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Notes on Generalized Quasi Einstein Manifolds

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Abstract The Ricci tensor of a generalized quasi-Einstein manifold M has the form $R_{ij} = \alpha g_{ij} + \beta A_i A_j + \gamma B_i B_j$ where A and B are two orthonormal forms. It is proved that, among other results, the two generators are closed if and if the Ricci tensor is a Codazzi tensor. M reduces to be quasi-Einstein if M is Ricci symmetric and the second generator is not parallel. M is Einstein if its Ricci tensor is symmetric and the two generators are not parallel. Sufficient conditions on a generalized quasi-Einstein GRW space-time to be either Einstein or nearly quasi-Einstein are given. Finally, a generalized quasi-Einstein pseudo-Ricci symmetric manifold are investigated.

1 Introduction

A pseudo-Riemannian manifold M is called Einstein if its Ricci tensor satisfies

$$R_{ij} = \alpha g_{ij},$$

where α is a constant. It is well-known that $\alpha = \frac{R}{n}$ where R is the scalar curvature of M [3]. A quasi-Einstein manifold was introduced by Chaki and R. K. Maity[5] as a generalization of Einstein manifolds. Such manifolds have a Ricci tensor of the form

$$R_{ij} = \alpha g_{ij} + \beta A_i A_j,$$

where α and β are scalars and A is a unit $1-$form. Quasi-Einstein Lorentzian manifolds are called perfect fluid space-times whenever A is time-like[20]. Some basic geometric properties of quasi-Einstein space-time were obtained in[19]. De and Ghosh investigated generalized quasi-Einstein manifold in [7] as a new generalization of Einstein manifolds and an extension of Chaki's notion. In a generalized quasi-Einstein manifold, the Ricci tensor satisfies

$$R_{ij} = \alpha g_{ij} + \beta A_i A_j + \gamma B_i B_j, \tag{1.1}$$

where α, β, γ are scalars, A and B are two orthonormal $1-$forms[2, 8, 12]. The forms A and B are called the generators of generalized quasi-Einstein manifolds. De and Gazi in [9] introduced the notion of a nearly quasi-Einstein manifold in which Ricci curvature tensor is of the form

$$R_{ij} = \alpha g_{ij} + \beta E_{ij},$$

where E_{ij} is $(0, 2)$ symmetric tensor. A pseudo-Ricci symmetric manifold, denoted by (PRS)$_n$, is a pseudo-Riemannian manifold whose Ricci tensor satisfies

$$(\nabla_X R)(Y, Z) = 2A(X) R(Y, Z) + A(Y) R(X, Z) + A(Z) R(Y, X),$$

where A is a non-zero $1-$form and ∇ is the Livi-Civita connection[4]. Locally it is,

$$\nabla_k R_{ij} = 2A_k R_{ij} + A_i R_{kj} + A_j R_{ik}.$$

Generalized quasi-Einstein manifolds have been investigated in mathematics and physics literature. These manifolds portray a generalization of Einstein manifolds and an extension of quasi-Einstein manifolds. Perfect fluid space-times are pictured out as quasi-Einstein manifolds. In [11], the authors studied generalized quasi-Einstein manifolds admitting W_2-curvature tensor whereas generalized quasi-Einstein manifolds admitting the conharmonic curvature tensor are studied by Prakasha and Venkatesha in [16]. Non-trivial examples of generalized quasi-Einstein manifolds were considered in Section 6 of [11]. Sular et al gave some classification results of generalized quasi-Einstein manifolds[21]. Recently, in [13], the authors proved that a generalized quasi-Einstein pseudo Ricci symmetric manifolds can not admit a Codazzi Ricci tensor and the generators of these manifolds are not parallel.

In the present paper, some geometric properties of generalized quasi-Einstein manifolds are investigated. Second, Results on generalized quasi-Einstein warped product manifold are discussed. Next, it is proved that a generalized quasi-Einstein GRW space-time is either Einstein or nearly quasi-Einstein. Finally, a generalized quasi-Einstein pseudo-Ricci symmetric manifold is considered.

