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On relative quasi hyperideals and relative bi-hyperideals in ordered semihypergroups

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Abstract. In this paper, we introduce relative bi-hyperideals, relative m- left hyperideal, relative n-right hyperideal, relative quasi hyperideal, relative regularity, and relative intra-regularity of ordered semihypergroup H. We prove that every $(S_1, \varnothing)$-$(m, 0)$ left hyperideal and every$(\varnothing, S_2)$-$(0, n)$ right hyperideal of H have the (S_1, S_2)-(m, n) intersection property, but not for (S_1, S_2)-(m, n) quasi-hyperideals of arbitrary ordered semihypergroups. Then, we go on to prove that every (S_1, S_2)-(m, n) quasi-hyperideal of a regular ordered semihypergroup has the ordered (S_1, S_2)-(m, n) intersection property. We also prove several other results characterizing ordered semihypergroups, relative intra-regular and relative regular ordered semihypergroups.

1 Introduction

The notion of bi-ideal or $(1, 1)$-ideal was introduced by Good and Hughes [24]. Lajos [26] generalized this notion of bi-ideal by introducing (m, n)-ideal. The concept of quasi-deals was introduced by Steinfeld [20] both in rings and in semigroups. The intersection property of quasi-ideals in rings and in semigroups was studied by Zhang et al. [15]. For general properties of quasi-ideals, we refer to the monograph [21]. Iseki [11], [12], [13] and Christoph [7] studied quasi-ideals in semirings. Weinert [10] studied quasi-ideals in rings. The notion of quasi-ideals in ternary semigroups was studied by Dixit et al [29]. The concept of ordered semigroups was studied by many authors, viz, Lee [8], Alimov [16], and Saito [27], [28]. The concept of ordered quasi-ideals in ordered semigroups was introduced by Kehayopulu in her paper [18]. The concept of T-ideal (or relative ideal) in semigroup S (resp. left, right relative ideals), where $T \subseteq S$, was introduced by Wallace [1], [2]. Hrmova [25] generalized this notion of T-ideal (or relative ideal) by defining the notion of a relative (S_1, S_2)-ideal of a semigroup S, where S_1, S_2 are subsets of S, and using these concepts, he generalized and extended the results of Wallace [2]. Recently, Khan et al. [17] studied relative ideals in ordered semigroups.

The concept of hyperstructures was given by Marty [9]. Hila et al [14] studied quasi-hyperideals in semihypergroups. Heideri et al [6] studied ordered hyperstructures. Recently, Basar et al [3] studied relative hyperideals in ordered ternary semihypergroups. For detailed theory and applications of semihypergroups and hyperstructures, one can refer to the monographs by Corsini, Leoreanu and Davvaz [5], [22], [23]. As a matter of fact, from the definitions introduced in the next section of the present paper, it becomes clear that the class of relative (S_1, S_2)-(m, n)-bi-hyperideals is a generalization of the class of relative (S_1, S_2)-bi-hyperideals which is a generalizations of the class of relative (S_1, S_2)-(m, n)-quasi hyperideals which is a generalization of the class of relative (S_1, S_2)-quasi hyperideals which generalizes the class of m-left and n-right relative hyperideals which is a generalization of the class of one sided relative hyperideals which is a generalization of the class of relative hyperideals which is a generalization of the class of hyperideals(without relation) which generalizes the class of one sided hyperideals(without relation) in ordered semihypergroups (and not confined to these only). Also, the class of relative (S_1, S_2)-intra-regularity as well as the relative regularity in ordered semihypergroups introduced in the present paper is a generalization of the class of intra-regularity, and the class of regularity(without relation) in rings, semigroups, ordered semigroups, and in other algebraic structures. Accordingly, we deduce and announce some results in the new settings of these definitions which strengthen the previous results in semigroups, ternary semigroups, ordered semigroups, rings and have feasibly possible applications in semihypergroups(without order), and in other algebraic structures. We hope that this constructive algebraic framework which is also known as hopestructures will be useful in further studies of relative ideals with realistic possibility of opening up new direction of research in hyperstructures as well as in other algebraic structures.

2 Fundamental Notions and Definitions

In the current section, we give the preliminaries which will be used throughout this article. A hyperstructure H is a nonvoid set equipped with an hyperoperation "$\circ$" on H defined as follows:

$$\circ : H \times H \to \mathcal{P}^*(H) \mid (x, y) \to (x \circ y)$$

and an operation " $*$ " on $\mathcal{P}^*(H)$ defined as follows:

$$* : \mathcal{P}^*(H) \times \mathcal{P}^*(H) \to \mathcal{P}^*(H) \mid (X, Y) \to X * Y$$

such that

$$X * Y = \bigcup_{(x,y)\in X\times Y} (x \circ y)$$

for any $X, Y \in \mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ denotes the nonempty subsets of H. A hyperoperation "$\circ$" on H gives rise to an operation "$*$" on $\mathcal{P}^*(H)$. Conversely, an operation "$*$" on $\mathcal{P}^*(H)$ gives rise to a hyperoperation "$\circ$" on H, defined as follows: $x \circ y = \{x\} * \{y\}$. Therefore, a semihypergroup $(H, \circ, *)$ can be identified by $(H, \circ)$ because of the interdependency of the operation " $*$ " and the hyperoperation " $\circ$ ". Clearly, we have $X \subseteq Y \Rightarrow X * D \subseteq Y * D, D * X \subseteq D * Y$ for any $X, Y, D \in \mathcal{P}^*(H)$ and $H * H \subseteq H$. For a subset X of an semihypergroup H, we define by $(X]$ the subset of H as follows:

$$(X] = \{s \in H :\mid s \leq x \text{ for some x} \in \text{X}\}.$$

If " $\leq$ " is an order relation on a semihypergroup H, we define the order relation " $\preceq$ " on $\mathcal{P}^*(H)$ as follows:

$$\preceq := \{(X, Y) \mid \forall x \in X\ \exists\ y \in Y \text{ such that x} \leq \text{y}\}.$$

Therefore, for $X, Y \in \mathcal{P}^*(H)$, we denote $X \preceq Y$ if for every $x \in X$, there exists $y \in Y$ such that $x \leq y$. This is indeed, a reflexive and transitive relation on $\mathcal{P}^*(H)$.
A hyperstructure $(H, \circ)$ is called a semihypergroup if for all $x, y, z \in H$, $(x \circ y) \circ z = x \circ (y \circ z)$, i. e.,

$$\bigcup_{m\in x\circ y} m \circ z = \bigcup_{n\in y\circ z} x \circ n.$$

A nonempty subset A of a semihypergroup $(H, \circ)$ is called a subsemihypergroup of H if $A * A \subseteq A$. An semihypergroup $(H, \circ)$ equipped with a partial order " $\leq$ " on H that is compatible with semihypergroup operation " $\preceq$ " such that for all $x, y, z \in H$,

$$x \leq y \Rightarrow z \circ x \preceq z \circ y \text{ and x} \circ \text{z} \preceq \text{y} \circ \text{z},$$

ia called an ordered semihypergroup. Throughout this paper, H will denote an semihypergroup(semihypergroup) unless otherwise stated. Let $(H, \circ, \leq)$ be an ordered semihypergroup, $S \subseteq H$ and let X, Y, Z be nonempty subsets of S, then we easily have the following:

(1) If $x \in X * Y$, then $x \in x^{'} \circ y$ for some $x^{'} \in X$, $y \in Y$;

(2) If $x \in X$, $y \in Y$, then $x \circ y \subseteq X * Y$;

(3) $X \subseteq (X]_S$;

(4) If $X \subseteq Y$, then $(X]_S \subseteq (Y]_S$;

(5) $(X]_S * (Y]_S \subseteq (X * Y]_S$;

