A NOTE ON S-ZARISKI TOPOLOGY

Noômen Jarboui

Communicated by Ayman Badawi


Keywords and phrases: Prime spectrum; S-Zariski topology; Zariski topology.

Abstract In [E. Yildiz, B. A. Ersoy, Ü. Tekir and S. Koc, On S-Zariski topology, Comm. Algebra, 49 (3) (2021), 1212-1224], the authors have constructed a topology on \(\text{Spec}_S(R)\) (the set of \(S\)-prime ideals of a commutative ring \(R\) with identity) called the \(S\)-Zariski topology. They have proved that (1) \(\text{Spec}_S(R)\) is an irreducible space if and only if \(N_S(R) \in \text{Spec}(R)\) and (2) \(\text{Spec}_S(R)\) is compact. The aim of this short note is to improve the above mentioned results. More precisely, we show that \(\text{Spec}_S(R)\) is an irreducible space if and only if \(N(R) \in \text{Spec}_S(R)\). Moreover, we demonstrate that any basis element of the \(S\)-Zariski topology is compact. In particular, \(\text{Spec}_S(R)\) is compact. As a consequence, we show that any set \(X\) satisfying \(\{P \in \text{Spec}(R) \mid P \cap S = \emptyset\} \subseteq X \subseteq \text{Spec}_S(R)\) is compact.

1 Introduction

All rings considered below are commutative with identity. Let \(R\) be a ring and \(S\) a multiplicative closed subset of \(R\) (briefly, m.c.s). An ideal \(P\) of \(R\) is called \(S\)-prime if \(P \cap S = \emptyset\) and there exists an (fixed) \(s \in S\) such that whenever \(ab \in R\) for some \(a,b \in R\), then either \(sa \in P\) or \(sb \in P\) (cf. [2, 5]). We let \(\text{Spec}_S(R)\) denote the set of all \(S\)-prime ideals of \(R\). In [7], the authors have constructed a topology on \(\text{Spec}_S(R)\) called the \(S\)-Zariski topology. Any closed set of \(\text{Spec}_S(R)\) has the form \(V_S(E) := \{P \in \text{Spec}_S(R) \mid sE \subseteq P\} \subseteq \text{Spec}_S(R)\), where \(E\) is a subset of \(R\). The collection of all \(D^S_a := \{P \in \text{Spec}_S(R) \mid sa \notin P\} \subseteq \text{Spec}_S(R)\) is a basis for the \(S\)-Zariski topology. The irreducibility and the compactness of \(\text{Spec}_S(R)\) were investigated in [7]. More precisely, [7, Theorem 5] states that \(\text{Spec}_S(R)\) is irreducible if and only if \(N_S(R)\) is a prime ideal, where \(N_S(R) = \{a \in R \mid sa^n = 0\} \subseteq \text{Spec}_S(R)\). Moreover, \(\text{Spec}_S(R)\) is a compact topological space (cf. [7, Theorem 6]). Our purpose here is to sharpen these two results. Our main result in Section 2 is Theorem 2.2, which shows that \(\text{Spec}_S(R)\) is irreducible if and only if \(N(R)\) is an \(S\)-prime ideal, where \(N(R)\) is the nilradical of \(R\) (the set of nilpotent elements of \(R\)). In Section 3, we prove that each basis element \(D^S_a\) is compact. In particular, \(\text{Spec}_S(R) = D^S_a\) is compact (see Theorem 3.2). As a consequence, we show that any set \(X\) such that \(\{P \in \text{Spec}(R) \mid P \cap S = \emptyset\} \subseteq X \subseteq \text{Spec}_S(R)\) is compact (see Corollary 3.3). As usual \(\text{Spec}(R)\) denotes the set of all prime ideals of the ring \(R\). Any unexplained terminology is standard as in [1] and [4].

2 Irreducibility of \(\text{Spec}_S(R)\)

Recall that a topological space \(X\) is called irreducible if \(X\) is nonempty and cannot be expressed as the union of two proper closed subsets, or equivalently, any two nonempty open subsets of \(X\) intersect (cf. [3, 4]). It is worth noticing that \(\text{Spec}_S(R)\) is always nonempty because \((0) \cap S = \emptyset\) and so there exists a prime ideal \(Q\) of \(R\) disjoint from \(S\) according to [6, Theorem 3.44]. Therefore, \(Q \in \text{Spec}_S(R)\). Recall that Theorem 5 in [7] states that \(\text{Spec}_S(R)\) is irreducible if and only if \(N_S(R)\) is a prime ideal, where \(N_S(R) = \{a \in R \mid sa^n = 0\} \subseteq \text{Spec}_S(R)\). Our goal in this section is to provide another characterization of the irreducibility of \(\text{Spec}_S(R)\) by means of the nilradical of \(R\).