2 On generalized quasi-Einstein manifolds

Proposition 2.1. *The scalar curvature of a generalized quasi-Einstein manifold with generators A and B is given by*

$$R = (n-2)\,\alpha + \left(A^i A^j + B^i B^j\right) R_{ij}.$$

Proof. Let M be a generalized quasi-Einstein manifold, that is,

$$R_{ij} = \alpha g_{ij} + \beta A_i A_j + \gamma B_i B_j,$$

where $A_i A^i = 1$, $B_i B^i = 1$ and $A_i B^i = 0$. A contraction with g^{ij} implies

$$R = n\alpha + \beta + \gamma, \tag{2.1}$$

which means that M has a constant scalar curvature R. Contracting Equation (1.1) with $A^i A^j$ and then by $B^i B^j$, one gets

$$\begin{aligned} A^i A^j R_{ij} &= \alpha + \beta, && (2.2)\\ B^i B^j R_{ij} &= \alpha + \gamma && (2.3) \end{aligned}$$

Thus,

$$R = (n-2)\,\alpha + \left(A^i A^j + B^i B^j\right) R_{ij}.$$

□

Theorem 2.2. *Let M be a generalized quasi-Einstein manifold. Then the generators A and B are closed if the Ricci tensor is a Codazzi tensor. Moreover, the covariant derivative of the Ricci curvature tensor is given by*

$$\nabla_j R_{ik} = 2\beta A_k \left(\nabla_j A_i\right) + 2\gamma B_k \left(\nabla_j B_i\right).$$

Proof. The covariant derivative of Equation (1.1) implies

$$\nabla_k R_{ij} = \beta \left(\nabla_k A_i\right) A_j + \beta A_i \left(\nabla_k A_j\right) + \gamma \left(\nabla_k B_i\right) B_j + \gamma B_i \left(\nabla_k B_j\right).$$

Also,

$$\nabla_i R_{kj} = \beta \left(\nabla_i A_k\right) A_j + \beta A_k \left(\nabla_i A_j\right) + \gamma \left(\nabla_i B_k\right) B_j + \gamma B_k \left(\nabla_i B_j\right).$$

Then, the Codazzi deviation tensor D is

$$\begin{aligned} D_{kij} &= \nabla_k R_{ij} - \nabla_i R_{kj} \\ &= \beta \left(\nabla_k A_i\right) A_j + \beta A_i \left(\nabla_k A_j\right) + \gamma \left(\nabla_k B_i\right) B_j + \gamma B_i \left(\nabla_k B_j\right) \\ &\quad - \left[\beta \left(\nabla_i A_k\right) A_j + \beta A_k \left(\nabla_i A_j\right) + \gamma \left(\nabla_i B_k\right) B_j + \gamma B_k \left(\nabla_i B_j\right)\right]. \end{aligned}$$

Two contractions withA^j and B^j give

$$\begin{aligned} A^j D_{kij} &= \beta \left[\nabla_k A_i - \nabla_i A_k\right], \\ B^j D_{kij} &= \gamma \left[\nabla_k B_i - \nabla_i B_k\right]. \end{aligned}$$

If R_{ij} is a Codazzi tensor, then the generators are closed, that is, $\nabla_k A_i = \nabla_i A_k$ and $\nabla_k B_i = \nabla_i B_k$.

Conversely, assume that A and B are closed, then

$$\begin{aligned} 0 &= \beta A_i (\nabla_k A_j) + \gamma B_i (\nabla_k B_j) - [\beta A_k (\nabla_i A_j) + \gamma B_k (\nabla_i B_j)] \\ &= \nabla_j (\beta A_k A_i + \gamma B_k B_i) - 2\beta A_k (\nabla_j A_i) - 2\gamma B_k (\nabla_j B_i) \\ &= \nabla_j R_{ik} - 2\beta A_k (\nabla_j A_i) - 2\gamma B_k (\nabla_j B_i), \end{aligned}$$

Thus the covariant derivative of R_{ik} is

$$\nabla_j R_{ik} = 2\beta A_k (\nabla_j A_i) + 2\gamma B_k (\nabla_j B_i).$$

□

Theorem 2.3. *A Ricci symmetric generalized quasi-Einstein manifold M reduces to be quasi-Einstein if B is not parallel. M reduces to be Einstein if the two generators are note parallel together.*

Proof. The covariant derivative of the Ricci tensor for generalized quasi-Einstein is

$$\beta (\nabla_i A_k) A_j + \beta A_k (\nabla_i A_j) + \gamma (\nabla_i B_k) B_j + \gamma B_k (\nabla_i B_j) = \nabla_i R_{kj}.$$

Assume that the Ricci curvature tensor is symmetric that is, $\nabla_i R_{kj} = 0$. Thus

$$\beta (\nabla_i A_k) A_j + \beta A_k (\nabla_i A_j) + \gamma (\nabla_i B_k) B_j + \gamma B_k (\nabla_i B_j) = 0.$$