(6) $((X]_S * (Y]_S]_S = (X * Y]_S$;

(7) For every left (resp. right) S-hyperideal I of S, $(I]_S = I$;

(8) $(X \cap Y) \subseteq (X]_S \cap (Y]_S$;

(9) $X \cup Y = (X]_S \cup (Y]_S$;

(10) $((X_S]_S]_S = (X]_S$;

(11) $X * (Y \cap Z) \subseteq X * Y \cap X * Z$;

(12) $X * (Y \cup Z) = X * Y \cup X * Z$;

(13) If A, B are S-hyperideals of H such that $A \cap B \neq \varnothing$, then $(A * B]_S$, $A \cap B$ are S-hyperideals of H;

(14) If I is a sub-semihypergroup of H, then $(I * s * I]_I$ is a I-hyperideal of H for each $s \in H$;

(15) If H is a semihypergroup and $X, Y, Z \in \mathcal{P}^\star(H)$, then we define $(X * Y) * Z = X * (Y * Z)$; and if H is an ordered semihypergroup and $A, B \subseteq H$, then

$$\Big(\bigcup_{i \in I} A_i\Big) * B = \bigcup_{i \in I} (A_i * B) \text{ and } B * \Big(\bigcup_{i \in I} A_i\Big) = \bigcup_{i \in I} (B * A_i).$$

Definition 2.1. Suppose that $(H, \circ, \leq)$ is an ordered semihypergroup and $S \subseteq H$. Then, a nonempty subset I of H is called a right (resp., left) S-hyperideal (or relative hyperideal) of H if

(1) $I * S \subseteq I$(resp., $S * I \subseteq I$); and

(2) if $x \in I$ and $S \ni y \leq x$, then $y \in I$, i. e., if $(I]_S = I$.

A subset of H which is both a right and left S-hyperideal of H is called an S-hyperideal(or relative hyperideal) of H. We observe that $I * S \subseteq I(resp.,\ S * I \subseteq I) \iff x \circ s \subseteq I(resp.,\ s \circ x \subseteq I)$ for every $x \in I$, and every $s \in S$. Obviously, every right(resp., left) S-hyperideal of an ordered semihypergroup H is a sub-semihypergroup of H.

Definition 2.2. [4] Suppose that $(H, \circ, \leq)$ is an ordered semihypergroup and S_1, S_2 are nonempty subsets of H. A nonempty subset S of H is called an (S_1, S_2)-hyperideal or a relative hyperideal of H if $S_1 * S \subseteq S$, $S * S_2 \subseteq S$ and $S_1 \cup S_2 \ni a \leq b$ for some $b \in S \Rightarrow a \in S$. If $S_1 = \varnothing$ or $S_2 = \varnothing$, then the (S_1, S_2)-hyperideal coincides with one sided relative hyperideal of H.

Definition 2.3. Suppose that $(H, \circ, \leq)$ is an ordered semihypergroup, and let $S \subseteq H$. A nonempty subset Q of H is called a relative S-quasi hyperideal of H if

(1) $(S * Q]_S \cap (Q * S]_S \subseteq Q$; and

(2) $p \in Q, S \ni q \leq p \Rightarrow q \in Q$, i. e., $(Q]_S = Q$.

Definition 2.4. Suppose that $(H, \circ, \leq)$ is an ordered semihypergroup, $S \subseteq H$ and m, n are non-negative integers. A nonempty subset Q of H is called a relative S-(m, n)-quasi hyperideal of H if

(1) $(S^m * Q]_S \cap (Q * S^n]_S \subseteq Q$; and

(2) $p \in Q, S \ni q \leq p \Rightarrow q \in Q$, i. e., $(Q]_S = Q$.

Definition 2.5. [4] Suppose that $(H, \circ, \leq)$ is an ordered semihypergroup and let $S_1, S_2 \subseteq H$. A nonempty subset Q of H is called a relative (S_1, S_2)-quasi hyperideal of H if

(1) $(Q * S_2]_S \cap (S_1 * Q]_S \subseteq Q$, where $S = S_1 \cup S_2$; and

(2) $m \in Q, S \ni n \leq m \Rightarrow n \in Q$, i. e., $(Q]_S = Q$.

Definition 2.6. Suppose that $(H, \circ, \leq)$ is an ordered semihypergroup and let $S_1, S_2 \subseteq H$. A nonempty subset Q of H is called and a relative (S_1, S_2)-(m, n)-quasi hyperideal of H if

(1) $(S_1^m * Q]_S \cap (Q * S_2^n]_S \subseteq Q$, where $S = S_1 \cup S_2$, and m, n are non-negative integers; and

(2) $p \in Q, S \ni q \leq p \Rightarrow q \in Q$, i. e., $(Q]_S = Q$.

If $S_1 = \varnothing$ or $S_2 = \varnothing$, then the (m, n)-(S_1, S_2)-quasi hyperideal of H becomes one sided relative n right -hyperideal, i. e., $(\varnothing, S_2)$-$(0, n)$-quasi hyperideal of H or $(S_1, \varnothing)$-$(m, 0)$-quasi hyperideal, i. e., one sided relative m-left hyperideal of H.

Definition 2.7. Let $(H, \circ, \leq)$ be an ordered semihypergroup and let S, B be any nonempty subsets of H. Then, B is said to be a S-bi-hyperideal of H if

(1) $B * S * B \subseteq B$; and

(2) for all $t \in B$, $S \ni g \leq t \Rightarrow g \in B$.

Definition 2.8. Let $(H, \circ, \leq)$ be an ordered semihypergroup and let S, B be any nonempty subsets of H. Then, B is said to be a S-(m, n)-bi-hyperideal of H, where m, n are nonnegative integers if

(1) $B^m * S * B^n \subseteq B$; and

(2) for all $t \in B$, $S \ni g \leq t \Rightarrow g \in B$.

Definition 2.9. [4] Suppose that $(H, \circ, \leq)$ is an ordered semihypergroup and S_1, S_2 are nonempty subsets of H. Then, $S(\neq \varnothing)$ is called an (S_1, S_2)-bi-hyperideal or a relative bi-hyperideal of H if

(1) $S * (S_1 \cup S_2) * S = S * S_1 * S \cup S * S_2 * S \subseteq S$; and

(2) for all $s \in S$, $S_1 \cup S_2 \ni k \leq s \Rightarrow k \in S$.

Definition 2.10. Suppose that $(H, \circ, \leq)$ is an ordered semihypergroup, S_1, S_2 are nonempty subsets of H and m, n are non-negative integers. Then, a nonempty subset B of H is called an (S_1, S_2)-(m, n)-bi-hyperideal or a relative bi-hyperideal of H if

(1) $B^m * (S_1 \cup S_2) * B^n = B^m * S_1 * B^n \cup B^m * S_2 * B^n \subseteq B$; and

(2) for all $s \in B$, $S_1 \cup S_2 (= S) \ni k \leq s \Rightarrow k \in S$, i. e., $(B]_S = B$.

We denote an (S_1, S_2)-bi-hyperideal $B_R(s)$ and (S_1, S_2)-quasi hyperideal $Q_R(s)$ of H generated by an element s of H as follows: $B_R(s) = (s \cup s^2 \cup s * S * s]_S$ and $Q_R(s) = (s \cup ((s * S_2]_S \cap (S_1 * s]_S)]_S$ respectively, where $S = S_1 \cup S_2$.

Definition 2.11. [4] Suppose that $(H, \circ, \leq)$ is an ordered semihypergroup and $S_1, S_2 \subseteq H$. Then H is called (S_1, S_2)-regular (or relative regular) if for every $s \in S$, there exist $k \in S_1 \cup S_2$ such that $\{s\} \preceq s \circ k \circ s$. Equivalently: for all $B \subseteq S, B \subseteq (S * B^2 * S]_S$. Equvalently:

(1) $s \in (s * S * s]_S$ for all $s \in S = S_1 \cup S_2$; and

(2) $B \subseteq (B * S * B]_S$ for all $B \subseteq S$.

