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ON LIE ALGEBRAS DEFINED BY A VECTOR-VALUED 1-FORM

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Abstract In this paper, we study various algebraic properties of the Lie algebras of vector fields on the tangent bundle of a differentiable manifold. In addition we particulary think to investigate the nullity space of the associated curvature to a vector-valued 1-form and the first space of Chevalley-Eilenberg's cohomology of these Lie algebras. To avoid the problem of solving partial differential equations, with superior dimension, related to this subject, we choose the algebraic approach using the formalism of Frölicher-Nijenhius. We use Maple software as a computer and resolution tool.

1 Introduction

The theory of the analytical approach oriented by Mathematicians like H. Rund, O. Loos, etc., is not satisfactorily established as fundamental tools for solving mathematical problems in differential equations. Several attempts were made to build an adequate approach. The only most interesting in this direction is the theory of the algebraic approach. This theory is based essentially on the formalism of Frölicher-Nijenhuis. This tool uses differential operators called finite order vector-valued forms on a finite dimension differentiable manifold. Indeed, it is the fundamental tool of our work in the case where the vector-valued forms are of degree 1 on a tangent bundle of a differentiable manifold of finite dimension. Operators called almost tangent structures, almost produced or complex structures and connections (in other words, connections in the sense of Grifone) are particular vector-valued forms of degree 1 (or vector-valued 1-forms). M. Anona in [1] generalized the almost tangent structure by considering a vector-valued 1-form satisfying certain conditions on a differentiable manifold. He investigated the cohomology induced by a vector-valued 1-form on a manifold and generalized some results of Grifone. N. L. Youssef adopted from the point of view of M. Anona in [14] a generalization of the results of J.Grifone on non-linear connections by considering a vector-valued 1-form of constant rank of zero torsion on a differentiable manifold. He found that this structure has properties similar to an almost tangent structure. In [9], M. Anona, P. Randriambololondrantomalala and H. S. G. Ravelonirina studied some properties of a vector-valued 1-form having an almost product structure in the sense of Grifone on a differentiable manifold. They studied in particular certain Lie algebras which attach to it. Therefore, some results have not yet been found when considering a vector-valued 1-form verifying under other conditions. This allowed us to develop, expand and generalize the studies for a vector-valued 1-form verifying certain conditions on a differentiable manifold.

Let M be a differentiable manifold of dimension n and class C^{∞}. We define by Γ a vector-valued 1-form on the tangent fiber TM such that Γ is class C^{∞} in $TM-\{0\}$ that $J\Gamma = J$ and $\Gamma J = -J$. This vector-valued 1-form is a connection within the sense of Grifone on M and Γ has a product structure on M. The data of this connection permit us to decompose TTM of the tangent bundle of TM in a direct sum of horizontal space $h\left(TM\right)$ and of vertical space $v\left(TM\right)$ where h and v are respectively the horizontal projector and vertical projector of the corresponding connection Γ to the respective eigenvalues 1 and -1. We see in [9] that in one part, the Lie algebra $\mathfrak{A}_\Gamma$ of vector fields on $TM - \{0\}$ which the corresponding Lie derivative of Γ is zero and in other part, the Lie algebra of the horizontal nullity space $\mathfrak{N}_R^h$ of the curvature R. In addition, they found some properties of the connection from normalizer of $\mathfrak{N}_R^h$ and the one of the cohomology space of Chevalley-Eilenberg of $\mathfrak{A}_\Gamma$. In this paper, we propose some properties of a vector-valued 1-form on the tangent bundle TM of M submissive to an almost product (resp. complex) structure in studying some Lie algebras which is attached to it and the involution of the nullity space of the associated Nijenhuis torsion R to L. Given L a vector-valued 1-form on the tangent bundle TM of the differentiable manifold M with dimension n, $\mathfrak{A}_L$ a Lie algebra of the vector fields on TM such that the corresponding Lie derivative to L is zero, $\mathfrak{A}_{L,\lambda_i}, i = 1, \ldots, n$ the corresponding eigen subspaces of $\mathfrak{A}_L$ to eigenvalues λ_i of L; $\mathfrak{N}_R$ the space of vector fields X on TM which the interior product in comparison to X of the associated curvature R is zero. The subspaces $\mathfrak{A}_{L,\lambda_i}, i = 1, \ldots, n$ are ideals of $\mathfrak{A}_L \cap \mathfrak{N}_R$ and $\mathfrak{A}_L$. We prove that a vector field X on M is an element of $\mathfrak{A}_L$ if and only if X leaves invariant the defined generalised distributions by the eigen subspaces $\mathfrak{A}_{L,\lambda_i}, i = 1, \ldots, n$ of L. If L is diagonalisable and the eigenvalues $\lambda_i, i = 1, \ldots, n$ are constants then $\mathfrak{A}_L \cap \mathfrak{N}_R$ constitute a direct product of ideals $\mathfrak{A}_{L,\lambda_i}, i = 1, \ldots, n$. Consequently, the Lie algebra $\mathfrak{A}_L$ equals to the direct product of $\mathfrak{A}_{L,\lambda_i}, i = 1, \ldots, n$ when the associated curvature R to L is null and all derivation of $\mathfrak{A}_L$ is inner. In other words, the first space of Chevalley-Eilenberg's cohomology of $\mathfrak{A}_L$ is null. Moreover, if the $\lambda_i, i = 1, \ldots, n$ are functions on M which rank of each λ_i is constant, the first space of Chevalley-Eilenberg's cohomology of $\mathfrak{A}_L$ is null. Otherwise, if all nullity fields of the associated curvature R to the vector-valued 1-form L are generated by $\mathfrak{A}_L \cap \mathfrak{N}_R$ then the nullity space $\mathfrak{N}_R$ is involutive. A classical result of [9] and [6] prove that all flat vector-valued 1-form L on TM has an involutive nullity space and we find the same result if L is a projection on TM, that is, $L^2 = L$. On the one hand, using a result of [10], if the nullity space $\mathfrak{N}_R$ of R is involutive then the derived ideal of $\mathfrak{N}_R$ is itself and the first spaces of Chevalley-Eilenberg's cohomology of $\mathfrak{N}_R$ and its normalizer $\mathcal{N}_R$ are null. On the other hand, if the nullity space $\mathfrak{N}_R$ of R is involutive which L is an almost product structure, that is $L^2 = I$, we prove that the image of the bracket of two nullity fields of the curvature R by L also belongs to $\mathfrak{N}_R$. Particular, all involutive nullity space has a nonzero centralizer. We get some examples to illustrate our results.

