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A NOTE ON S-ZARISKI TOPOLOGY

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Abstract In [E. Yildiz, B. A. Ersoy, Ü. Tekir and S. Koc, On S-Zariski topology, Comm. Algebra, 49 (3) (2021), 1212-1224], the authors have constructed a topology on $\text{Spec}_S(R)$ (the set of S-prime ideals of a commutative ring R with identity) called the S-Zariski topology. They have proved that (1) $\text{Spec}_S(R)$ is an irreducible space if and only if $N_S(R) \in \text{Spec}(R)$ and (2) $\text{Spec}_S(R)$ is compact. The aim of this short note is to improve the above mentioned results. More precisely, we show that $\text{Spec}_S(R)$ is an irreducible space if and only if $N(R) \in \text{Spec}_S(R)$. Moreover, we demonstrate that any basis element of the S-Zariski topology is compact. In particular, $\text{Spec}_S(R)$ is compact. As a consequence, we show that any set X satisfying $\{P \in \text{Spec}(R) \mid P \cap S = \emptyset\} \subseteq X \subseteq \text{Spec}_S(R)$ is compact.

1 Introduction

All rings considered below are commutative with identity. Let R be a ring and S a multiplicative closed subset of R (briefly, m.c.s). An ideal P of R is called S-prime if $P \cap S = \emptyset$ and there exists an (fixed) $s \in S$ such that whenever $ab \in R$ for some $a, b \in R$, then either $sa \in P$ or $sb \in P$ (cf. [2, 5]). We let $\text{Spec}_S(R)$ denote the set of all S-prime ideals of R. In [7], the authors have constructed a topology on $\text{Spec}_S(R)$ called the S-Zariski topology. Any closed set of $\text{Spec}_S(R)$ has the form $V_S(E) := \{P \in \text{Spec}_S(R) \mid sE \subseteq P \text{ for some } s \in S\}$, where E is a subset of R. The collection of all $D_a^S := \{P \in \text{Spec}_S(R) \mid sa \notin P \text{ for all } s \in S\}$, where $a \in R$, forms a basis for the S-Zariski topology. The irreducibility and the compactness of $\text{Spec}_S(R)$ were investigated in [7]. More precisely, [7, Theorem 5] states that $\text{Spec}_S(R)$ is irreducible if and only if $N_S(R)$ is a prime ideal, where $N_S(R) = \{a \in R \mid sa^n = 0 \text{ for some } s \in S \text{ and } n \in \mathbb{N}\}$. Moreover, $\text{Spec}_S(R)$ is a compact topological space (cf. [7, Theorem 6]). Our purpose here is to sharpen these two results. Our main result in Section 2 is Theorem 2.2, which shows that $\text{Spec}_S(R)$ is irreducible if and only if $N(R)$ is an S-prime ideal, where $N(R)$ is the nilradical of R (the set of nilpotent elements of R). In Section 3, we prove that each basis element D_a^S is compact. In particular, $\text{Spec}_S(R) = D_1^S$ is compact (see Theorem 3.2). As a consequence, we show that any set X such that $\{P \in \text{Spec}(R) \mid P \cap S = \emptyset\} \subseteq X \subseteq \text{Spec}_S(R)$ is compact (see Corollary 3.3). As usual $\text{Spec}(R)$ denotes the set of all prime ideals of the ring R. Any unexplained terminology is standard as in [1] and [4].

2 Irreducibility of $\text{Spec}_S(R)$

Recall that a topological space X is called *irreducible* if X is nonempty and cannot be expressed as the union of two proper closed subsets, or equivalently, any two nonempty open subsets of X intersect (cf. [3, 4]). It is worth noticing that $\text{Spec}_S(R)$ is always nonempty because $(0) \cap S = \emptyset$ and so there exists a prime ideal Q of R disjoint from S according to [6, Theorem 3.44]. Therefore, $Q \in \text{Spec}_S(R)$. Recall that Theorem 5 in [7] states that $\text{Spec}_S(R)$ is irreducible if and only if $N_S(R)$ is a prime ideal, where $N_S(R) = \{a \in R \mid sa^n = 0 \text{ for some } s \in S \text{ and } n \in \mathbb{N}\}$. Our goal in this section is to provide another characterization of the irreducibility of $\text{Spec}_S(R)$ by means of the nilradical of R.