Definition 2.12. Suppose that $(H, \circ, \leq)$ is an ordered semihypergroup and $S \subset H$. Then, H is called S-intra-regular (or relative intra-regular) if for every $g \in S$, there exist $a, b \in S$ such that $\{g\} \preceq (a \circ g) * (g \circ b)$, that is, for every $g \in S$, there exist $a, b, c \in S$ such that $c \in (a \circ g) * (g \circ b)$ and $g \leq c$. Equivalently: For all $B \subseteq S$, $B \subseteq (S * B^2 * S]_S$.

Definition 2.13. Suppose that $(H, \circ, \leq)$ is an ordered semihypergroup and $S_1, S_2 \subseteq H$. Then, H is said to be (S_1, S_2)-intra-regular if for every $s \in S_1 \cup S_2, \exists\, s_1, s_2 \in S = S_1 \cup S_2$ such that $\{s\} \preceq s_1 \circ s^2 \circ s_2$. Equivalently: For all $B \subseteq S = S_1 \cup S_2, B \subseteq (S * B^2 * S]_S$.

If $S_1 = \varnothing$ or $S_2 = \varnothing$, then H is said to be $(\varnothing, S_2)$-intra-regular or $(S_1, \varnothing)$-intra-regular.

Definition 2.14. A nonempty subset Q of an ordered semihypergroup $(H, \circ, \leq)$ has the (S_1, S_2)-(m, n)-intersection property if Q is the intersection of a $(S_1, \varnothing)$-$(m, 0)$ left quasi hyperideal and a $(\varnothing, S_2)$-$(0, n)$ right quasi hyperideal of H.

From the definitions, it is clear that, relative ordered hyperideals, relative ordered bi-hyperideals, relative ordered quasi-hyperideals, relative ordered intra-regularity and relative ordered regularity of H is a generalization of hyperideals, bi-hyperideals, quasi-hyperideals, intra-regularity and regularity of H (without relation and without order), respectively.

3 Relative (m, n)-quasi hyperideals of ordered semihypergroups

We start with the following:

Lemma 3.1. *Suppose that $(H, \circ, \leq)$ is an ordered semihypergroup and $S \subseteq H$. For a nonempty subset G of S, the following nonempty subset*

$$Q = (G \cup ((G * S]_S \cap (S * G]_S))]_S$$

of H is the relative S-quasi hyperideal of H generated by G.

Proof. We have

$$(Q * S]_S \cap (S * Q]_S \subseteq Q; \text{and}$$

$$\begin{aligned}
Q * S &= (G \cup ((G * S]_S \cap (S * G]_S))]_S * S \\
&= (G \cup ((G * S]_S \cap (S * G]_S))]_S * (S]_S \\
&\subseteq (G \cup (G * S]_S]_S * (S]_S \\
&\subseteq ((G \cup (G * S]_S) * S]_S \\
&= (G * S \cup (G * S]_S]_S \\
&\subseteq (G * S \cup (G * S]_S]_S \\
&= ((G * S]_S]_S \\
&= (G * S]_S.
\end{aligned}$$

Then, we conclude that

$$(Q * S]_S \subseteq ((G * S]_S]_S = (G * S]_S.$$

In a similar fashion, we obtain $(S * Q]_S \subseteq (S * G]_S$. Therefore, we have

$$\begin{aligned}
(Q * S]_S \cap (S * Q]_S &\subseteq (G * S]_S \cap (S * G]_S \\
&\subseteq G \cup ((G * S]_S \cap (S * G]_S) \\
&\subseteq (G \cup ((G * S]_S \cap (S * G]_S))]_S = Q.
\end{aligned}$$

Further, we obtain $(Q]_S = Q$. In fact, as $Q = (I]_S$ for some $I \subseteq S$, we get

$$(Q]_S = ((I]_S]_S = (I]_S = Q.$$

Finally, if M is a relative quasi hyperideal of H such that $X \subseteq M$, then we have

$$Q = (G \cup ((G * S]_S \cap (S * G]_S))]_S \subseteq (M \cup ((M * S]_S \cap (S * M]_S))]_S = (M]_S = M.$$

This completes the proof. □

Lemma 3.2. *Suppose that $(H, \circ, \leq)$ is an ordered semihypergroup and $S \subseteq H$. Let for every relative S-right hyperideal R and every relative left hyperideal L and relative quasi-hyperideal Q of H, we have*

$$R \cap Q \cap L \subseteq (L * Q * R]_S.$$

Then, H is S-intra-regular.

Proof. Let $R \subseteq S$. If $R = \varnothing$, then obviously, $R \subseteq (S * R^2 * S]_S$. Suppose that $R \neq \varnothing$. Let us denote by $Q_r(R), Q_l(R), Q_q(R)$, the relative S-right hyperideal, relative S-left hyperideal and the relative S-quasi hyperideal of H, respectively, generated by R. Then, by the given assumption, we have

$$\begin{aligned} R &\subseteq Q_r(R) \cap Q_q(R) \cap Q_l(R) \\ &\subseteq (Q_l(R) * Q_q(R) * Q_r(R)]_S \\ &= ((R \cup S * R]_S * (R \cup ((R * S]_S \cap (S * R]_S)]_S * (R \cup R * S]_S]_S \\ &= ((R \cup S * R) * (R \cup ((R * S]_S \cap (S * R]_S)) * (R \cup R * S)]_S \\ &\subseteq ((R \cup S * R) * (R \cup (R * S]_S) * (R \cup R * S)]_S \\ &= ((R^2 \cup S * R^2 \cup R * (R * S]_S \cup S * R * (R * S]_S) * (R \cup R * S)]_S \end{aligned}$$

Since

$$R * (R * S]_S \subseteq (R]_S * (R * S]_S \subseteq (R^2 * S]_S,$$

and

$$S * R * (R * S]_S \subseteq (S * R]_S * (R * S]_S \subseteq (S * R^2 * S]_S,$$

we obtain the following:

$$\begin{aligned} R &\subseteq ((R^2 \cup S * R^2 \cup (R^2 * S]_S \cup (S * R^2 * S]_S) * (R \cup R * S)]_S \\ &= (R^3 \cup S * R^3 \cup (R^2 * S]_S * R \cup (S * R^2 * S]_S * R \cup R^3 * S \cup S * R^3 * S \cup (R^2 * S]_S * \\ &\quad R * S \cup (S * R^2 * S]_S * R * S]_S. \end{aligned}$$

As

$$(R^2 * S]_S * R \subseteq (R^2 * S]_S * (R]_S \subseteq (R^2 * S * R]_S \subseteq (R^2 * S]_S,$$
$$(S * R^2 * S]_S * R \subseteq (S * R^2 * S]_S * (R]_S \subseteq (S * R^2 * S * R]_2 \subseteq (S * R^2 * S]_S,$$
$$(R^2 * S]_S * R * S \subseteq (R^2 * S]_S * (R * S]_S \subseteq (R^2 * S * R * S]_S \subseteq (R^2 * S]_S,$$

and

$$\begin{aligned} (S * R^2 * S]_S * R * S &\subseteq (S * R^2 * S]_S * (R * S]_S \\ &\subseteq (S * R^2 * S * R * S]_S \\ &\subseteq (S * R^2 * S]_S, \end{aligned}$$

we receive

$$\begin{aligned} R &\subseteq (R^3 \cup S * R^2 * S \cup (R^2 * S]_S \cup (S * R^2 * S]_S]_S \\ &= (R^3 \cup (S * R^2 * S]_S \cup (R^2 * S]_S]_S. \end{aligned}$$