2 Preliminary

In the next, let M be a differentiable manifold of dimension n and class C^∞, TM the tangent fiber of M. All subjects supposed are class C^∞ on M or TM. The set $\chi(M)$ (resp. $\chi(TM)$) denotes Lie algebra of vector fields on M (resp. TM). A distribution on M (resp. TM) is a $F(M)-$submodule of $\chi(M)$ (resp. $F(TM)-$submodule of $\chi(TM)$).

Definition 2.1. [3] A vector-valued 1-form L on TM is a linear application on TM itself, that is, an endomorphism of TM. Let's consider L, H two vector-valued 1-forms on TM, we define the bracket of L with H by $[L,H](X,Y) = [LX,H]Y - [LY,H]X - L[X,H] + L[Y,H]$ for all vector fields X, Y on TM.

Definition 2.2. To the vector-valued 1-form L, we define Lie algebra $\mathfrak{A}_L$ of vector fields on TM which the corresponding Lie derivative of L is zero. A vector field X is then an element of $\mathfrak{A}_L$ if and only if $[X, LY] = L[X,Y]$ for all $Y \in \chi(TM)$. Localy, cf. [7], if $(x^i)_{i=1,\ldots,2n}$ the local coordinates on the tangent bundle TM we get $X = X^i \frac{\partial}{\partial x^i}$, $L = L^i_j dx^j \otimes \frac{\partial}{\partial x^i}$. $[X, L] = 0$ is equivalent to

$$X^i \frac{\partial L^j_k}{\partial x^i} - L^j_i \frac{\partial X^i}{\partial x^k} + L^i_k \frac{\partial X^j}{\partial x^i} = 0. \tag{2.1}$$

The relation (2.1) is a system of $4n^2$ linear partial differential equations of the first order of n-unknown functions X^i of C^∞ class for $i \in \{1, \ldots, 2n\}$.

Definition 2.3. [3] Let L be a vector-valued 1-form on TM. We define Nijenhuis torsion of L the vector-valued 2-form noted R by $R = \frac{1}{2}[L,L]$ such that for all $X, Y \in \chi(TM)$, we have $R(X,Y) = [LX, LY] + L^2[X,Y] - L[LX,Y] - L[X,LY]$. If $R = 0$ then L is flat. So we can define a foliation $\mathfrak{F}$ with finite dimension. Let's denote by $\mathfrak{A}_L^{\mathfrak{F}}$ the Lie algebra of infinitesimal automorphisms of the foliation $\mathfrak{F}$, that is, $X \in \mathfrak{A}_L^{\mathfrak{F}} \Longleftrightarrow [X, \mathfrak{A}_L] \subset \mathfrak{A}_L$. Consequentely, $\mathfrak{A}_L$ is a Lie subalgebra of $\mathfrak{A}_L^{\mathfrak{F}}$. In local coordinates (x^i, y^i) of TM, R is written by $R = \frac{1}{2} R^i_{jk} dx^j \wedge dx^k \otimes \frac{\partial}{\partial y^i}$ where $R^i_{jk} = L^i_l (\frac{\partial L^l_j}{\partial x^k} - \frac{\partial L^l_k}{\partial x^j}) + L^l_j \frac{\partial L^i_k}{\partial y^l} - L^l_k \frac{\partial L^i_j}{\partial y^l}$. The nullity space of associated curvature R to L is a set $\mathfrak{N}_R = \{X \in \chi(TM)$ such that $i_X R = 0\}$ where i_X denotes the interior product in comparison to a vector fields X.

Definition 2.4. [2] Let $\mathfrak{A}$ be a Lie algebra on manifold M. An ideal $I \subset \mathfrak{A}$ is a submodule of $\mathfrak{A}$ stable for all inner derivation.

Proposition 2.5. *$L\mathfrak{A}_L$ is an ideal of $\mathfrak{A}_L$ where $L\mathfrak{A}_L$ denotes the image of $\mathfrak{A}_L$ by L. In particular, if $L^2 = 0$ then $L\mathfrak{A}_L$ is a commutative ideal.*

Proof. Let $X \in \mathfrak{A}_L$. By definition of $\mathfrak{A}_L$, $[X, L\mathfrak{A}_L] \subset L\mathfrak{A}_L$. The nullity of the curvature R is written by $[LX, LY] = L[LX, Y] + L[X, LY] - L^2[X, Y]$ $\forall X, Y$ vector fields. If $X \in \mathfrak{A}_L$, the above relation becomes $[LX, LY] = L[LX, Y]$ $\forall Y$. So LX belongs to $\mathfrak{A}_L$ and we get $L\mathfrak{A}_L$. Similarly, if $Y \in \mathfrak{A}_L$ we have $[LX, LY] = L[LX, Y] = L^2[X, Y]$. Which gives us $[L\mathfrak{A}_L, L\mathfrak{A}_L] \subset L\mathfrak{A}_L$, that is, $L\mathfrak{A}_L$ is stable by bracket. And if $L^2 = 0$, $[LX, LY] = 0$ for all $X, Y \in \mathfrak{A}_L$. Thus LX and LY are commutated. □

Proposition 2.6. *The Lie algebra $\mathfrak{A}_L$ of L makes stable the nullity space $\mathfrak{N}_R$ of the curvature R.*

Proof. It's immediate. Let $X \in \mathfrak{A}_L \Longleftrightarrow [X, L] = 0$. According to Jacobi's identity, we have $[X, [L, L]] = [L, [X, L]] + [L, [X, L]] = 0$ because $[X, L] = 0$. Then we get $[X, R] = 0$. In other words, $[X, R](Y, Z) = [X, R(Y, Z)] - R([X, Y], Z) - R(Y, [X, Z]) = 0$. So X belongs to $\mathfrak{N}_R$. □

3 Some properties where L is diagonalisable

Let L be a vector-valued 1-form on TM, diagonalisable to constant eigenvalues $\lambda_1, \ldots, \lambda_p$ where the reals λ_i, $i = 1, \ldots, p$ are assumed to be two by two distincts. At any point $x \in M$, the tangent space $T_x M$ in x of M is written $T_x M = V_{\lambda_1}(x) \oplus \cdots \oplus V_{\lambda_p}(x)$ where $V_{\lambda_i}(x)$ denotes the corresponding vector eigenspace of $T_x M$ to the eigenvalue λ_i.

Proposition 3.1. *A vector field X on M is an element of $\mathfrak{A}_L$ if and only if X makes invariant the generalised distributions defined by the eigen subspaces of L.*

Proof. Let's assume $X \in \mathfrak{A}_L$ and $Y \in V_{\lambda_i}$. We get $[X, LY] = L[X, Y]$. That implies $L[X, Y] = \lambda_i [X, Y]$ because $[X, \lambda_i Y] = \lambda_i [X, Y]$. In other words, $[X, Y] \in V_{\lambda_i}$. Then X preserves the subspaces of L, that is, X leaves invariant the generalised distributions defined by the eigen subspaces of L. Conversely, let Y be a vector field on M. At each point $x \in M$, Y decomposes in an unique manner on the $V_{\lambda_i}(x)$ and we write $Y = Y^1 + \cdots + Y^p$ where $Y^i \in V_{\lambda_i}(x), i = 1, \ldots, p$. Since X preserves the eigen subspaces, we get $[X, LY] - L[X, Y] = 0$ for all $Y \in \chi(M)$. In other words, $X \in \mathfrak{A}_L$. □

Remark 3.2. This proposition generalizes the Proposition 4.1 of [9] within the case of a connection in the sense of Grifone.