We start our investigation with the following straightforward result.

Lemma 2.1. *Let R be a commutative ring and S a m.c.s of R. Then*

$$a \in N_S(R) \Longleftrightarrow \exists\ s \in S \text{ such that } sa \in N(R).$$

Proof. "$\Longrightarrow$" Since $a \in N_S(R)$, then there exist $s \in S$ and $n \in \mathbb{N}$ such that $sa^n = 0$. Thus, $(sa)^n = s^{n-1}(sa^n) = 0$. This shows that $sa \in N(R)$.
"$\Longleftarrow$" By assumption, there exist $s \in S$ and $n \in \mathbb{N}$ such that $(sa)^n = 0$. Put $s^* = s^n \in S$. Clearly, $s^*a^n = 0$. Hence, $a \in N_S(R)$. □

We present now the titular result of this section which improves [7, Theorem 5].

Theorem 2.2. *Let R be a commutative ring and S a m.c.s of R. Then the following statements are equivalent:*
(1) $\operatorname{Spec}_S(R)$ *is an irreducible space;*
(2) $N(R) \in \operatorname{Spec}_S(R)$*;*
(3) $N_S(R) \in \operatorname{Spec}(R)$.

Proof. (1)$\Longrightarrow$(2) Firstly, notice that $N(R) \cap S = \emptyset$ since $0 \notin S$. Assume by way of contradiction that $N(R) \notin \operatorname{Spec}_S(R)$. Then, for any $s \in S$, there exist $a, b \in R$ such that $ab \in N(R)$ but $sa, sb \notin N(R)$. As $sa \notin N(R)$ for any $s \in S$, it follows from Lemma 2.1 that $a \notin N_S(R)$. Thus, $D_a^S \neq \emptyset$ by virtue of [7, Proposition 6]. A similar argument shows that $D_b^S \neq \emptyset$. Since $ab \in N(R)$, then $ab \in N_S(R)$. Another appeal to [7, Proposition 6] guarantees that $D_{ab}^S = \emptyset$. Since $D_a^S \cap D_b^S = D_{ab}^S$ according to [7, Proposition 6], we get readily $D_a^S \cap D_b^S = \emptyset$. This shows that $\operatorname{Spec}_S(R)$ is not irreducible, the desired contradiction.
(2)$\Longrightarrow$(3) Let $a, b \in R$ such that $ab \in N_S(R)$. According to Lemma 2.1, there exists $s \in S$ such that $sab \in N(R)$. As $N(R)$ is an S-prime ideal, then there exists $s^* \in S$ such that either $s^*sa \in N(R)$ or $s^*b \in N(R)$. Using again Lemma 2.1, we deduce that $a \in N_S(R)$ or $b \in N_S(R)$.
(3)$\Longrightarrow$(1) Follows readily from [7, Theorem 5]. □

Lemma 2.3. *Let R be a commutative ring and S a m.c.s of R. Then*

$$P \in \operatorname{Spec}_S(R) \Longrightarrow \sqrt{P} \in \operatorname{Spec}_S(R).$$

Proof. Let $a, b \in R$ such that $ab \in \sqrt{P}$. Then there exists $n \in \mathbb{N}$ such that $a^n b^n \in P$. As $P \in \operatorname{Spec}_S(R)$, then there exists $s \in S$ such that either $sa^n \in P$ or $sb^n \in P$. Hence, either $sa \in \sqrt{P}$ or $sb \in \sqrt{P}$. This completes the proof. □

Recall from [5] that a ring R is said to be an *S-integral domain* if there exists a fixed $s \in S$ such that whenever $ab = 0$ for some $a, b \in R$, then either $sa = 0$ or $sb = 0$. This concept generalizes that of an integral domain. It is not difficult to check that R is an S-integral domain if and only if $(0) \in \operatorname{Spec}_S(R)$. Next we recover [7, Corollary 1].