As $R^3 \subseteq R^2 * S \subseteq (R^2 * S]_S$, we receive

$$R \subseteq ((S * R^2 * S]_S \cup (R^2 * S]_S]_S,$$

therefore, we have

$$\begin{aligned} R^2 &\subseteq R * ((S * R^2 * S]_S \cup (R^2 * S]_S]_S \\ &\subseteq (R]_S * ((S * R^2 * S]_S \cup (R^2 * S]_S]_S \\ &\subseteq (R * ((S * R^2 * S]_S \cup (R^2 * S]_S)]_S \\ &= (R * (S * R^2 * S]_S \cup R * (R^2 * S]_S]_S \\ &\subseteq ((R]_S * (S * R^2 * S]_S \cup (R]_S * (R^2 * S]_S *]_S \\ &\subseteq ((R * S * R^2 * S]_S \cup (R^3 * S]_S]_S \\ &\subseteq ((S * R^2 * S]_S]_S \\ &= (S * R^2 * S]_S, \end{aligned}$$

$$\begin{aligned} R^2 * S &\subseteq (S * R^2 * S]_S * S \\ &= (S * R^2 * S]_S * (S]_S \\ &\subseteq (S * R^2 * S^2]_S \\ &\subseteq (S * R^2 * S]_S, \end{aligned}$$

and

$$(R^2 * S]_S \subseteq ((S * R^2 * S]_S]_S = (S * R^2 * S]_S.$$

This yields that

$$R \subseteq ((S * R^2 * S]_S \cup (R^2 * S]_S]_S = ((S * R^2 * S]_S]_S = (S * R^2 * S]_S.$$

Hence, H is S-intra-regular. □

Lemma 3.3. *Suppose that $(H, \circ, \leq)$ is an ordered semihypergroup and $\{S_i \mid i \in I\}$ is a nonempty family of hypersubsets of H. Then, $\bigcap_{i \in I} S_i = \varnothing$ or $\bigcap_{i \in I} S_i$ is a nonempty hypersubset of H.*

Proof. It is straightforward. □

Lemma 3.4. *Suppose that $(H, \circ, \leq)$ is an ordered semihypergroup and $S \subseteq H$. Then, $S^m \subseteq S$ for all nonnegative integer m.*

Proof. It is straightforward. □

Lemma 3.5. *Suppose that $(H, \circ, \leq)$ is an ordered semihypergroup, and let $S_1, S_2 \subseteq H$, $A_1, A_2 \subseteq A \subseteq S = S_1 \cup S_2$. Further suppose that Q is an (m, n)-(S_1, S_2)-quasi hyperideal of H. Then, $A \cap Q = \varnothing$ or $A \cap Q$ is (m, n)-(A_1, A_2)-quasi hyperideal of A, where m, n are nonnegative integers.*

Proof. Suppose that $A \cap Q \neq \varnothing$. Since, $Q \subseteq H$, $A \subseteq S \subseteq H$; $Q,\ A \neq \varnothing$. Therefore, $A \cap Q$ is a nonempty subset of $A = A_1 \cup A_2 \subseteq S = S_1 \cup S_2$. Therefore, we obtain

$$\begin{aligned} (A_1^m * (A \cap Q]_A \cap ((A \cap Q) * A_2^n]_A \cap A &\subseteq A \cap (A_1^m * Q]_A \cap (Q * A_2^n]_A \\ &\subseteq A \cap (S_1^m * Q]_S \cap (Q * S_2^n]_S \\ &\subseteq A \cap Q. \end{aligned}$$

and

$$(A \cap Q]_S \cap A \subseteq A \cap (A]_S \cap (Q]_S \subseteq A \cap (Q]_S = A \cap Q.$$

Hence, $A \cap Q$ is an ordered (m, n)-(A_1, A_2)-quasi hyperideal of A. □

Proposition 3.6. *Suppose that $(H, \circ, \leq)$ is an ordered semihypergroup, $S_1, S_2 \subseteq H$, $S = S_1 \cup S_2$ and $\{Q_i \mid i \in I\}$ is a nonempty family of (S_1, S_2)-quasi hyperideals of H. Then, $\bigcap_{i \in I} Q_i = \varnothing$ or $\bigcap_{i \in I} Q_i$ is (S_1, S_2)-(m, n)-quasi hyperideal of H, where m, n are nonnegative integers.*

Proof. Let $\bigcap_{i \in I} Q_i \neq \varnothing$. By Lemma 3.3, we have $\bigcap_{i \in I} Q_i$ is a nonempty hypersubset of H. For all $i \in I$, we obtain

$$(S_1^m * (\bigcap_{i \in I} Q_i)]_S \cap ((\bigcap_{i \in I} Q_i) * S_2^n]_S \subseteq (S_1^m * Q_i]_S \cap (Q_i * S_2^n]_S \subseteq Q_i.$$

Therefore, we receive

$$(S_1^m * (\bigcap_{i \in I} Q_i)]_S \cap ((\bigcap_{i \in I} Q_i) * S_2^n]_S \subseteq \bigcap_{i \in I} Q_i,$$

and

$$(\bigcap_{i \in I} Q_i]_S \subseteq \bigcap_{i \in I} (Q_i]_S = \bigcap_{i \in I} Q_i.$$

Hence, $\bigcap_{i \in I} Q_i$ is (S_1, S_2)-(m, n) quasi hyperideal of H. □

Theorem 3.7. *Suppose that $(H, \circ, \leq)$ is an ordered semihypergroup, and $S_1, S_2 \subseteq H$, $S = S_1 \cup S_2$. Then the following assertions are true:*

(1) Suppose that $\{L_i \mid i \in I\}$ is a nonempty family of $(S_1, \varnothing)$-$(m, 0)$ left hyperideal of H. Then, $\bigcap_{i\in I} L_i = \varnothing$ or $\bigcap_{i\in I} L_i$ is $(S_1, \varnothing)$-$(m, 0)$ left hyperideal of H.

(2) Suppose that $\{R_i \mid i \in I\}$ is a nonempty family of $(\varnothing, S_2)$-$(0, n)$ right hyperideal of H. Then, $\bigcap_{i\in I} R_i = \varnothing$ or $\bigcap_{i\in I} R_i$ is $(\varnothing, S_2)$-$(0, n)$ right hyperideal of H.

Proof. (1) Suppose that $\{L_i \mid i \in I\}$ is a nonempty family of $(S_1, \varnothing)$-(m, o) left hyperideal of H and $\bigcap L_i \neq \varnothing$. By Lemma 3.3, $\bigcap_{i\in I} L_i$ is a nonempty hypersubset of H. For all $i \in I$, we obtain

$$S_1^m * (\bigcap_{i\in I} L_i) \subseteq S_1^m * L_i \subseteq L_i.$$

Therefore, $S_1^m * (\bigcap_{i\in I} L_i) \subseteq \bigcap_{i\in I} L_i$, and $(\bigcap_{i\in I} L_i]_S \subseteq \bigcap_{i\in I} (L_i]_S = \bigcap_{i\in I} L_i$. Hence, $\bigcap_{i\in I} L_i$ is $(S_1, \varnothing)$-$(m, 0)$ left hyperideal of H.
(2) Similarly, one can prove that $\bigcap_{i\in I} R_i$ is $(\varnothing, S_2)$-$(0, n)$ right hyperideal of H. □

Lemma 3.8. *Suppose that $(H, \circ, \leq)$ is an ordered semihypergroup, and $S_1, S_2 \subseteq H, S = S_1 \cup S_2$, and Q is a nonempty subset of H. Then the following assertions are true:*

(1) $(S_1^m * Q]_S$ is $(S_1, \varnothing)$-$(m, 0)$ left hyperideal of H.