Proposition 3.3. *The corresponding eigen subspace $\mathfrak{A}_{L,\lambda}$ of $\mathfrak{A}_L$ to the constant eigenvalue λ of L is an ideal of $\mathfrak{A}_L \cap \mathfrak{N}_R$ and $\mathfrak{A}_L$. For an other constant $\lambda' \neq \lambda$, we get the direct product $\mathfrak{A}_{L,\lambda} \otimes \mathfrak{A}_{L,\lambda'}$.*

Proof. It's obvious that $\mathfrak{A}_{L,\lambda}$ is a Lie algebra. For $X \in \mathfrak{A}_{L,\lambda}$ and $Y \in \mathfrak{A}_L$, we have $L[Y, X] = [Y, LX] = [Y, \lambda X] = \lambda [Y, X]$. So $[Y, X] \in \mathfrak{A}_{L,\lambda}$. Which proves that $\mathfrak{A}_{L,\lambda}$ is an ideal of $\mathfrak{A}_L$. Since $\mathfrak{A}_{L,\lambda} \in \mathfrak{A}_L \cap \mathfrak{N}_R$ and $\mathfrak{A}_{L,\lambda} \subset \mathfrak{A}_L$ is an ideal then $\mathfrak{A}_{L,\lambda}$ is an ideal of $\mathfrak{A}_L \cap \mathfrak{N}_R$. If we suppose that there is $\lambda' \neq \lambda$, it's immediate that $\mathfrak{A}_{L,\lambda} \cap \mathfrak{A}_{L,\lambda'} = \{0\}$. Let's assume two eigenvectors X and Y such that $X \in \mathfrak{A}_{L,\lambda}$ and $Y \in \mathfrak{A}_{L,\lambda'}$. We can write successively that $L[X, Y] = [LX, Y] = [X, LY] = \lambda [X, Y] = \lambda' [X, Y]$. So $(\lambda - \lambda')[X, Y] = 0$. Which is only possible if $[X, Y] = 0$. □

Remark 3.4. The Proposition 3.10 of [9] is a particular case of the Proposition 3.3.

Lemma 3.5. *Let V a subspace of $\mathbb{R}$- vector space W; $k_1, \dots, k_p$ the two by two distinct reals; $X^1, \dots, X^p$ the vector fields of W who verify $X^1 + \cdots + X^p \in V$ and that $k_1X^1 + \cdots + k_pX^p \in V$. Then every X^i belongs to V for all $i \in [1, p]$.*

Proof. It's immediate. □

Proposition 3.6. *$\mathfrak{A}_L \cap \mathfrak{N}_R$ is a direct product of ideals $\mathfrak{A}_{L,\lambda_i}$, $i = 1, \dots, p$. That is,*

$$\mathfrak{A}_L \cap \mathfrak{N}_R = \mathfrak{A}_{L,\lambda_1} \otimes \cdots \otimes \mathfrak{A}_{L,\lambda_p}.$$

Proof. According to Proposition 3.3 for all $\lambda_i \neq \lambda_j$ with $i \neq j$ $(i, j \in [1, p])$, the direct product of ideals $\mathfrak{A}_{L,\lambda_i}$ with $i = 1, \dots, p$, is an ideal of $\mathfrak{A}_L \cap \mathfrak{N}_R$. Then $\mathfrak{A}_{L,\lambda_1} \otimes \cdots \otimes \mathfrak{A}_{L,\lambda_p} \subset \mathfrak{A}_L \cap \mathfrak{N}_R$. Since $X \in \mathfrak{A}_L \cap \mathfrak{N}_R$ is equivalent to $X \in \mathfrak{A}_L$ and $LX \in \mathfrak{A}_L$. By decomposing X on the eigen subspaces V_{λ_i} $(i = 1, \dots, p)$, that $X = X^1 + \cdots + X^p$ we get $X^1 + \cdots + X^p \in \mathfrak{A}_L$ and $\lambda_1X^1 + \cdots + \lambda_pX^p \in \mathfrak{A}_L$ where $X^i \in V_{\lambda_i}$, $i = 1, \dots, p$. According to the previous Lemma 3.5 we have $X^i \in \mathfrak{A}_L$ for $i = 1, \dots, p$. Thus $X^i \in \mathfrak{A}_{L,\lambda_i}$, $i = 1, \dots, p$ and that $X \in \mathfrak{A}_{L,\lambda_1} \otimes \cdots \otimes \mathfrak{A}_{L,\lambda_p}$. Hence the reverse inclusion. □

Corollary 3.7. *If L is flat then we get $\mathfrak{A}_L = \mathfrak{A}_{L,\lambda_1} \otimes \cdots \otimes \mathfrak{A}_{L,\lambda_p}$*

Proof. Immediate. □

Definition 3.8. Let $\mathfrak{A}$ be a Lie algebra on M. We define the centralizer $\mathfrak{C}$ of $\mathfrak{A}$ the set of vector fields $X \in \mathfrak{A}$ such that $[X, Y] = 0$ for all $Y \in \chi(M)$.

Proposition 3.9. *We get the following assertions:*

(i) The centralizer of $\mathfrak{A}_L \cap \mathfrak{N}_R$(resp. $\mathfrak{A}_L$) in $\chi(M)$ is the direct sum of centralizers of $\mathfrak{A}_L \cap \mathfrak{N}_R$(resp. $\mathfrak{A}_L$) in $\chi(M)$ contained in eigen subspaces of L.

(ii) The center of $\mathfrak{A}_L \cap \mathfrak{N}_R$ is the direct product of centers of $\mathfrak{A}_{L,\lambda_i}$, $i = 1, \dots, p$.

(iii) The center of $\mathfrak{A}_L$ commutes with all projection of $\mathfrak{A}_L$ on the eigen subspaces of L.