Corollary 2.4. *Let R be a commutative ring and S a m.c.s of R. If R is an S-integral domain, then* $\operatorname{Spec}_S(R)$ *is an irreducible space.*

Proof. As R is an S-integral domain, then $(0) \in \operatorname{Spec}_S(R)$. Hence, $N(R) = \sqrt{(0)} \in \operatorname{Spec}_S(R)$ by virtue of Lemma 2.3. Thus, $\operatorname{Spec}_S(R)$ is an irreducible space by Theorem 2.2. □

3 Compactness of $\operatorname{Spec}_S(R)$

Recall from [4] that a topological space X is called *compact* if any open cover of X has a finite subcover. Recall also [7, Definition 1] that if R is a commutative ring, S is a m.c.s of R and I is an ideal of R, then the S-radical of I is defined by:

$$\sqrt[S]{I} := \{a \in R \mid sa^n \in I \text{ for some } s \in S \text{ and } n \in \mathbb{N}\}.$$

Lemma 3.1. *Let R be a commutative ring and S a m.c.s of R. If I and J are ideals of R. Then*

$$V_S(I) \subseteq V_S(J) \Longleftrightarrow \sqrt[S]{J} \subseteq \sqrt[S]{I}.$$

Proof. Follows immediately by combining [7, Remark 2], [7, Proposition 5] and [7, Proposition 7]. □

The next theorem improves [7, Theorem 6].

Theorem 3.2. *Let R be a commutative ring and S a m.c.s of R. Then D_a^S is compact for any $a \in R$. In particular,* $\text{Spec}_S(R) = D_1^S$ *is compact.*

Proof. Assume that $D_a^S \subseteq \bigcup_{i\in\Delta} D_{a_i}^S$, where $a_i \in R$ for any $i \in \Delta$. Then, by using [7, Theorem 1], we obtain $V_S(\langle a\rangle) \supseteq \bigcap_{i\in\Delta} V_S(\langle a_i\rangle) = V_S(I)$, where $I = \langle a_i, i \in \Delta\rangle$ is the ideal of R generated by the a_i's. It follows from Lemma 3.1 that $\sqrt[S]{\langle a\rangle} \subseteq \sqrt[S]{I}$. Hence, $a \in \sqrt[S]{I}$. Thus, there exist $s \in S$ and $n \in \mathbb{N}$ such that $sa^n \in I$. Therefore, there exists a finite subset δ of Δ such that $sa^n \in \langle a_i, i \in \delta\rangle$. This shows that $a \in \sqrt[S]{\langle a_i, i \in \delta\rangle}$. Thus, $V_S(\sqrt[S]{\langle a_i, i \in \delta\rangle}) \subseteq V_S(a)$ by virtue of Lemma 3.1. Hence, $V_S(\langle a_i, i \in \delta\rangle) \subseteq V_S(a)$ ([7, Proposition 7]); or equivalently, $\bigcap_{i\in\delta} V_S(a_i) \subseteq V_S(a)$. It follows that $D_a^S \subseteq \bigcup_{i\in\delta} D_{a_i}^S$. This completes the proof. □

As a consequence, we derive the following corollary.

Corollary 3.3. *Let R be a commutative ring and S a m.c.s of R. Assume that $\{P \in \text{Spec}(R) \mid P \cap S = \emptyset\} \subseteq X \subseteq \text{Spec}_S(R)$. Then X is compact.*

Proof. Using Theorem 3.2, it is enough to show that if O is an open subset of $\text{Spec}_S(R)$ containing X, then $O = \text{Spec}_S(R)$. To this end, let $P \in \text{Spec}_S(R)$. As $P \cap S = \emptyset$, then it follows from [6, Theorem 3.44] that there exists $Q \in \text{Spec}(R)$ such that $P \subseteq Q$ and $Q \cap S = \emptyset$. Since $Q \in X \subseteq O$, then there exists $a \in R$ such that $Q \in D_a^S \subseteq O$. Thus, for any $s \in S$, $sa \notin Q$. In particular, for any $s \in S$, $sa \notin P$. This proves that $P \in D_a^S \subseteq O$. Therefore, we have proved that $\text{Spec}_S(R) \subseteq O$. As the reverse inclusion is obvious, we get $O = \text{Spec}_S(R)$. The proof is complete. □

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