Two different contractions with A^k and B^k imply

$$\begin{aligned} \beta (\nabla_i A_j) &= 0, \\ \gamma (\nabla_i B_j) &= 0. \end{aligned}$$

Assume that B is not parallel, then $\gamma = 0$, and hence Equation (1.1) leads to

$$R_{ij} = \alpha g_{ij} + \beta A_i A_j,$$

which means that M is quasi-Einstein. Moreover, if A and B are not parallel, then $\gamma = \beta = 0$ and hence Equation (1.1) reduces to

$$R_{ij} = \alpha g_{ij}.$$

Thus, M is Einstein. □

3 Generalized quasi-Einstein warped product manifold

Let $(\bar{M}, \overline{g})$ and $(\tilde{M}, \tilde{g})$ be two pseudo-Riemannian manifolds with dimensions dim $\bar{M} = \bar{n}$, dim $\tilde{M} = \tilde{n}$ and let $F : \bar{M} \to (0, \infty)$ be a smooth positive function on $\bar{M}$. Consider the product manifold $\bar{M} \times \tilde{M}$ with its natural projections $\pi : \bar{M} \times \tilde{M} \to \bar{M}$ and $\eta : \bar{M} \times \tilde{M} \to \tilde{M}$. Then the warped product manifold $M = \bar{M} \times_F \tilde{M}$ is the manifold $\bar{M} \times \tilde{M}$ furnished with metric

$$g = \overline{g} \oplus f^2 \tilde{g}.$$

The manifold $\bar{M}$ is called the base manifold of M whereas $\tilde{M}$ is called the fiber manifold of M[6].

Now, Let $a, b, c, d, ... \in \{1, ..., \bar{n}\}$ denote the basis vector fields on a neighborhood $\bar{U}$ of the base manifold $\bar{M}$ whereas $\alpha, \beta, \gamma, \delta, ... \in \{\bar{n}+1, ..., n\}$ denote the basis vector fields on a neighborhood $\tilde{U}$ of the fiber manifold $\tilde{M}$. Likewise, $i, j, k, l, ... \in \{1, ..., n\}$ denote the basis vector fields on a neighborhood $\bar{U} \times \tilde{U}$ of the warped product manifold. The local components of the metric $g = \overline{g} \times_F \tilde{g}$, are $g_{ab} = \overline{g}_{ab}$, $g_{\alpha j} = 0$ and $g_{\alpha\beta} = F\tilde{g}_{\alpha\beta}$. The local components Γ^h_{ij} of the Livi-Civita connection on the warped product $M = \bar{M} \times_F \tilde{M}$ are as follows

$$\begin{aligned} \Gamma^a_{bc} &= \overline{\Gamma}^a_{bc}, \quad \Gamma^\alpha_{\beta\gamma} = \tilde{\Gamma}^\alpha_{\beta\gamma}, \\ \Gamma^a_{\alpha\beta} &= \frac{1}{2}\overline{g}^{ab} F_b \tilde{g}_{\alpha\beta}, \quad \Gamma^\alpha_{a\beta} = \frac{1}{2F} F_a \delta^\alpha_\beta, \\ \Gamma^\alpha_{ab} &= \Gamma^a_{\alpha b} = 0 \end{aligned}$$

where $F_a = \partial_a F = \frac{\partial F}{\partial x^\alpha}$.

The local components of the Riemannian curvature tensor of the warped product $M = \bar{M} \times_F \tilde{M}$ are given as

$$\begin{aligned} R_{abcd} &= \bar{R}_{abcd}, \\ R_{\alpha\beta\gamma\delta} &= F\tilde{R}_{\alpha\beta\gamma\delta} - \frac{1}{4}P\tilde{G}_{\alpha\beta\gamma\delta}, \\ R_{\alpha a\beta b} &= \frac{-1}{2}T_{ab}\tilde{g}_{\alpha\beta}, \end{aligned}$$

where $P = \bar{g}^{ab}F_aF_b$, and $\tilde{G}_{\alpha\beta\gamma\delta} = \tilde{g}_{\alpha\gamma}\tilde{g}_{\beta\delta} - \tilde{g}_{\alpha\delta}\tilde{g}_{\beta\gamma}$. T is a $(0,2)$ tensor and its local components are

$$T_{ab} = \bar{\nabla}_b F_a - \frac{1}{2F}F_aF_b.$$

Locally, the Ricci curvature R_{ij} of the warped product $M = \bar{M} \times_F \tilde{M}$ has the following components

$$R_{ab} = \bar{R}_{ab} - \frac{\tilde{n}}{2F}T_{ab}, \tag{3.1}$$

$$R_{\alpha\beta} = \tilde{R}_{\alpha\beta} - \left[tr\,(T) + \frac{\tilde{n}-1}{2F}\bar{\Delta}F\right]\tilde{g}_{\alpha\beta}, \tag{3.2}$$

where $tr\,(T) = \bar{g}^{ab}T_{ab}$.

