcal{R}$. Hence by condition (B) it follows that $A/(\mathcal{R}(A[x]) \cap A) \in S_{\mathcal{R}_1}$. But $A \in \mathcal{R}_1$ and so $A/(\mathcal{R}(A[x]) \cap A) \in \mathcal{R}_1$. Thus $A = \mathcal{R}(A[x]) \cap A$, and so by condition (C), we have $A[x]/\mathcal{R}(A[x]) = 0$, that is, $A[x] \in \mathcal{R}$. Thus conditions (A) is satisfied. □

Corollary 3.17. Let $\mathcal{R}_1$ and $\mathcal{R}$ be radicals such that $\mathcal{R}_1 \subseteq \mathcal{R}$ and $\mathcal{R}$ has the Amitsur property. Then conditions (A) and (B) are equivalent. In particular, a radical $\mathcal{R}$ with Amitsur property is polynomially extensible if and only if it satisfies $A \in S_\mathcal{R}$ for all $A[x] \in S_\mathcal{R}$.

Proposition 3.18. Let $\mathcal{R}$ and $\mathcal{R}_1$ be radicals. The relation $\mathcal{R}_x \subseteq \mathcal{R}_1$ is equivalent to $A \in \mathcal{R}_1$ for all $A[x] \in \mathcal{R}$. Δ

Proof. Straightforward. □

Proposition 3.19. Let $\mathcal{R}$ and $\mathcal{R}_1$ be radicals such that $\mathcal{R}_x \subseteq \mathcal{R}_1 \subseteq \mathcal{R}$. If $\mathcal{R}$ has the Amitsur property, then the semisimple class $S_{\mathcal{R}_1}$ is polynomially extensible: $A \in S_{\mathcal{R}_1}$ implies $A[x] \in S_{\mathcal{R}_1}$.

Proof. Assume that $A \in S_{\mathcal{R}_1}$. Since $\mathcal{R}$ has the Amitsur property, we have $(\mathcal{R}(A[x]) \cap A)[x] = \mathcal{R}(A[x]) \in \mathcal{R}$ and so $\mathcal{R}(A[x]) \cap A \in \mathcal{R}_1$ by condition (Δ). Hence $\mathcal{R}(A[x]) \cap A \subseteq \mathcal{R}_1(A) = 0$. Therefore by Proposition 3.9 we conclude that $\mathcal{R}(A[x]) = 0$, that is, $A[x] \in S_{\mathcal{R}}$. Taking into account that $\mathcal{R}_1 \subseteq \mathcal{R}$ it follows that $A[x] \in S_{\mathcal{R}_1}$. □

The next theorem characterizes the Amitsur property of a radical in terms of semisimple classes.

Theorem 3.20. A radical $\mathcal{R}$ has the Amitsur property if and only if

$$A[x] \in S_{\mathcal{R}} \; for \; all \; A \in S_{\mathcal{R}_x}. \qquad (D)$$

Proof. Assume that $\mathcal{R}$ has the Amitsur property, and let $A \in S_{\mathcal{R}_x}$ Since $\mathcal{R}$ has the Amitsur property, we have $(\mathcal{R}(A[x]) \cap A)[x] = \mathcal{R}(A[x]) \in \mathcal{R}$. Hence by the definition of $\mathcal{R}_x$ we infer that $\mathcal{R}(A[x]) \cap A \in \mathcal{R}_x$ and so $\mathcal{R}(A[x]) \cap A \subseteq \mathcal{R}_x(A) = 0$. Thus by Proposition 3.9 we conclude that $\mathcal{R}(A[x]) = 0$, that is, $A[x] \in S_{\mathcal{R}}$. Thus condition (D) is satisfied.

Conversely, assume the validity of condition (D). Suppose that $\mathcal{R}(A[x]) \cap A = 0$ for a semiring A. Since $\mathcal{R}_x(A) \in \mathcal{R}_x$ it follows that $\mathcal{R}_x(A)[x] \in \mathcal{R}$. This implies $\mathcal{R}_x(A)[x] \subseteq \mathcal{R}(A[x])$, and so $\mathcal{R}_x(A) \subseteq \mathcal{R}_x(A)[x] \cap A \subseteq \mathcal{R}(A[x]) \cap A = 0$. Hence $A \in S_{\mathcal{R}_x}$. Applying condition (D) we have $\mathcal{R}(A[x]) = 0$. Now Proposition 3.9 yields that $\mathcal{R}$ has the Amitsur property. □

Proposition 3.21. If a radical $\mathcal{R}$ has the Amitsur property, then the semisimple class $S_{\mathcal{R}}$ is polynomially extensible.

Proof. Claim that $S_{\mathcal{R}}$ is polynomially extensible. Assume that $A \in S_{\mathcal{R}}$. Since $\mathcal{R}$ has the Amitsur property, we have $(\mathcal{R}(A[x]) \cap A)[x] = \mathcal{R}(A[x]) \in \mathcal{R}$. Therefore $\mathcal{R}(A[x]) \cap A \subset \mathcal{R}(A) = 0$. Therefore $\mathcal{R}(A[x]) = 0$. Thus $A \in \mathcal{R} \Rightarrow \mathcal{R}(A[x]) = 0 \Rightarrow A[x] \in S_{\mathcal{R}}$. □

Proposition 3.22. For a hereditary radical $\mathcal{R}$, the following conditions are equivalent:

(i) $(f(x))_{A[x]} \in \mathcal{R} \Rightarrow (f(0))_{A[x]} \in \mathcal{R}$.

(ii) $f(x) \in \mathcal{R}(A[x]) \Rightarrow f(0) \in \mathcal{R}(A[x])$, for all semiring A.

Proposition 3.23. For any semiring A and any A-radical $\mathcal{R}$, we have $\mathcal{R}(A[x]) = \mathcal{R}(A)[x]$.

Proof. Let $\theta : (A[x])^+ \rightarrow \bigoplus \sum_{k=i}^{\infty} F_k$, where each $F_k \simeq A^+$ be the isomorphism defined as $\theta(a_0 + a_1x + a_2x^2 + a_3x^3 + ...) = (a_0, a_1, a_2, a_3, ...)$. Then $\mathcal{R}(A[x]) = \mathcal{R}(A)[x]$. □

Proposition 3.24. Every A- radical $\mathcal{R}$ has the Amitsur property.

Proof. By above Proposition 3.23, $\mathcal{R}(A[x]) = \mathcal{R}(A)[x]$. Also, we know that a radical $\mathcal{R}$ has Amitsur property if and only if $\mathcal{R}(A[x])$ is a polynomial semiring in x. Hence the Proposition. □

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