Proof. (i) Let Y be an element of the centralizer of $\mathfrak{A}_L \cap \mathfrak{N}_R$ (resp. of $\mathfrak{A}_L$) in $\chi(M)$. For all $X \in \mathfrak{A}_L \cap \mathfrak{N}_R$ (resp. $X \in \mathfrak{A}_L$) we have $[Y, X] = 0$. By decomposing Y on the eigen subspaces of L we have $Y = Y^1 + \cdots + Y^p$. We get

$$\left[Y^1, X\right] + \cdots + \left[Y^p, X\right] = 0, \textit{ where } Y^i \in V_{\lambda_i}, i = 1, \dots, p. \tag{3.1}$$

Applying L to the relation (3.1) we obtain

$$L(\left[Y^1, X\right] + \cdots + L\left[Y^p, X\right]) = \lambda_1\left[Y^1, X\right] + \cdots + \lambda_p\left[Y^p, X\right] = 0 \tag{3.2}$$

According to Lemma 3.5, two relations (3.1) and (3.2) give $[Y^i, X] = 0$ (here $V = \{0\}$) for all $X \in \mathfrak{A}_L \cap \mathfrak{N}_R$ (resp. $X \in \mathfrak{A}_L$) with $i \in [1, p]$. Hence the first assertion.

(ii) Given Y^i an element of the center $\mathfrak{A}_L \cap \mathfrak{N}_R$. By definition Y^i commutes with $\mathfrak{A}_{L,\lambda_i}$, $i = 1, \dots, p$ and an element of $\mathfrak{A}_{L,\lambda_i}$. For each $j \neq i$, $\mathfrak{A}_{L,\lambda_j}$ is to direct product with $\mathfrak{A}_{L,\lambda_i}$ (according to Proposition 3.3). Therefore Y^i commutes with $\mathfrak{A}_{L,\lambda_j}$, $j \neq i$. Then Y^i commutes with the direct product $\mathfrak{A}_{L,\lambda_i} \otimes \mathfrak{A}_{L,\lambda_j}$. According to Propostion 3.6, Y^i commutes with $\mathfrak{A}_L \cap \mathfrak{N}_R$. In other words, Y^i, $i = 1, \dots, p$, is an element of the center of $\mathfrak{A}_L \cap \mathfrak{N}_R$. If Y is an element of the center of $\mathfrak{A}_L \cap \mathfrak{N}_R$, Y commutes with $\mathfrak{A}_L \cap \mathfrak{N}_R$. But, $\mathfrak{A}_L \cap \mathfrak{N}_R$ is to direct product of $\mathfrak{A}_{L,\lambda_i}$ for $i = 1, \dots, p$. The component Y^i of Y commutes with $\mathfrak{A}_{L,\lambda_i}$. We get the second assertion.

(iii) Let Y be an element of the center of $\mathfrak{A}_L$ and X the one that of $\mathfrak{A}_L$, we have $[Y, X] = 0$. In writing $X = X^1 + \cdots + X^p$ where $X^i \in V_{\lambda_i}$, we obtain

$$\left[Y, X^1\right] + \cdots + \left[Y, X^p\right] = 0 \tag{3.3}$$

Applying L to the relation (3.3) and taking into account Y is an element to $\mathfrak{A}_L$, we get

$$\lambda_1\left[Y, X^1\right] + \cdots + \lambda_p\left[Y, X^p\right] = 0. \tag{3.4}$$

According to Lemma 3.5, $[Y, X^i] = 0$ for $i \in [1, p]$ and for all $X \in \mathfrak{A}_L$. The center of $\mathfrak{A}_L$ commutes therefore with all projection X^i of $\mathfrak{A}_L$ on the eigen subspaces of L.

□

Definition 3.10. Let $\mathfrak{A}$ be a Lie algebra. We define the normalizer $\mathcal{N}$ of $\mathfrak{A}$ the set of vector fields $X \in \mathfrak{A}$ such that $[X, \mathfrak{A}] \subset \mathfrak{A}$.

Proposition 3.11. *We suppose $\mathcal{N}_{L,\lambda_i}$ (resp. $\mathcal{C}_{L,\lambda_i}$) the normalizer (resp. centralizer) of $\mathfrak{A}_{L,\lambda_i}$ in $\chi(M)$, $\mathcal{N}^j_{L,\lambda_i} = \mathcal{N}_{L,\lambda_i} \cap V_{\lambda_j}$, $\mathcal{C}^j_{L,\lambda_i} = \mathcal{C}_{L,\lambda_i} \cap V_{\lambda_j}$ with $i, j \in [1, p]$. Then we have*

(i) $\mathcal{N}_{L,\lambda_i} = \mathcal{N}^i_{L,\lambda_i} + \sum_{j\neq i}^p \mathcal{C}^j_{L,\lambda_i}$;

(ii) $\cap_{i=1}^p \mathcal{N}_{L,\lambda_i} = \sum_{i=1}^p \mathcal{N}^i_{L,\lambda_i}(\cap_{j\neq i}^p \mathcal{C}^j_{L,\lambda_i})$;

(iii) if $\cap_{i=1}^p \mathcal{C}_{L,\lambda_i} = \{0\}$ *, so we get* $\cap_{i=1}^p \mathcal{N}_{L,\lambda_i} = \Pi_{i=1}^p \mathcal{N}^i_{L,\lambda_i}(\cap_{j\neq i}^p \mathcal{C}^j_{L,\lambda_i})$ *and* $\mathfrak{A}_L = \Pi_{i=1}^p \mathfrak{A}_{L,\lambda_i}$.

Proof. (i) We consider $X^{i_0} \in \mathfrak{A}_{L,\lambda_{i_0}}$ and $Y \in \mathcal{N}_{L,\lambda_{i_0}}$ for an $i_0 \in [1, p]$ we obtain $[Y, X^{i_0}] \in \mathfrak{A}_{L,\lambda_{i_0}}$. By decomposing Y on the eigen subspaces of L we have $Y = Y^1 + \cdots + Y^p$ with $Y^i \in V_{k_i}$, $i = 1, \ldots, p$. We get

$$\left[Y^1, X^{i_0}\right] + \cdots + \left[Y^p, X^{i_0}\right] \in \mathfrak{A}_{L,\lambda_{i_0}}. \tag{3.5}$$

Applying L to the relation (3.5), we have

$$\lambda_1 \left[Y^1, X^{i_0}\right] + \cdots + \lambda_p \left[Y^p, X^{i_0}\right] \in \mathfrak{A}_{L,\lambda_{i_0}}. \tag{3.6}$$