Theorem 3.1. *Let $M = \bar{M} \times_F \tilde{M}$ be a generalized quasi-Einstein warped product manifold, then the base manifold $\bar{M}$ is generalized quasi-Einstein.*

Proof. Let M be a warped product manifold. Then

$$R_{ab} = \bar{R}_{ab} - \frac{\tilde{n}}{2F}T_{ab}.$$

Assume that M is G(QE)$_n$, thus

$$R_{ab} = \alpha\bar{g}_{ab} + \beta\bar{A}_a\bar{A}_b + \gamma\bar{B}_a\bar{B}_b.$$

The last two Equations imply

$$\bar{R}_{ab} - \frac{\tilde{n}}{2F}T_{ab} = \alpha\bar{g}_{ab} + \beta\bar{A}_a\bar{A}_b + \gamma\bar{B}_a\bar{B}_b.$$

If $T_{ab} = 0$, then

$$\bar{R}_{ab} = \alpha\bar{g}_{ab} + \beta\bar{A}_a\bar{A}_b + \gamma\bar{B}_a\bar{B}_b,$$

which illustrates that $\bar{M}$ is generalized quasi-Einstein manifold. □

Theorem 3.2. *If $M = \bar{M} \times_F \tilde{M}$ is a generalized quasi-Einstein warped product manifold, then the fiber manifold $\tilde{M}$ is generalized quasi-Einstein.*

Proof. Let M be a warped product manifold. Thus

$$R_{\alpha\beta} = R_{\alpha\beta} = \tilde{R}_{\alpha\beta} - \left[tr\,(T) + \frac{\tilde{n}-1}{2F}\bar{\Delta}F\right]\tilde{g}_{\alpha\beta}.$$

Now, assume that M is G(QE)$_n$, then

$$R_{\alpha\beta} = \tilde{g}_{\alpha\beta} + \beta\tilde{A}_\alpha\tilde{A}_\beta + \gamma\tilde{B}_\alpha\tilde{B}_\beta.$$

Using the last two Equations one gets

$$\tilde{R}_{\alpha\beta} = \left[tr\,(T) + \frac{\tilde{n}-1}{2F}\bar{\Delta}F + \alpha\right]\tilde{g}_{\alpha\beta} + \beta\tilde{A}_\alpha\tilde{A}_\beta + \gamma\tilde{B}_\alpha\tilde{B}_\beta,$$

which shows that $\tilde{M}$ is G(QE)$_n$ warped product manifold. □

4 Generalized quasi-Einstein GRW space-times

A generalized Robertson-Walker space-time (or a GRW space-time) is the warped product $M = I \times_f M^*$ of an open connected interval $\left(I, -dt^2\right)$ with warping function $f : I \to R^+$. A Lorentzian manifold M is a generalized Robertson-Walker space-time if and only if M possess a unit time-like vector field u_i such that

$$\nabla_k u_i = \varphi\left(g_{ki} + u_k u_i\right), \quad (4.1)$$

$$R_{ij} u^j = \xi u_i, \quad (4.2)$$

where φ and ξ are scalar functions. Vector fields satisfying Equation (4.1) are called torse-forming and those obeying Equation (4.2) are eigenvectors of the Ricci tensor with eigenvalue ξ [14, 15].

Theorem 4.1. *The following statements are true in a generalized quasi-Einstein GRW space-time M,*

(i) M reduces to be a nearly quasi-Einstein manifold if u^i is not orthogonal to the both generators,

(ii) M reduces to be Einstein if u^i is orthogonal to both the generators, provided $\varphi \neq 0$ and $\xi = cons\tan t$.