(2) $(Q * S_2^n]$ is $(\varnothing, S_2)$-$(0, n)$ right hyperideal of H.

Proof. By Lemma 3.4, we receive

$$\begin{aligned}
(S_1^m * Q]_S * (S_1^m * Q]_S &\subseteq ((S_1^m * Q) * (S_1^m * Q)]_S \\
&\subseteq ((S_1^m * S_1) * S_1^m * Q]_S \\
&\subseteq (S_1 * (S_1 * S_1^{m-1} * Q)]_S \\
&= ((S_1 * S_1) * (S_1^{m-1} * Q)]_S \\
&\subseteq (S_1 * (S_1^{m-1} * Q)]_S \\
&= ((S_1 * S_1^{m-1}) * Q]_S \\
&= (S_1^m * Q]_S.
\end{aligned}$$

Therefore, $(S_1^m * Q]_S$ is a nonempty subset of H. We observe that

$$\begin{aligned}
S_1^m * (S_1^m * Q]_S &\subseteq (S_1 * S_1^{m-1} * Q]_S \\
&= (S_1]_S * (S_1 * S_1^{m-1} * Q]_S \\
&\subseteq (S_1 * (S_1 * S_1^{m-1} * Q)]_S \\
&= ((S_1 * S_1) * (S_1^{m-1} * Q)]_S \\
&\subseteq (S_1 * (S_1^{m-1} * Q)]_S \\
&= ((S_1 * S_1^{m-1}) * Q]_S \\
&= (S_1^m * Q]_S,
\end{aligned}$$

and $((S_1^m * Q]_S]_S = (S_1^m * Q]_S$. Hence, $(S_1^m * Q]_S$ is $(S_1, \varnothing)$-$(m, 0)$- left hyperideal of H.
(2). In a similar fashion, one can prove that $(Q * S_2^n]_S$ is $(\varnothing, S_2)$-$(0, n)$ right hyperideal of H. □

Lemma 3.9. *Suppose that $(H, \circ, \leq)$ is an ordered semihypergroup, and $S_1, S_2 \subseteq H$, $S = S_1 \cup S_2$. Then the following assertions are true:*

(1) Every $(S_1, \varnothing)$-$(m, 0)$ left hyperideal of H is (S_1, S_2)-(m, n)-quasi hyperideal of H for all positive integers m, n.

(2) Every $(\varnothing, S_2)$-$(0, n)$ right hyperideal of H is (S_1, S_2)-(m, n)-quasi hyperideal of H for all positive integers m, n.

Proof. (1) Suppose that L is $(S_1, \varnothing)$-$(m, 0)$ left hyperideal of H. Then, L is a nonempty hypersubset of H. Therefore,

$$(S_1^m * L]_S \cap (L * S_2^n]_S \subseteq (S_1^m * L]_S \subseteq (L]_S \subseteq L.$$

Hence, L is (S_1, S_2)-(m, n) quasi-hyperideal of H for all positive integers m, n.
(2) In a similar vein, one can prove that every $(\varnothing, S_2)$-$(0, n)$ right quasi hyperideal of H is (S_1, S_2)-(m, n) quasi hyperideal of H for all nonnegative integers m, n. □

Theorem 3.10. *Suppose that $(H, \circ, \leq)$ is an ordered semihypergroup, $S_1, S_2 \subseteq H, S = S_1 \cup S_2$. Let L be $(S_1, \varnothing)$-$(m, 0)$ quasi hyperideal of H and R be an $(\varnothing, S_2)$-$(0, n)$ right quasi hyperideal of H. Then, $L \cap R = \varnothing$ or $L \cap R$ is (S_1, S_2)-(m, n) quasi hyperideal of H.*

Proof. Suppose that $L \cap R \neq \varnothing$. Then, $L \cap R$ is the nonempty hypersubset of H. Therefore,

$$\begin{aligned}(S_1^m * (L \cap R)]_S \cap ((L \cap R) * S_2^n]_S &\subseteq (S_1^m * L]_S \cap (R * S_2^n]_S \\ &\subseteq (L]_S \cap (R]_S \\ &= L \cap R,\end{aligned}$$

and $(L \cap R] \subseteq (L] \cap (R] = L \cap R$. Hence, $L \cap R$ is (S_1, S_2)-(m, n) quasi hyperideal of H. □

Theorem 3.11. *Suppose that $(H, \circ, \leq)$ is an ordered semihypergroup, $S_1, S_2 \subseteq H, S = S_1 \cup S_2$. Let Q be an (S_1, S_2)-(m, n) quasi hyperideal of H. Then the following assertions are true:*

(1) Q has the (S_1, S_2)-(m, n) intersection property;

(2) $(Q \cup S_1^m * Q]_S \cap (Q \cup Q * S_2^n]_S = Q$;

(3) $(S_1^m * Q]_S \cap (Q \cup Q * S_2^n]_S \subseteq Q$;

(4) $(Q \cup S_1^m * Q]_S \cap (Q * S_2^n]_S \subseteq Q$.

Proof. $(1) \Rightarrow (2)$. Suppose that Q has (S_1, S_2)-(m, n) intersection property. As,

$$Q \subseteq Q \cup (S_1^m * Q]_S = (Q]_S \cup (S_1^m * Q] = (Q \cup S_1^m * Q],$$

and

$$Q \subseteq Q \cup (Q * S_2^n]_S = (Q]_S \cup (Q * S_2^n]_S = (Q \cup Q * S_2^n]_S,$$

we obtain

$$Q \subseteq (Q \cup S_1^m * Q]_S (Q \cup Q * S_2^n]_S.$$

As, Q has the (S_1, S_2)-(m, n)-intersection property, there exists $(S_1, \varnothing)$-$(m, 0)$ left quasi hyperideal of H and $(\varnothing, S_2)$-$(0, n)$ right quasi hyperideal R of H such that $Q = L \cap R$. Therefore, $Q \subseteq L$ and $Q \subseteq R$, thus

$$(S_1^m * Q]_S \subseteq (S_1^m * L]_S \subseteq (L]_S = L,$$

and

$$(Q * S_2^m]_S \subseteq (R * S_2^n]_S \subseteq (R]_S = R.$$

Therefore,

$$(Q \cup S_1^m * Q]_S = (Q]_S \cup (S_1^m * Q]_S = Q * \cup (S_1^m * Q]_S \subseteq L$$

and

$$(Q \cup Q * S_2^n]_S = (Q]_S \cup (Q * S_2^n]_S = Q \cup (Q * S_2^n]_S \subseteq R.$$

So,

$$(Q \subseteq S_1^m * Q]_S \cap (Q \cup Q * S_2^n]_S \subseteq L \cap R = Q.$$

Hence,

$$(Q \cup S_1^m * Q]_S \cap (Q \cup Q * S_2^n]_S = Q.$$

(2) $\Rightarrow$ (1). Suppose that

$$(Q \cup S_1^m * Q]_S \cap (Q \cup Q * S_2^n]_S = Q.$$

We will prove that $(Q \cup S_1^m * Q]_S$ is an $(S_1, \varnothing)$-$(m, 0)$ left quasi hyperideal of H and $(Q \cup Q * S_2^n]_S$ is an $(\varnothing, S_2)$-$(0, n)$ right quasi hyperideal of H. By Lemma 3.8, we receive $(S_1^m * Q]_S$ is an $(S_1, \varnothing)$-$(m, 0)$ left quasi hyperideal of H and $(Q * S_2^n]_S$ is an $(\varnothing, S_2)$-$(0, n)$ right quasi hyperideal of H, and therefore, $(S_1^m]_S$ and $(Q * S_2^n]_S$ are nonempty subsets of H. We have

$$\begin{aligned}
& (Q \cup S_1^m * Q]_S * (Q \cup S_1^m * Q]_S \\
= \ & (Q \cup (S_1^m * Q]_S) * (Q \cup (S_1^m * Q]_S) \\
= \ & Q * Q \cup (S_1^m * Q]_S \cup Q * (S_1^m * Q]_S \cup (S_1^m * Q]_S * (S_1^m * Q]_S \\
\subseteq \ & Q * Q \cup (S_1^m * Q]_S * (Q]_S \cup (S_1]_S * (S_1^m * Q]_S \cup (S_1^m * Q]_S * (S_1^m * Q]_S \\
\subseteq \ & Q * Q \cup (S_1^m * Q * Q]_S \cup (S_1 * S_1^m * Q]_S \cup (S_1^m * Q * S_1^m * Q]_S \\
\subseteq \ & Q \cup (S_1^m * Q]_S \cup (S_1^m * Q]_S \cup (S_1^m * Q]_S \\
= \ & Q \cup (S_1^m * Q]_S \\
= \ & (Q \cup S_1^m * Q]_S.
\end{aligned}$$