We start our investigation with the following straightforward result.
Lemma 2.1. Let $R$ be a commutative ring and $S$ a m.c.s of $R$. Then
\[ a \in N_S(R) \iff \exists s \in S \text{ such that } sa \in N(R). \]

Proof. “$\iff$” Since $a \in N_S(R)$, then there exist $s \in S$ and $n \in \mathbb{N}$ such that $sa^n = 0$. Thus, $(sa)^n = s^{n-1}(sa^n) = 0$. This shows that $sa \in N(R)$. “$\implies$” By assumption, there exist $s \in S$ and $n \in \mathbb{N}$ such that $(sa)^n = 0$. Put $s^* = s^n \in S$. Clearly, $s^*a^n = 0$. Hence, $a \in N_S(R)$.

We present now the titular result of this section which improves [7, Theorem 5].

Theorem 2.2. Let $R$ be a commutative ring and $S$ a m.c.s of $R$. Then the following statements are equivalent:
\begin{enumerate}
  \item $\text{Spec}_S(R)$ is an irreducible space;
  \item $N(R) \in \text{Spec}_S(R)$;
  \item $N_S(R) \in \text{Spec}(R)$.
\end{enumerate}

Proof. (1)$\implies$(2) Firstly, notice that $N(R) \cap S = \emptyset$ since $0 \notin S$. Assume by way of contradiction that $N(R) \notin \text{Spec}_S(R)$. Then, for any $s \in S$, there exist $a, b \in R$ such that $ab \in N(R)$ but $sa, sb \notin N(R)$. As $sa \notin N(R)$ for any $s \in S$, it follows from Lemma 2.1 that $a \notin N_S(R)$. Thus, $D_S^a \neq \emptyset$ by virtue of [7, Proposition 6]. A similar argument shows that $D_S^b \neq \emptyset$. Since $ab \in N(R)$, then $ab \in N_S(R)$. Another appeal to [7, Proposition 6] guarantees that $D_{ab} = \emptyset$. Since $D_S^a \cap D_S^b = D_{ab}$ according to [7, Proposition 6], we get readily $D_S^a \cap D_S^b = \emptyset$. This shows that $\text{Spec}_S(R)$ is not irreducible, the desired contradiction.

(2)$\implies$(3) Let $a, b \in R$ such that $ab \in N_S(R)$. According to Lemma 2.1, there exists $s \in S$ such that $sab \in N(R)$. As $N(R)$ is an S-prime ideal, then there exists $s^* \in S$ such that either $s^*sa \in N(R)$ or $s^*b \in N(R)$. Using again Lemma 2.1, we deduce that $a \in N_S(R)$ or $b \in N_S(R)$.

(3)$\implies$(1) Follows readily from [7, Theorem 5].

Lemma 2.3. Let $R$ be a commutative ring and $S$ a m.c.s of $R$. Then
\[ P \in \text{Spec}_S(R) \implies \sqrt{P} \in \text{Spec}_S(R). \]

Proof. Let $a, b \in R$ such that $ab \in \sqrt{P}$. Then there exists $n \in \mathbb{N}$ such that $a^n b^n \in P$. As $P \in \text{Spec}_S(R)$, then there exists $s \in S$ such that either $sa^n \in P$ or $sb^n \in P$. Hence, either $sa \in \sqrt{P}$ or $sb \in \sqrt{P}$. This completes the proof.

Recall from [5] that a ring $R$ is said to be an $S$-integral domain if there exists a fixed $s \in S$ such that whenever $ab = 0$ for some $a, b \in R$, then either $sa = 0$ or $sb = 0$. This concept generalizes that of an integral domain. It is not difficult to check that $R$ is an $S$-integral domain if and only if $(0) \in \text{Spec}_S(R)$. Next we recover [7, Corollary 1].

Corollary 2.4. Let $R$ be a commutative ring and $S$ a m.c.s of $R$. If $R$ is an $S$-integral domain, then $\text{Spec}_S(R)$ is an irreducible space.