Proof. Assume that M is a generalized quasi−Einstein manifold, that is

$$R_{ij} = \alpha g_{ij} + \beta A_i A_j + \gamma B_i B_j.$$

Multiplying by u^i, one gets

$$\left(\xi - \alpha\right) u_j = \beta u^i A_i A_j + \gamma u^i B_i B_j$$

The two Contractions of the both sides with A^i and B^i imply

$$\left(\xi - \alpha - \beta\right) u^i A_i = 0,$$

$$\left(\xi - \alpha - \gamma\right) u^i B_i = 0.$$

There are two different possible cases

(i) If u^i is not orthogonal to the both generator that is, $u^i A_i = u^i B_i \neq 0$, then $\beta = \gamma$ and hence the Ricci tensor becomes.

$$R_{ij} = \alpha g_{ij} + \beta\left(A_i A_j + B_i B_j\right).$$

Since $\left(A_i A_j + B_i B_j\right) = E_{ij}$, we have

$$R_{ij} = \alpha g_{ij} + \beta E_{ij},$$

where $\left(A_i A_j + B_i B_j\right) = E_{ij}$ is a symmetric tensor.

(ii) If u^i is orthogonal to the both generators, that is, $u^i A_i = u^i B_i = 0$, and hence

$$u^i R_{ij} = \alpha u_j,$$

The covariant derivative of the Ricci tensor of M is

$$\nabla_k R_{ij} = \beta\left[A_i\left(\nabla_k A_j\right) + A_j\left(\nabla_k A_i\right)\right] + \gamma\left[B_i\left(\nabla_k B_j\right) + B_j\left(\nabla_k B_i\right)\right]$$

contracting with u^i and using the condition $u^i A_i = u^i B_i = 0$, then

$$u^i \nabla_k R_{ij} = 0,$$

$$\nabla_k u^i R_{ij} - R_{ij} \nabla_k u^i = 0$$

If M is generalized Robertson-Walker space-time, then the following two equations hold

$$\nabla_k u^i = \varphi\left(\delta^i_k + u_k u^i\right),$$

$$R_{ij} u^i = \xi u_j$$

Thus

$$\nabla_k\left(\xi u_j\right) - R_{ij}\varphi\left(\delta^i_k + u_k u^i\right) = 0$$

$$u_j \nabla_k\left(\xi\right) + \xi \nabla_k u_j - \varphi \delta^i_k R_{ij} - \varphi u_k u^i R_{ij} = 0$$

$$\xi\varphi\left(g_{kj} + u_k u_j\right) - \varphi R_{kj} - \xi\varphi u_k u_j = 0$$

$$u_j \nabla_k\left(\xi\right) + \varphi\left(\xi g_{kj} - R_{kj}\right) = 0.$$

If $\xi = cons\tan t$, then the above Equation becomes

$$\varphi\left(\xi g_{kj} - R_{kj}\right) = 0.$$

If $\varphi \neq 0$, then M is an Einstein manifold.

□

5 Generalized quasi-Einstein (PRS)$_n$ manifold

Assume that M is generalized quasi-Einstein, then

$$R_{ij} = \alpha g_{ij} + \beta B_i B_j + \gamma D_i D_j, \tag{5.1}$$

By contracting with g^{ij}, one gets

$$R = n\alpha + \beta + \gamma, \tag{5.2}$$

which means that the scalar curvature is constant.

Let M be a pseudo-Ricci symmetric manifold, then the covariant derivative of the Ricci tensor is

$$\nabla_k R_{ij} = 2A_k R_{ij} + A_i R_{kj} + A_j R_{ik}, \tag{5.3}$$

Multiplying the both sides of Equation (5.3) by g^{kj}, one may get the divergence of the Ricci tensor as follows

$$\nabla_k R^k_i = 3A^j R_{ij} + A_i R.$$

It is well-known that

$$\nabla_k R^k_i = \frac{1}{2}\nabla_i R.$$

Thus,

$$\nabla_i R = 6A^j R_{ij} + 2A_i R. \tag{5.4}$$

A different contraction of Equation (5.3) by g^{ij}, it is

$$\nabla_k R = 2A_k R + 2A^j R_{kj}, \tag{5.5}$$

By solving these two equations, one gets

$$A^j R_{kj} = 0, \tag{5.6}$$

$$\nabla_k R = 2A_k R. \tag{5.7}$$

Equations (5.7) and (5.2) imply

$$R = 0.$$

Theorem 5.1. *The scalar curvature of a generalized quasi- Einstein (PRS)$_n$ manifold is zero.*

Assume that the Ricci tensor of a generalized quasi-Einstein (PRS)$_n$ manifold M is cyclic parallel[1, 10], that is,

$$\nabla_j R_{ki} + \nabla_k R_{ji} + \nabla_i R_{kj} = 0.$$

Equation (5.3) implies

$$A_k R_{ij} + A_j R_{ik} + A_i R_{jk} = 0. \tag{5.8}$$

Contracting Equation (5.8) with A^k and use Equation (5.6), one may obtain

$$aR_{ij} = 0, \tag{5.9}$$

where $a = A^k A_k$. Thus

$$R_{ij} = 0,$$

which means that M is Ricci flat. Then we have the following theorem.