Therefore, $(Q \cup S_1^m * Q]_S$ is a nonempty hypersubset of H. Thus, we have

$$\begin{aligned}
S_1^m * (Q \cup S_1^m * Q]_S &= S_1^m * (Q \cup (S_1^m * Q]_S) \\
&= S_1^m * Q \cup S_1^m * (S_1^m * Q]_S \\
&\subseteq S_1^m * Q \cup (S_1^m * Q]_S \cup (S_1^m * Q]_S \\
&\subseteq (Q]_S \cup (S_1^m * Q]_S \\
&= (Q \cup S_1^m * Q]_S,
\end{aligned}$$

and $((Q \cup S_1^m * Q]_S]_S = (Q \cup S_1^m * Q]_S$. Hence, $(Q \cup S_1^m * Q]_S$ is an $(S_1, \varnothing)$-$(m, 0)$ left quasi hyperideal of H.
Similarly, one can prove that $(Q \cup Q * S_2^n]_S$ is an $(\varnothing, S_2)$-$(0, n)$ right quasi hyperideal of H. Hence, Q has the (S_1, S_2)-(m, n)-intersection property.
(2) $\Rightarrow$ (3). Suppose that

$$(Q \cup S_1^m * Q]_S \cap (Q \cup Q * S_2^n]_S = Q.$$

As,

$$(S_1^m * Q]_S \subseteq (Q]_S \cup (S_1^m * Q]_S = (Q \cup S_1^m * Q]_S,$$

we have

$$(S_1^m * Q]_S \cap (Q \cup Q * S_2^n]_S \subseteq (Q \cup S_1^m * Q]_S \cap (Q \cup Q * S_2^n]_S = Q.$$

Hence, $(S_1^m * Q]_S \cap (Q \cup Q * S_2^n]_S \subseteq Q$.
(3) $\Rightarrow$ (2). Suppose that $(S_1^m * Q]_S \cap (Q \cup Q * S_2^n]_S \subseteq Q$. Since, $Q \subseteq Q \cup (S_1^m * Q]_S = (Q \cup S_1^m * Q]_S$, and

$$Q \subseteq Q \cup (Q * S^n]_S = (Q \cup S_1^m * Q]_S,$$

we have

$$Q \subseteq (Q \cup S_1^m * Q]_S \cap (Q \cap Q * S_2^n]_S.$$

We have

$$\begin{aligned}
& (Q \cup S_1^m * Q]_S \cap (Q \cup Q * S_2^n]_S \\
= \ & (Q \cup (S_1^m * Q]_S) \cap (Q \cup (Q * S_2^n]_S) \\
= \ & (Q \cap (Q \cup (Q * S_2^n]_S)) \cup ((S_1^m * Q]_S \cap (Q \cup Q * S_2^n]_S) \\
\subseteq \ & Q \cup Q = Q.
\end{aligned}$$

Hence, $(Q \cup S_1^m * Q]_S \cap (Q \cup Q * S_2^n]_S = Q$.
(2) $\Rightarrow$ (4). The proof proceeds similar to (1) $\Rightarrow$ (3).
(4) $\Rightarrow$ (2). The proof follows similar to (3) $\Rightarrow$ (2). □

Lemma 3.12. *Suppose that $(H,\circ,\leq)$ is an ordered semihypergroup, and $S_1,S_2 \subseteq H, S = S_1 \cup S_2$. Then, every $(S_1,\varnothing)$-$(m,0)$ left quasi hyperideal and every $(\varnothing,S_2)$-$(0,n)$ right quasi hyperideal of ordered semihypergroup H have the (S_1,S_2)-(m,n)-intersection property.*

Proof. Suppose that L is $(S_1,\varnothing)$–$(S_1,\varnothing)$-$(m,0)$ left quasi hyperideal and R is $(\varnothing,S_2)$-$(0,n)$ right quasi hyperideal of H. By Lemma 3.9, it follows that L is (S_1,S_2)-(m,n) quasi hyperideal of H. Then, we have

$$\begin{aligned}(S_1^m * L]_S \cup (L \cup L * S_2^n]_S &= (S_1^m * L]_S * (L \cup (L * S_2^n]_S)\\ &= ((S_1^m * L]_S \cap L) \cup ((S_1^m * L]_S \cap (L * S_2^n]_S)\\ &\subseteq L \cup L = L.\end{aligned}$$

By Theorem 3.11, L has the (S_1,S_2)-(m,n)-intersection property. In a similar fashion, one can prove that R has the (S_1,S_2)-(m,n) intersection property. □

Proposition 3.13. *Suppose that $(H,\circ,\leq)$ is an ordered semihypergroup and Q is an (S_1,S_2)-(m,n)- quasi hyperideal of H, where $S_1,S_2 \subseteq H, S = S_1 \cup S_2$, m,n are nonnegative integers. If $S_1^m * Q \subseteq Q * S_2^n$ or $Q * S_2^n \subseteq S_1^m * Q$, then Q has the (S_1,S_1)-(m,n) intersection property.*

Proof. Suppose that $S_1^m * Q \subseteq Q * S_2^n$. Then, $(S_1^m * Q]_S \subseteq (Q * S_2^n]_S$. As, Q is (S_1,S_2)-(m,n)-quasi hyperideal of H, we have

$$S_1^m * Q \subseteq (S_1^m * Q]_S = (S_1^m * Q]_S \cap (Q * S_2^n]_S \subseteq Q.$$

Therefore, Q is $(S_1,\varnothing)$-$(m,0)$ left quasi hyperideal of H. By Lemma 3.12, we find that Q is the (S_1,S_2)-(m,n)-intersection property. Similarly, one can prove that Q has the (S_1,S_2)-(m,n) intersection property. □

Lemma 3.14. *Suppose that $(H,\circ,\leq)$ is a regular ordered semihypergroup; $S_1,S_2 \subseteq H, S = S_1 \cup S_2$ and B is a nonempty subset of H. Then, we have*

(1) $B \subseteq (S_1^m * B]_S$ for all positive integer m.

(2) $B \subseteq (B * S_2^n]_S$ for all positive integer n.