Proof. As $R$ is an $S$-integral domain, then $(0) \in \text{Spec}_S(R)$. Hence, $N(R) = \sqrt{(0)} \in \text{Spec}_S(R)$ by virtue of Lemma 2.3. Thus, $\text{Spec}_S(R)$ is an irreducible space by Theorem 2.2.

3 Compactness of $\text{Spec}_S(R)$

Recall from [4] that a topological space $X$ is called compact if any open cover of $X$ has a finite subcover. Recall also [7, Definition 1] that if $R$ is a commutative ring, $S$ is a m.c.s of $R$ and $I$ is an ideal of $R$, then the $S$-radical of $I$ is defined by:
\[ \sqrt{S} := \{ a \in R \mid sa^n \in I \text{ for some } s \in S \text{ and } n \in \mathbb{N} \}. \]

Lemma 3.1. Let $R$ be a commutative ring and $S$ a m.c.s of $R$. If $I$ and $J$ are ideals of $R$. Then
\[ V_S(I) \subseteq V_S(J) \iff \sqrt{J} \subseteq \sqrt{I}. \]
Proof. Follows immediately by combining [7, Remark 2], [7, Proposition 5] and [7, Proposition 7].

The next theorem improves [7, Theorem 6].

Theorem 3.2. Let \( R \) be a commutative ring and \( S \) a m.c.s of \( R \). Then \( D_S^S \) is compact for any \( a \in R \). In particular, \( \text{Spec}_S(R) = D_S^S \) is compact.

Proof. Assume that \( D_S^S \subseteq \bigcup_{i \in A} D_{a_i}^S \), where \( a_i \in R \) for any \( i \in \Delta \). Then, by using [7, Theorem 1], we obtain \( V_S((a)) \supseteq \bigcap_{i \in \Delta} V_S(a_i) = V_S(I) \), where \( I = (a_i, i \in \Delta) \) is the ideal of \( R \) generated by the \( a_i \)'s. It follows from Lemma 3.1 that \( \sqrt{(a)} \subseteq \sqrt{I} \). Hence, \( a \in \sqrt{I} \). Thus, there exist \( s \in S \) and \( n \in \mathbb{N} \) such that \( sa^n \in I \). Therefore, there exists a finite subset \( \delta \) of \( \Delta \) such that \( sa^n \in (a_i, i \in \delta) \). This shows that \( a \in \sqrt{(a_i, i \in \delta)} \). Thus, \( V_S(\sqrt{(a_i, i \in \delta)}) \subseteq V_S(a) \) by virtue of Lemma 3.1. Hence, \( V_S((a_i, i \in \delta)) \subseteq V_S(a) \) [7, Proposition 7]); or equivalently, \( \bigcap_{i \in \delta} V_S(a_i) \subseteq V_S(a) \). It follows that \( D_S^S \subseteq \bigcup_{i \in \delta} D_{a_i}^S \). This completes the proof. □

As a consequence, we derive the following corollary.

Corollary 3.3. Let \( R \) be a commutative ring and \( S \) a m.c.s of \( R \). Assume that \( \{ P \in \text{Spec}(R) \mid P \cap S = \emptyset \} \subseteq X \subseteq \text{Spec}_S(R) \). Then \( X \) is compact.

Proof. Using Theorem 3.2, it is enough to show that if \( O \) is an open subset of \( \text{Spec}_S(R) \) containing \( X \), then \( O = \text{Spec}_S(R) \). To this end, let \( P \in \text{Spec}_S(R) \). As \( P \cap S = \emptyset \), then it follows from [6, Theorem 3.4.4] that there exists \( Q \in \text{Spec}(R) \) such that \( P \subseteq Q \) and \( Q \cap S = \emptyset \). Since \( Q \in X \subseteq O \), then there exists \( a \in R \) such that \( Q \in D_S^S \subseteq O \). Thus, for any \( s \in S \), \( sa \notin Q \). In particular, for any \( s \in S \), \( sa \notin P \). This proves that \( P \in D_S^S \subseteq O \). Therefore, we have proved that \( \text{Spec}_S(R) \subseteq O \). As the reverse inclusion is obvious, we get \( O = \text{Spec}_S(R) \). The proof is complete. □

References


Author information

Noômen Jarboui, Department of Mathematics, Faculty of Sciences, University of Sfax, P. O. Box 1171, Sfax 3038, Tunisia.
E-mail: noomenjarboui@yahoo.fr

Received: June 1, 2021
Accepted: December 19, 2021