Theorem 5.2. *A generalized quasi-Einstein (PRS)n manifold with cyclic parallel Ricci tensor is Ricci flat.*

References

[1] Abu-Donia, Hassan, Sameh Shenawy, and Abdallah Abdelhameed Syied. "The W-curvature Tensor on Relativistic Space-times." KYUNGPOOK Math. J **60**(2020): 185-195.

[2] Bejan, Cornelia-Livia, and Tran Quoc Binh. "*Generalized Einstein manifolds*." In Differential geometry and its applications, pp. 57-64. 2008.

[3] Besse, A. L. "*Einstein manifolds*: Ergeb. Math. Grenzgeb. 3. Folge, Bd. 10." (1987).

[4] Chaki, M. C. "*On pseudo Ricci symmetric manifolds*." Bulgarian journal of physics **15**(1988): 526-531.

[5] Chaki, M. C. "*On quasi Einstein manifolds*." Publicationes Mathematicae, Debrecen **57**(2000): 297-306.

[6] Bang-yen, Chen. *Differential geometry of warped product manifolds and submanifolds*. World Scientific, 2017.

[7] De, Uday Chand, and Gopal Chandra Ghosh. "*On generalized quasi Einstein manifolds*." Kyungpook mathematical journal **44**(2004): 607-607.

[8] De, U. C. "*On conformally flat special quasi Einstein manifolds*." Publ. Math., Debrecen, Hungary **66**(2005): 129-136.

[9] De, Uday Chand, and Abul Kalam Gazi. "*On nearly quasi-Einstein manifolds*." Novi Sad J. Math **38** (2008): 115-121.

[10] De, Uday Chand, H. M. Abu-Donia, Sameh Shenawy, and Abdallah Abdelhameed Syied. "On generalized projective P-curvature tensor." Journal of Geometry and Physics **159**(2021): 103952.

[11] Hui, Shyamal Kumar, and Richard S. Lemence. "*On generalized quasi-Einstein manifold admitting W2-curvature tensor*." International Journal of Mathematical Analysis **6**(2012): 1115-1121.

[12] Hui, Shyamal Kumar, and Richard S. Lemence. "Ricci pseudosymmetric generalized quasi-Einstein manifolds." Sut J. Math **51**(2015): 195-213.

[13] Kırık, Bahar, and Füsun Özen Zengin. "*Applications of a Special Generalized Quasi-Einstein Manifold*." Bulletin of the Iranian Mathematical Society **45**(2019): 89-102.

[14] Mantica, Carlo Alberto, and Luca Guido Molinari. "*On the Weyl and Ricci tensors of Generalized Robertson-Walker space-times*." Journal of Mathematical Physics **57**(2016): 102502.

[15] Mantica, Carlo Alberto, and Luca Guido Molinari. "*Generalized Robertson–Walker spacetimes—a survey*." International Journal of Geometric Methods in Modern Physics **14**(2017): 1730001.

[16] Prakasha D. G. and Venkatesha H., Some results on generalised quasi-Einstein manifolds,

[17] Chinese Journal of Mathematics (Hindawi Publishing Corporation) (2014), Art. Id. 563803, 5 pages.

[18] Shaikh, Absos Ali, and Shyamal Kumar Hui. "*On some classes of generalized quasi-Einstein manifolds*." Communications of the Korean Mathematical Society **24**(2009): 415-424.

[19] A. A. Shaikh, D. W. Yoon and S. K. Hui, On quasi-Einstein spacetimes, *Tsukuba J. Math.*, **33** **(**2009), 305-326.

[20] Shaikh, Absos Ali, Young-Ho Kim, and Shyamal Kumar Hui. "On Lorentzian quasi-Einstein manifolds." Journal of the Korean Mathematical Society **48**(2011): 669-689.

[21] Sular, Sibel, and Cihan Özgür. "*Characterizations of generalized quasi-Einstein manifolds*." Analele Universitatii" Ovidius" Constanta-Seria Matematica **20**(2012): 407-416.

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