According to Lemma 3.5, $[Y^j, X^{i_0}] \in \mathfrak{A}_{L,\lambda_{i_0}}$ for $i \in [1, p]$ and for an $i_0 \in [1, p]$. In other words, the component of Y belongs to $\mathcal{N}_{L,\lambda_{i_0}}$. Since the components of Y are elements of $V_{\lambda_i}, i = 1, \ldots, p$ then they belong to $\mathcal{N}^{i_0}_{L,\lambda_{i_0}}$ as well as Y. The relation (3.6) is the image of (3.5). Taking into account that $X^{i_0} \in \mathfrak{A}_L$, we get $L\left[Y^1, X^{i_0}\right] + \cdots + L\left[Y^p, X^{i_0}\right] = \lambda_{i_0}\left(\left[Y^1, X^{i_0}\right] + \cdots + \left[Y^p, X^{i_0}\right]\right)$. So we have equality $\lambda_1 \left[Y^1, X^{i_0}\right] + \cdots + \lambda_p \left[Y^p, X^{i_0}\right] = \lambda_{i_0}(\left[Y^1, X^{i_0}\right] + \cdots +$
$+ \left[Y^p, X^{i_0}\right])$ for $i \neq i_0$. We derive the following equation

$$\sum_{i\neq i_0}^p (\lambda_i - \lambda_{i_0}) \left[Y^i, X^{i_0}\right] = 0. \tag{3.7}$$

Applying L back to the relation (3.7), we have

$$\sum_{i\neq i_0}^p (k_i - k_{i_0}) L \left[Y^i, X^{i_0}\right] = 0 = \sum_{i\neq i_0}^p (k_i - k_{i_0}) k_i \left[Y^i, X^{i_0}\right]. \tag{3.8}$$

Thus

$$\sum_{i\neq i_0}^p \lambda_i (\lambda_i - \lambda_{i_0}) \left[Y^i, X^{i_0}\right] = 0. \tag{3.9}$$

According to Lemma 3.5, the relations (3.7) and (3.9) give $(k_i - k_{i_0}) \left[Y^i, X^{i_0}\right] = 0$ for all $i \neq i_0$. For $i \neq i_0$, $[Y^i, X^{i_0}] = 0$. The components $Y^i, i \neq i_0$, of Y, belong to $\mathcal{C}_{L,\lambda_{i_0}}$. Since $Y^i \in V_{\lambda_i}, i = 1, \ldots, p$, so we have $Y^i \in \mathcal{C}^i_{L,\lambda_{i_0}}, i \neq i_0$. Hence $Y \in \mathcal{N}^{i_0}_{L,\lambda_{i_0}} + \sum_{i\neq i_0}^p \mathcal{C}^i_{L,\lambda_{i_0}}$. Reverse inclusion is obvious.

(ii) This is an immediate consequence of the first assertion.

(iii) We suppose that $\cap_{i=1}^p \mathcal{C}_{L,\lambda_i} = 0$. We consider $Y^i \in \mathcal{N}^i_{L,\lambda_i}(\cap_{j\neq i}^p \mathcal{C}^j_{L,\lambda_i})$ and $Y^1 \in \mathcal{N}^1_{L,\lambda_1}(\cap_{j\neq 1}^p \mathcal{C}^j_{L,\lambda_1})$ with $i \neq 1$. According to Jacobi's identity, we have

$$\left[\left[Y^i, Y^1\right], X^{i_0}\right] = \left[\left[X^{i_0}, Y^1\right], Y^i\right] + \left[\left[Y^i, X^{i_0}\right], Y^1\right]. \tag{3.10}$$

Taking into account that $i \neq 1$, two terms of second member of the equality (3.10) are nulls. So $\left[Y^i, Y^1\right] \in \mathcal{C}_{L,\lambda_{i_0}}$ for $i_0 \in [1, p]$ and thus $\left[Y^i, Y^1\right] \in \cap_{i_0=1}^p \mathcal{C}_{L,\lambda_{i_0}}$. If $\cap_{i_0=1}^p \mathcal{C}_{L,\lambda_{i_0}} = \{0\}$, we have $\left[Y^i, Y^1\right] = 0$ for $i \neq 1$. In other words we get the direct product of

$\mathcal{N}^i_{L,\lambda_i}(\cap^p_{j\neq i}\mathcal{C}^j_{L,\lambda_i})$ with $\mathcal{N}^1_{L,\lambda_1}(\cap^p_{j\neq 1}\mathcal{C}^j_{L,\lambda_1})$. We suppose $Y^i, Z^i \in \mathcal{N}^i_{L,\lambda_i}(\cap^p_{j\neq i}\mathcal{C}^j_{L,\lambda_i})$, we have $[Y^i, Z^i] \in \mathcal{N}^i_{L,\lambda_i}(\cap^p_{j\neq i}\mathcal{C}^j_{L,\lambda_i})$. And $[Y^i, Z^i]$ also belongs to the eigen subspace V_{λ_i}. Hence the result $\cap^p_{i=1}\mathcal{N}_{L,\lambda_i} = \Pi^p_{i=1}\mathcal{N}^i_{L,\lambda_i}(\cap^p_{j\neq i}\mathcal{C}^j_{L,\lambda_i})$. Finally, it's obvious that $\mathfrak{A}_L \subset \cap^p_{i=1}\mathcal{N}_{L,\lambda_i}$. By definition, $\mathfrak{A}_{L,\lambda_i} = \mathfrak{A}_L \cap V_{\lambda_i}$. By the expression of $\cap^p_{i=1}\mathcal{N}_{L,\lambda_i}$,that is, $\cap^p_{i=1}\mathcal{N}_{L,\lambda_i} = \Pi^p_{i=1}\mathcal{N}^i_{L,\lambda_i}(\cap^p_{j\neq i}\mathcal{C}^j_{L,\lambda_i})$ and $\mathfrak{A}_L \subset \cap^p_{i=1}\mathcal{N}_{L,\lambda_i}$. We obtain $\mathfrak{A}_L \subset \Pi^p_{i=1}\mathfrak{A}_{L,\lambda_i}$. Hence $\mathfrak{A}_L = \Pi^p_{i=1}\mathfrak{A}_{L,\lambda_i}$.

□

Theorem 3.12. *If L is flat, diagonalizable, with constant eigenvalues $\lambda_i, i = 1, \dots, p$ such that the multiplicity order of λ_i was constant. Then $\mathfrak{A}_L = \mathfrak{A}_{L,\lambda_1} \otimes \dots \otimes \mathfrak{A}_{L,\lambda_p}$ and all derivation of $\mathfrak{A}_L$ is inner. So the first space of Chevalley-Eilenberg's cohomology $\mathfrak{A}_L$ is null.*