Proof. (1) Suppose that $b \in B$. As, H is regular, $\exists\ s \in S$ such that $\{b\} \preceq b \circ s \circ b$. Since, $b \circ s \subseteq S$, we receive

$$\{b\} \preceq b \circ s \circ b = (b \circ s) \circ b \subseteq S_1 * B,$$

and therefore, $B \subseteq (S_1 * B]_S$. Suppose that m is a positive integer such that $B \subseteq (S_1^m * B]_S$. Thus, $S_1 * B \subseteq S_1 * (S_1^m * B]_S = (S_1]_S * (S_1^m * B]_S \subseteq (S_1 * (S_1^m * B)]_S = (S_1^{m+1} * B]_S$. Hence, $B \subseteq (S_1^m * B]_S$ for all positive integer m.
(2) In a similar vein, one can prove that $B \subseteq (B * S_2^n]_S$ for all positive integer n. □

Theorem 3.15. *Every (S_1,S_2)-(m,n) quasi hyperideal of a regular ordered semihypergroup $(H,\circ,\leq)$ has (S_1,S_2)-(m,n)-intersection property, where $S_1,S_2 \subseteq H, S = S_1 \cup S_2$.*

Proof. Suppose that Q is an (S_1,S_2)-(m,n) quasi hyperideal of a regular ordered semihypergroup H. By Lemma 3.14, we receive $Q \subseteq (Q * S_2^n]_S$, and

$$(Q \cup Q * S_2^n]_S = Q \cup (Q * S_2^n]_S = (Q * S_2^n]_S.$$

Therefore,

$$(S_1^m * Q]_S \cap (Q \cup Q * S_2^n]_S = (S_1^m * Q]_S \cap (Q * S_2^n]_S \subseteq Q.$$

By Theorem 3.11, we receive that Q has the (S_1,S_2)-(m,n)-intersection property. □

Theorem 3.16. *Suppose that $(H,\circ,\leq)$ is a regular ordered semihypergroup and B is a nonempty subset of H. Also, $S(= S_1 \cup S_2) \subseteq H$ and m,n are nonnegative integers. Then, B is (S_1,S_2)-(m,n) quasi hyperideal of H if and only if $B = (S_1^m * B]_S \cap (B * S_2^n]_S$.*

Proof. $\Leftarrow$: Suppose that B is (S_1,S_2)-(m,n) quasi hyperideal of H. Then, $(S_1^m * B]_S \cap (B * S_2^n]_S \subseteq B$. By Lemma 3.14, we have $B \subseteq (S_1^m * B]_S$, and $B \subseteq (B * S_2^n]_S$. Thus, $B \subseteq (S_1^m * B]_S \cap (B * S_2^n]_S$. Hence, $B = (S_1^m * B]_S \cap (B * S_2^n]_S$.
$\Rightarrow$: Let $B = (S_1^m * B]_S \cap (B * S_2^n]_S$. By Lemma 3.8, we receive $(S_1^m * B]_S$ is $(S_1,\varnothing)$-$(m,0)$ left quasi hyperideal and $(B * S_2^n]_S$ is $(\varnothing,S_2)$-$(0,n)$ right quasi hyperideal of H. By Theorem 3.10, we have B is (S_1,S_2)-(m,n) quasi hyperideal of H. □

4 Characterization of relative intra-regular ordered semihypergroups

Kehayopulu et al. [19] and Khan et al. [17] gave many characterizations of intra-regular ordered semigroups. In this part, we give some new characterizations of relative intra-regular ordered semihypergroups through relative bi-hyperideals, relative quasi-hyperideals, relative left hyperideals and relative right hyperideals of ordered semihypergroups.

Theorem 4.1. *Suppose that $(H, \circ, \leq)$ is an ordered semihypergroup and S_1, S_2 are sub-semihypergroups of H such that $S_1, S_2 \subseteq S_1 \cup S_2$ and $S_2 * S_1, S_1 * S_2 \subseteq S_1 \cup S_2 (= S)$. Then, the following statements are true:*

(1) H is (S_1, S_2)-intra-regular if and only if for an (S_1, S_2)-bi-hyperideal B contained in S and an (S_1, S_2)-quasi hyperideal Q of H implies $B \cap Q \subseteq (S_1 * B * Q * S_2]_S$.

(2) H is (S_1, S_2)-intra-regular if and only if for an (S_1, S_2)-bi-hyperideal B contained in S and an (S_1, S_2)-quasi hyperideal Q of H implies $B \cap Q \subseteq (S_1 * Q * B * S_2]_S$.

Proof. $\Rightarrow$: Suppose that $\{s\} \subseteq B \cap Q \subseteq S_1 \cup S_2$. Since, H is (S_1, S_2)-intra-regular, there exists $s_1 \in S_1, s_2 \in S_2$ such that $\{s\} \preceq s_1 \circ s^2 \circ s_2$. Now,

$$\begin{aligned}
\{s\} \preceq s_1 \circ s^2 \circ s_2 &\preceq s_1 \circ s \circ (s_1 \circ s^2 \circ s_2) \circ s_2 \\
&= s_1 \circ (s \circ s_1 \circ s) \circ s \circ s_2^2 \\
&\subseteq S_1 * (B * S_1 * B) * Q * S_2 \\
&\subseteq S_1 * B * Q * S_2.
\end{aligned}$$

Hence, $B \cap Q \subseteq (S_1 * B * Q * S_2]_S$.
$\Leftarrow$ Consider $s \in S$. Since, $s \in S$, we have either $s \in S_1$ or $s \in S_2$. Let $s \in S_1$. Therefore, since $B_R(s)$ and $Q_R(s)$ are (S_1, S_2)-bi-hyperideal and (S_1, S_2)-quasi hyperideal generated by s respectively, we obtain

$$\begin{aligned}
\{s\} &\subseteq B_R(s) \cap Q_R(s) \\
&\subseteq (S_1 * B_R(s) * Q_R(s) * S_2]_S \\
&= (S_1 * (s \cup s^2 \cup s * S_1 * s \cup s * S_2 * s]_S * (s \cup ((s * S_2]_S \cap (S_1 * s]_S * S_2]_S \\
&\subseteq ((S_1 * s \cup S_1 * s^2 \cup S_1 * s * S_1 * s \cup S_1 * s * S_2 * s]_S * (s \cup (s * S_2]_S]_S * S_2]_S \\
&\subseteq ((S_1 * s \cup S_1 * S_2 * s]_S * (s \cup (s * S_2]_S]_S * S_2]_S \\
&\subseteq ((S * s \cup S * s]_S * (s * S_2 \cup (s * S_2^2]_S]_S \\
&\subseteq ((S * s]_S * (s * S \cup (s * S]_S]_S \\
&\subseteq ((S * s^2 * S]_S \cup (S * s^2 * S]_S]_S \\
&\subseteq ((S * s^2 * S]_S]_S \\
&= (S * s^2 * S]_S.
\end{aligned}$$

Similarly, one can show that $s \in S_2$. Hence, H is (S_1, S_2)-intra-regular.
$\Rightarrow$: Suppose that $\{s\} \subseteq B \cap Q \subseteq S_1 \cup S_2$. Since, H is (S_1, S_2)-intra-regular, there exists $s_1 \in S_1$, $s_2 \in S_2$ such that $\{s\} \preceq s_1 \circ s^2 \circ s_2$. Then, we have

$$\begin{aligned}
\{s\} \preceq s_1 \circ s^2 \circ s_2 &\preceq s_1 \circ (s_1 \circ s^2 \circ s_2) \circ s \circ s_2 \\
&= s_1^2 \circ s \circ (s \circ s_2 \circ s) \circ s_2 \\
&\subseteq S_1 * Q * (B * S_2 * B) * S_2 \\
&\subseteq S_1 * Q * B * S_2.
\end{aligned}$$

Hence, $B \cap Q \subseteq (S_1 * Q * B * S_2]_S$.
$\Leftarrow$ Let $B_R(s)$ and $Q_R(s)$ be (S_1, S_2)-bi hyperideal and (S_1, S_2)-quasi hyperideal generated by