Proof. If L is flat, we have $\mathfrak{A}_L = \mathfrak{A}_{L,\lambda_1} \otimes \dots \otimes \mathfrak{A}_{L,\lambda_p}$ according to Corollary 3.7. Then every $\mathfrak{A}_{L,\lambda_i}, i = 1, \dots, p$, is a characteristic ideal of $\mathfrak{A}_L$. In other words, the $\mathfrak{A}_{L,\lambda_i}, i = 1, \dots, p$, are the submodules of $\mathfrak{A}_L$ stables for all derivation of $\mathfrak{A}_L$.
Let X be an eigenfield of L associated to the eigenvalue $\lambda_i, i = 1 \dots p$, on M such that $X = 0$ on a domain U of the adapted chart to the foliation. We denote D the derivation of $\mathfrak{A}_{L,\lambda_i}, i = 1, \dots, p$. D is a local derivation on the adapted chart of domain U to the foliation, that is, $D(X) = 0$ on U. We say that $\mathfrak{A}_{L,\lambda_i}$ is a characteristic ideal of $\mathfrak{A}_L$. There is $Z \in \mathfrak{A}_{L,\lambda_i}$ such that $D_{|\mathfrak{A}_{L,\lambda_i}}(X) = [Z, X]$ for all $X \in \mathfrak{A}_{L,\lambda_i}$. Let's consider $Y \in \mathfrak{A}_L$ we have $D[X, Y] = [D(X), Y] + [X, D(Y)]$. Since $[X, Y] \in \mathfrak{A}_{L,\lambda_i}$ and $D_{|\mathfrak{A}_{L,\lambda_i}}$ is a derivation of $\mathfrak{A}_{L,\lambda_i}$ then we get $D[X, Y] = D_{|\mathfrak{A}_{L,\lambda_i}}[X, Y] = [Z, [X, Y]]$. The field X belongs to $\mathfrak{A}_{L,\lambda_i}$ so we obtain $D(X) = D_{|\mathfrak{A}_{L,\lambda_i}}(X) = [Z, X]$. And $[Z, [X, Y]] = [[Z, X], Y] + [X, D(Y)]$, that is

$$[Z, [X, Y]] + [Y, [Z, X]] = -[D(Y), X] \tag{3.11}$$

But according to Jacobi's identity, we have

$$[Z, [X, Y]] + [Y, [Z, X]] = [[Y, Z], X]. \tag{3.12}$$

The relations (3.11) and (3.12) give $-[D(Y), X] = [[Y, Z], X]$, which is equivalent to $[D(Y), X] + [[Y, Z], X] = 0$. We get $[D(Y) - [Z, Y], X] = 0$ for $X \in \mathfrak{A}_{L,\lambda_i}$. But the bracket is distributive with respect to $+$ then $D(Y) - [Z, Y] \in \mathcal{C}_{L,\lambda_i}, i = 1, \dots, p$. Which gives $D(Y) - [Z, Y] \in \cap^p_{i=1}\mathcal{C}_{L,\lambda_i}$ (and here $\cap^p_{i=1}\mathcal{C}_{L,\lambda_i} = \{0\}$). So, $D(Y) = [Z, Y]$ for all $Y \in \mathfrak{A}_L$, that is, D is inner. Consequentely, all derivation of $\mathfrak{A}_L$ is inner. In other words, its first space of Chevalley-Eilenberg's cohomology is null. □

The nullity space $\mathfrak{N}_R$ of the curvature R is a distribution of TM. In general, this space $\mathfrak{N}_R$ is not involutive.

4 On involution of the nullity space $\mathfrak{N}_R$ of the associated curvature R to L

4.1 Case where L assumes an almost product structure

In the next, we suppose that L defines an almost product structure on the tangent bundle TM of M, that is, $L^2 = I$.

Definition 4.1. [1] A distribution Ω is said involutive if $[\Omega, \Omega] \subset \Omega$. In other words, for all $X, Y \in \Omega$ we have $[X, Y] \in \Omega$.

Proposition 4.2. *If all elements of the nullity space $\mathfrak{N}_R$ are generated by the elements of $\mathfrak{A}_L \cap \mathfrak{N}_R$ then $\mathfrak{N}_R$ is involutive.*

Proof. Let X, Y be the nullity fields generated by $\mathfrak{A}_L \cap \mathfrak{N}_R$. For $X, Y \in \mathfrak{N}_R$, we get $R(X, Z) = 0 = R(Y, Z)$ for all $Z \in \chi(TM)$. If $X \in \mathfrak{A}_L \Leftrightarrow [X, L] = 0$; which implies that $[X, R] = 0$. Similarly for $Y \in \mathfrak{A}_L \Leftrightarrow [Y, L] = 0$ implies that $[Y, R] = 0$. And for all $Z \in \chi(TM)$, we obtain $[X, R](Y, Z) = 0 = [X, R(Y, Z)] - R([X, Y], Z) - R(Y, [X, Z])$. Therefore $R([X, Y], Z) = 0$. In other words $[X, Y] \in \mathfrak{N}_R$. In a similar way, we have $[Y, R] = 0$. □

Remark 4.3. The Proposition 3.2 of [9] is obtained by Proposition 4.2 in the case where L is a connection in the sense of Grifone.

Proposition 4.4. *If L is flat then the nullity space $\mathfrak{N}_R$ of the associated curvature R to L is involutive.*

Proof. This result is classic. □

Definition 4.5. A vector-valued 1-form L of TM is a projection if $L^2 = L$.

Theorem 4.6. *If L is a projection of TM then the nullity space $\mathfrak{N}_R$ of the associated curvature R to L is involutive.*

Proof. Let's consider $X, Y \in \mathfrak{N}_R$ and for $Z \in \chi(TM)$. By decomposition, $R([X,Y], Z) = [L[X,Y], LZ] + L[[X,Y], Z] - L[L[X,Y], LZ] - L[[X,Y], Z] = [L[X,Y], LZ] + L[[X,Y], Z] - L[[X,Y], L].Z - L[[X,Y], Z] - L[L[X,Y], Z] = [L[X,Y], LZ] - L[[X,Y], L].Z - L[L[X,Y], Z] = [L[X,Y], LZ] - L[[X,Y], L].Z - [L[X,Y], LZ] + L[[X,Y], L].Z = 0$. In other words, $[X,Y] \in \mathfrak{N}_R$. □

Corollary 4.7. *If L is a projection on TM, $\mathfrak{A}_L \cap \mathfrak{N}_R$ defines a Lie algebra on M.*

Proof. Immediate. □

Definition 4.8. Every R of L, we associate a vector-valued 2-form $\mathfrak{R}$ defined by $\mathfrak{R}(X,Y) = L^2 R(X,Y) + R(LX, LY) - LR(LX, Y) - LR(X, LY)$ for all $X, Y \in \chi(M)$.

Remark 4.9. We notice that if $L^2 = I$ then by simple calculation we have $R = \frac{1}{4}\mathfrak{R}$. Similarly, for $L^2 = -I$, we find $R = -\frac{1}{4}\mathfrak{R}$. Finally, the nullity space $\mathfrak{N}_R$ of the curvature R is the subspace of the one of curvature $\mathfrak{R}$.

Proposition 4.10. *If the nullity space $\mathfrak{N}_R$ of the associated curvature R to L is involutive, the nullity space $\mathfrak{N}_{\mathfrak{R}}$ of the curvature $\mathfrak{R}$ is also.*

Proof. We just use the above relation. □

Theorem 4.11. *Let $\mathcal{N}_R$ be the normalizer of $\mathfrak{N}_R$. If $\mathfrak{N}_R$ is involutive then it coincides with $\mathcal{N}_R$.*

Proof. It's obvious that $\mathfrak{N}_R \subset \mathcal{N}_R$. It remains to prove the reverse inclusion.
We consider $X \in \mathcal{N}_R$, we have $[X, \mathfrak{N}_R] \subset \mathfrak{N}_R$. That is, $[X, Y] \subset \mathfrak{N}_R$ for all $Y \in \mathfrak{N}_R$. Since $\mathfrak{N}_R$ is involutive then $X \in \mathfrak{N}_R$. Thus $\mathcal{N}_R \subset \mathfrak{N}_R$. □

Definition 4.12. [8] We call first space of Chevalley-Eilenberg cohomology of $\mathfrak{A}$ the quotient vector space $H^1(\mathfrak{A}) = Der(\mathfrak{A})/_{ad\mathfrak{A}}$ where $Der(\mathfrak{A})$ (resp. $ad\mathfrak{A}$) is Lie algebra of derivations (resp. inner derivations) of $\mathfrak{A}$.

Theorem 4.13. *If the nullity space $\mathfrak{N}_R$ is involutive, and if we denote by $\mathcal{N}_R$ its normalizer in $\chi(TM\backslash\{0\})$ then*

- *The derivative ideal of $\mathfrak{N}_R$ coincides with $\mathfrak{N}_R$,*
- *The first space of Chevalley-Eilenberg cohomology respectively of $\mathfrak{N}_R$ and $\mathcal{N}_R$ are reduced to zero.*

Proof. The first assertion results by the Theorem 4.11. For the second assertion, if $\mathfrak{N}_R$ is involutive then it is a Lie algebra on $\chi(M)$.
According to Theorem 2.32 de [10], we obtain that the first space of Chevalley-Eilenberg cohomology of $\mathfrak{N}_R$ is isomorphic to the quotient space $\mathcal{N}_R/_{\mathfrak{N}_R}$. From hypothesis, the spaces $\mathcal{N}_R$ and $\mathfrak{N}_R$ coincide. The first space of Chevalley-Eilenberg cohomology of $\mathfrak{N}_R$ is therefore reduced to zero. □

Corollary 4.14. *If the nullity space $\mathfrak{N}_R$ of the associated curvature R to L is involutive then the first space of Chevalley-Eilenberg cohomology of $\mathfrak{A}_L \cap \mathfrak{N}_R$ is null.*

Theorem 4.15. *Any vectors fields $X, Y \in \mathfrak{A}_L \cap \mathfrak{N}_R$ if and only if $R([X,Y], Z) = 0$ for all $Z \in \chi(TM)$.*

Proof. We consider $X, Y \in \mathfrak{A}_L \cap \mathfrak{N}_R$, that is, $LX, LY \in \mathfrak{A}_L$. According to the expression of R in the almost product structure, we get $R([X,Y], Z) = 0$ for all $Z \in \chi(M)$. Conversely, we suppose that $R([X,Y], Z) = 0$, that is, $[X,Y] \in \mathfrak{N}_R$. It remains to prove that $[X,Y] \in \mathfrak{A}_L$. By absurd, supposing that $[X,Y] \notin \mathfrak{A}_L$ and by definition $L[[X,Y], Z] \neq [[X,Y], LZ]$ then $R([X,Y], Z) \neq 0$, which contradicts to the hypothesis. Necessarily, $[X,Y] \in \mathfrak{A}_L$. □

Theorem 4.16. *Let $\mathfrak{C}_R$ be the centralizer of the nullity space $\mathfrak{N}_R$ of R. If the nullity space $\mathfrak{N}_R$ is involutive then its centralizer $\mathfrak{C}_R$ is not null.*

Proof. Let X be a vector field on M such that $[X, \chi(M)] \equiv \{0\}$. Since $\mathfrak{N}_R$ is involutive then $[\mathfrak{N}_R, \mathfrak{N}_R] \equiv \mathfrak{N}_R$ (by adapting theorem 4.13). We obtain $[X, \chi(M)] \subset [\mathfrak{N}_R, \mathfrak{N}_R] \equiv \mathfrak{N}_R$. A nullity field allows us to have $X \neq 0$ because $X \in \mathfrak{N}_R$. Hence the result. □

Example 4.17. Let $M = \mathbb{R}^3$ be a manifold of local coordinates $(x^i, y^i)_{i=1,\dots,3}$ on the tangent bundle $T\mathbb{R}^3$, L the following vector-valued 1-form: $L^1_2 = x^1$, $L^2_3 = x^1x^2$ and $L^i_j = 0$ otherwise. So, L is written by $L = x^1 dx^2 \otimes \frac{\partial}{\partial x^1} + x^1x^2 dx^3 \otimes \frac{\partial}{\partial x^2}$. A vector field $X = X^i \frac{\partial}{\partial x^i}$ belongs to $\mathfrak{A}_L$ if and only if X can be written under form $X = \frac{1}{x^1}\frac{\partial}{\partial x^1} + \left(\frac{x^2}{x^1} + P\left(x^2, x^3\right)\right)\frac{\partial}{\partial x^2} + k\frac{\partial}{\partial x^3} + Y^i\frac{\partial}{\partial y^i}$, where k is a constant. The components of associated curvature R to L are: $R^1_{12} = -1 = -R^1_{21}$, $R^1_{13} = -x^1x^2 = -R^1_{31}$; $R^1_{23} = -x^2 = -R^1_{32}$ and; other are null. The nullity space $\mathfrak{N}_R$ of R can to define as follow $\mathfrak{N}_R = \{(x^1)^2 P(x^i)\frac{\partial}{\partial x^1} + x^1x^2 Q(x^i)\frac{\partial}{\partial x^2} + Z(x^i)\frac{\partial}{\partial x^3} + Y^i\frac{\partial}{\partial y^i}, i = 1, \dots, 3\}$. We obtain $\mathfrak{A}_L \cap \mathfrak{N}_R = \{(x^1)^2 P^1(x^i, i \neq 3)\frac{\partial}{\partial x^1} + x^1x^2 Q^2(x^i)\frac{\partial}{\partial x^2} + x^1x^2 Z^3(x^i)\frac{\partial}{\partial x^3} + Y^i\frac{\partial}{\partial y^i},\ i = 1, \dots, 3\}$, that is a generator of the nullity space. The space $\mathfrak{N}_R$ is thus involutive.