$s \in S$. Then, either $s \in S_1$ or $s \in S_2$. Let $s \in S_2$. Then, we obtain

$$\begin{aligned}
\{s\} &\subseteq B_R(s) \cap Q_R(s) \\
&\subseteq (S_1 * Q_R(s) * B_R(s) * S_2]_S \\
&= (S_1 * (s \cup ((s * S_2]_S \cap (S_1 * s]_S))]_S * (s \cup s^2 \cup s * S_1 * s \cup s * S_2 * s]_S * S_2]_S \\
&\subseteq (S_1 * (s \cup (S_1 * s]_S))]_S * ((s * S_2 \cup s^2 * S_2 \cup s * S_1 * s * S_2 \cup s * S_2 * s * S_2]_S \\
&\subseteq ((S_1 * s \cup (S_1^2 * s]_S]_S * (s * S_2 \cup s * S_1 * S_2]_S]_S \\
&\subseteq ((S * s]_S * (s * S]_S]_S \\
&\subseteq ((S * s^2 * S]_S]_S \\
&= (S * s^2 * S]_S.
\end{aligned}$$

In case $s \in S_1$, one can prove in a similar manner. Hence, H is (S_1, S_2)-intra-regular. □

Theorem 4.2. *Suppose that $(H, \circ, \leq)$ is an ordered semihypergroup and S_1, S_2 are nonempty subsets of H such that $S_1 * S_2 \subseteq S$ and $S_2 * S_1 \subseteq S (= S_1 \cup S_2)$. Then, we have*

(1) H is (S_1, S_2)-intra regular if and only if for an $(S_1, \varnothing)$-hyperideal $L \subseteq S_1$ and an (S_1, S_2)-bi-hyperideal B of H implies $L \cap B \subseteq (L * B * S_2]_S$.

(2) H is (S_1, S_2)-intra-regular if and only if for an $(\varnothing, S_2)$-hyperideal $R \subseteq S_2$ and an (S_1, S_2)-bi hyperideal B of H implies $B \cap R \subseteq (S_1 * B * R]_S$.

Proof. (1) Let $\{s\} \subseteq L \cap B \subseteq S_1 \subseteq S_1 \cup S_2$. Since, H is (S_1, S_2)-intra-regular, there exists $s_1 \in S_1$, $s_2 \in S_2$ such that $\{s\} \preceq s_1 \circ s^2 \circ s_2$. Now,

$$\begin{aligned}
\{s\} \preceq s_1 \circ s^2 \circ s_2 &\preceq s_1 \circ (s_1 \circ s^2 \circ s_2) \circ s \circ s_2 \\
&= s_1^2 \circ s \circ (s \circ s_2 \circ s) \circ s_2 \\
&\subseteq S_1 * L * (B * S_2 * B) * S_2 \\
&\subseteq L * B * S_2.
\end{aligned}$$

Hence, $L \cap B \subseteq (L * B * S_2]_S$.
$\Leftarrow$ Suppose that $B_R(s)$ and $S_1(s) = (s \cup S_1 * s]_S * (S, \varnothing)$ is (S_1, S_2)-bi hyperideal and $(S_1, \varnothing)$-hyperideal of H generated by $s \in S$. Now, since $s \in S$, either $s \in S_1$ or $s \in S_2$. First suppose that $s \in S_1$. Then, we have

$$\begin{aligned}
\{s\} &\subseteq S_1(s) \cap B_R(s) \\
&\subseteq (S_1(s) * B_R(s) * S_2]_S \\
&= ((s \cup S_1 * s]_S * (s \cup s^2 \cup s * S_1 * s \cup s * S_2 * s]_S * S_2]_S \\
&\subseteq ((s \cup S_1 * s]_S * (s * S_2 \cup s^2 * S_2 \cup s * S_1 * s * S_2 \cup s * S_2 * s * S_2]_S]_S \\
&\subseteq ((s \cup S_1 * s]_S * (s * S_2 \cup s * S_1 * S_2 \cup s * S_1 * S_2 \cup s * S_2 s_1 * S_2]_S]_S \\
&\subseteq ((s \cup S * s]_S * (s * S]_S]_S \\
&= (s^2 * S \cup S * s^2 * S]_S.
\end{aligned}$$

Therefore, $s \leq v$ for some $v \in s^2 * S \cup S * s^2 * S$. If $v \in s^2 * S$, then $\{s\} \preceq s^2 \circ s_1$ for some $s_1 \in S$. Since, either, $s_1 \in S_1$, or, $s_1 \in S_2$, $\{s\} \preceq s^2 \circ s_1 \preceq s \circ (s^2 \circ s_1) \circ s_1 = s \circ s^2 \circ s_1^2 \subseteq S * s^2 * S_1 \Rightarrow s \in (S * s^2 * S_1]_S$ or $s \in (S * s^2 * S_2]_S$. Thus, $s \in (S * s^2 * S]_S$. On the other hand, when $v \in S * s^2 * S$, $s \in (S * s^2 * S]_S$. Hence, H is (S_1, S_2)-intra-regular. The situation when $s \in S_2$ is similar.
(2) Suppose that $\{s\} \subseteq B \cap R \subseteq S_2 \subseteq S_1 \cup S_2$. Since, H is (S_1, S_2)-intra-regular, there exists $s_1 \in S_1$, $s_2 \in S_2$ with

$$\begin{aligned}
\{s\} \preceq s_1 \circ s^2 \circ s_2 &\leq s_1 \circ s \circ (s_1 \circ s^2 \circ s_2) \circ s_2 \\
&= s_1 \circ (s \circ s_1 \circ s) \circ s \circ s_2^2 \\
&\subseteq S_1 * (B * S_1 * B) * R * S_2 \\
&\subseteq S_1 * B * R.
\end{aligned}$$

Hence, $B \cap R \subseteq (S_1 * B * R]_S$.
$\Leftarrow$ Consider $B_R(s)$, and $S_2(s) = (s \cup s * S_2]_S$, the (S_1, S_2)-bi-hyperideal and the $(\varnothing, S_2)$-hyperideal respectively generated by $s \in S$. Once, $s \in S$, we have either $s \in S_1$ or $s \in S_2$. First, let $s \in S_1$. We have

$$\begin{aligned}
\{s\} \subseteq B_R(s) \cap S_2(s) &\subseteq (S_1 * B_R(s) * S_2(s)]_S \\
&= (S_1 * (s \cup s^2 \cup s * S_1 * s \cup s * S_2 * s]_S * (s \cup s * S_2]_S]_S \\
&\subseteq ((S_1 * s \cup S_1 * s^2 \cup S_1 * s * S_1 * s \cup S_1 * sS_2 * s]_S * (s \cup s * S]_S]_S \\
&\subseteq ((S_1 * s \cup S_1 * S_2 * s]_S * (s \cup s * S_2]_S]_S \\
&\subseteq ((S * s]_S * (s \cup s * S]_S]_S \\
&\subseteq ((S * s^2 \cup S * s^2 * S]_S]_S \\
&= (S * s^2 \cup S * s^2 * S]_S.
\end{aligned}$$

Thus, $s \leq v$ for some $v \in (S * s^2 \cup S * s^2 * S)$. If $v \in S * s^2$, then $\{s\} \preceq s_1 \circ s^2$ for some $s_1 \in S$. Then, either $s_1 \in S_1$ or $s_1 \in S_2$. Now,

$$\{s\} \preceq s_1 \circ s^2 \preceq s_1 \circ (s_1 \circ s^2) = s_1^2 \circ s^2 \circ s \subseteq S_1 * s^2 * S \Rightarrow s \in (S_1 * s^2 * S]_S,$$

or $s \in (S_2 * s^2 * S]_S$. Hence, $s \in (S * s^2 * S]_S$. In either case, if $v \in S * s^2 * S$, then clearly, we have $s \in (S * s^2 * S]_S$. Therefore, in either case, we obtain $b \in (S * s^2 * S]_S$, i.e., H is S-intra-regular. In a similar way, one can prove the case when $s \in S_2$. □

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