Example 4.18. Let $M = \mathbb{R}^3$ be a manifold; $(x^i, y^i)_{i=1\dots3}$ the local coordinates of tangent fiber $T\mathbb{R}^3$. We define L by: $L^3_3 = 1, L^1_2 = e^{x^1y^1}, L^2_1 = e^{-x^1y^1}$ and that $L^i_j = 0$ otherwise. And the associated curvature R to L is null. In other words, L is flat. Thus $\mathfrak{N}_R = \chi(\mathbb{R}^3)$. Hence $\mathfrak{N}_R$ is then involutive.

Example 4.19. Case where $M = \mathbb{R}^3$, let $(x^i, y^i)_{i=1,\dots,3}$ be local coordinates of $T\mathbb{R}^3$. We define L by: $L^1_1 = 1, L^2_1 = x^1y^2$ and $L^i_j = 0$ otherwise. We have $L^2 = L$. All components of the associated curvature R to L are nulls, that is $R = 0$. Then $\mathfrak{N}_R = \chi(\mathbb{R}^3)$. Thus $\mathfrak{N}_R$ is involutive.

Example 4.20. We assume $M = \mathbb{R}^2$; $(x^i, y^j)_{i,j=1,2}$ the local coordinates of the tangent space $T\mathbb{R}^2$. The components of L are: $L^1_2 = e^{x^2}, L^2_1 = e^{-x^2}$ and others are null. We have $L^2 = I$ and the coefficients of the associated curvature R to L are null. In other words, $R = 0$ and $\mathfrak{N}_R = \chi(\mathbb{R}^2)$ which is involutive. Thus its normalizer is generated by vector fields $x^k\frac{\partial}{\partial x^i}$ and the $\frac{\partial}{\partial x^i} + Y^i(x^l, y^m)\frac{\partial}{\partial y^j}, i, j, k, l, m = 1, 2$.

Example 4.21. We take the previous example 4.18 in the case of $M = \mathbb{R}^3$ by considering (x, y, z, u, v, w) the local coordinates of the tangent bundle $T\mathbb{R}^3$. We get $L^2 = I$ and the nullity space $\mathfrak{N}_R$ of the associated curvature R to L equals to $\chi(M)$. Consider $\mathfrak{N}_R = \{X(x, y, z)\frac{\partial}{\partial x} + Y(x, y, z)\frac{\partial}{\partial y} + Z(x, y, z)\frac{\partial}{\partial z}\}$. Thus the centralizer $\mathfrak{C}_R$ of $\mathfrak{N}_R$ is generated by vector fields $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial u}$, $\frac{\partial}{\partial v}$, $\frac{\partial}{\partial w}$.

4.2 Case where L defines an almost complex structure

In the next, we suppose that L defines an almost complex structure on the tangent bundle TM of M, that is, $L^2 = -I$.

Proposition 4.22. *[1] A vector field X is an infinitesimal automorphism of L if and only if we have $[X, L] = 0$.*

Proposition 4.23. *If there is a subset of infinitesimal automorphism fields of L generating the nullity space $\mathfrak{N}_R$ of the curvature R then the space $\mathfrak{N}_R$ is involutive.*

Proof. Let's consider X, Y two nullity fields of curvature R, we have $R(X,Z) = 0 = R(Y,Z)$ for all $Z \in \chi(M)$. According to expression of R, taking into account that fields X and Y are infinitesimal automorphisms of L and the Proposition 4.22, we obtain $R([X,Y],Z) = [L[X,Y],LZ] - [[X,Y],Z] - L[[X,Y],LZ] - L[L[X,Y],Z] = 0$ for all $Z \in \chi(M)$. Thus the bracket $[X,Y]$ belongs to the space $\mathfrak{N}_R$. Hence the result. □

Remark 4.24. The Proposition 3.2 of [9] is a particular case of the previous Proposition in the case of L is a connection of Grifone.

Proposition 4.25. *If the nullity space $\mathfrak{N}_R$ is generated by infinitesimal automorphism fields then*

- $[\mathfrak{N}_R, \mathfrak{A}_L] \subset \mathfrak{A}_L$,
- $\mathfrak{N}_R$ *is a Lie algebra of the nullity fields of R.*

Proof.
- We consider $X \in \mathfrak{N}_R$ and $Y \in \mathfrak{A}_L$, we get $[[X,Y],L] = -[[Y,L],X] + [[X,L],Z]$ according to Jacobi's identity. So we have $[[X,Y],L] = [[X,L],Z] = 0$ because X is an infinitesimal automorphism nullity field of L. Thus $[X,Y] \subset \mathfrak{A}_L$.
- Since all nullity fields are infinitesimal automorphism nullity fields then $\mathfrak{N}_R$ is involutive, that is, $[X,Y] \in \mathfrak{N}_R$ for $X, Y \in \mathfrak{N}_R$.

□

Remark 4.26. This Proposition contains a result of [9] in the case where L is a connection within the sense of Grifone that has an almost product structure while its Lie algebra is generated by projectable fields.

Proposition 4.27. *We denote by $\mathfrak{A}_L^{\mathfrak{F}}$ the set of infinitesimal automorphism fields of L on TM. The nullity space $\mathfrak{N}_R$ is involutive if and only if $\mathfrak{N}_R = \mathfrak{A}_L^{\mathfrak{F}} \cap \chi(TM)$.*

Proof. If $\mathfrak{N}_R$ is involutive then all nullity fields are generated by elements of $\mathfrak{A}_L^{\mathfrak{F}}$ according to Proposition 4.23. Thus $\mathfrak{N}_R \subset \mathfrak{A}_L^{\mathfrak{F}}$. Hence the inclusion $\mathfrak{N}_R \subset \chi(TM) \cap \mathfrak{A}_L^{\mathfrak{F}}$. We consider $X \in \mathfrak{A}_L^{\mathfrak{F}} \cap \chi(TM)$ then this vector field X verifies $[X,L] = 0$ and for all Y vectors field of TM, we have $R(X,Y) = 0$. Therefore $X \in \mathfrak{N}_R$, hence the reverse inclusion. Conversely, supposing that $\mathfrak{N}_R = \mathfrak{A}_L^{\mathfrak{F}} \cap \chi(TM)$ then $\mathfrak{N}_R$ is involutive. □

Proposition 4.28. *We suppose that the nullity space $\mathfrak{N}_R$ is involutive and if we denote by $\mathfrak{C}_R$ its centralizer. The intersection of $\mathfrak{N}_R$ with its centralizer $\mathfrak{C}_R$ is not reduced to zero.*

Proof. If $\mathfrak{N}_R$ is involutive then $\mathfrak{C}_R$ is not reduced to zero according to Theorem 4.16. So the vectors field belonging to $\mathfrak{C}_R$ generates the nullity fields of R. □

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