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SURFACE RECOVERING BY A GIVE TOTAL AND MEAN CURVATURE IN ISOTROPIC SPACE R_3^2

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Abstract In this paper, we solve the question of the existence of a surface with a given total and mean curvature in the isotropic space R_3^2. For this, we first give the definition of an isotropic space and the necessary facts from surface theory in R_3^2.

The problem is solved in the class of transfer surfaces. Therefore, we present the total and mean curvature of the surface and its dual image. The duality of an isotropic space coincides with the polar map of an isotropic space. Transfer surfaces are divided into three types relative to their location in space. For each type, we define the total and mean curvature of the surface and its dual image. In each case, we prove the existence of a surface when the total curvature of the dual image is given. Moreover, the total curvature is given as a function of the divided variable.

In all cases, we give the equation for a surface that has this function as its total curvature. Also, we prove the existence of a surface with a given mean curvature for the three types considered in the work.

Moreover, it is proved that the solutions form a one-parameter λ set of $\vec{r} = \vec{r}_\lambda(u, v)$ surfaces.

1 Introduction

An isotropic space is a space with a degenerate metric. it differs significantly from Euclidean space. Therefore, the presentation of the basic elements of the theory of surfaces is necessary. Moreover, an isotropic space is self-dual in the sense of projective geometry.

Recovery is understood as the problem of finding the surface itself when its total curvature function is given. We consider the problem in an isotropic space, which is a representative of spaces with a degenerate metric [3, 17, 13, 6].

We first present the definition of an isotropic space and some facts from surface theory. Then we formulate the problem and give its solution.

Let there be given a three-dimensional affine space A_3, set by an affine coordinate system $Oxyz$.

Definition 1.1. [2, 7, 10, 11] If the dot product of vectors $X\{x_1, y_1, z_1\}$ and $Y\{x_2, y_2, z_2\}$ is given by the formula

$$\begin{cases} (X,Y)_1 = x_1x_2 + y_1y_2 & \text{if } (X,Y)_1 \neq 0, \\ (X,Y)_2 = z_1z_2 & \text{if } (X,Y)_1 = 0, \end{cases}$$

then the space is said to be an isotropic space and denoted by R_3^2 .

We define the norm of a vector and the distance between the points with the help of the dot product. More precisely, the vector norm $|\overrightarrow{X}| = \sqrt{(\overrightarrow{X}, \overrightarrow{X})}$, the distance between the points $M(X),\ N(Y)$, and
$MN = \sqrt{(Y - X, Y - X)}$.

In coordinates, they have the following form

$$MN_1{}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2, \tag{1.1}$$

if $MN_1 = 0$, then $MN_2 = |z_2 - z_1|$.

The motion, that is, a linear transformation that preserves the entered distance (1.1), has the form:

$$\begin{cases} x' = x \cdot \cos\alpha - y \cdot \sin\alpha + a, \\ y' = x \cdot \sin\alpha + y \cdot \cos\alpha + b, \\ x' = A \cdot x + B \cdot y + z + c, \end{cases}$$

and the transformation matrix is [8, 19, 12] $\begin{pmatrix} cos\alpha & -sin\alpha & 0 \\ sin\alpha & cos\alpha & 0. \\ A & B & 1 \end{pmatrix}$

There are two kinds of sphere in the isotropic space R_3^2, the sphere by distance $x^2 + y^2 = r^2$, and the sphere by curvature in the space

$$z = \frac{1}{2}\left(x^2 + y^2\right). \tag{1.2}$$

2 Surface Theory in Isotropic Space

Let a regular surface be given in R_3^2 by the vector equation

$$\overrightarrow{r}(u, v) = x(u, v) \cdot \overrightarrow{i} + y(u, v) \cdot \overrightarrow{j} + z(u, v) \cdot \overrightarrow{k}.$$

Then the first and second fundamental forms of the surface are determined by the following formulas [2]

$$I = ds^2 = Edu^2 + 2Fdudv + Gdv^2,$$
$$II = Ldu^2 + 2Mdudv + Ndv^2.$$

Although they are outwardly identical to the corresponding forms of Euclidean space, they differ significantly in the formula for calculating the coefficients.

$$\begin{cases} E = r_u^2 = x_u^2 + y_u^2 \\ F = r_u r_v = x_u x_v + y_u y_v \\ G = r_v^2 = x_v^2 + y_v^2 \end{cases}, \quad \text{and} \quad \begin{cases} L = (r_{uu}, n) \\ M = (r_{uv}, n) \\ N = (r_{vv}, n) \end{cases}.$$

Defined by analogy with Euclidean space, the total and mean curvature of the surface, respectively, has the following form

$$K = \frac{LN - M^2}{EG - F^2}, \quad \text{and} \quad 2H = \frac{EN - 2FM + GL}{EG - F^2}.$$

When we consider surfaces that are uniquely projected onto the plane Oxy, the formulas for the total and mean curvature of the surface are further simplified and have the following form

$$K = LN - M^2, \quad \text{and} \quad 2H = L + N.$$

The isotropic space R_3^2 is self-dual in the projective sense ([4]). Also, one can define in an isotropic space the polar correspondence with respect to the sphere (1.2). The polar correspondence of each plane, intersecting a sphere of an isotropic space, matches a point outside the sphere of that space.

Let the regular surface F, given by the equation $z = f(x, y)$, be contained inside the sphere, and its boundary is the intersection of the sphere with a plane α (see Fig. 1).

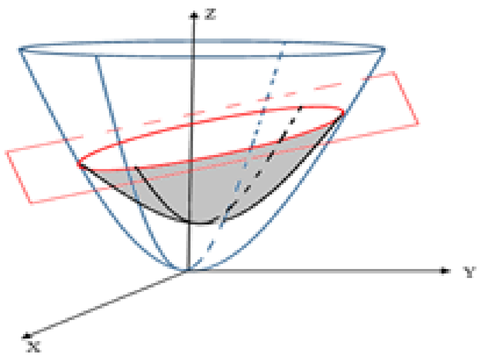

Fig. 1

Suppose that a point $M(x_0, y_0, z_0) \in F$ and π be the tangent plane of the surface at this point. Denote by M^* the point of the dual image of the plane π with respect to the sphere (1.2). When a point $M \in F$ changes on F, then its image M^* in the general case forms some surface F^*.

Definition 2.1. ([2]) The surface F^* is said to be the surface dual to the surface F with respect to the sphere (1.2). When F is regular and belongs to the class C^2, the dual surface has the following equation

$$\begin{cases} x^*(u,v) = f'_u(u,v) \\ y^*(u,v) = f'_v(u,v) \\ z^*(u,v) = u \cdot f'_u(u,v) + v \cdot f'_u(u,v) - f(u,v). \end{cases} \tag{2.1}$$

Calculating the total curvature K^* and the mean curvature H^* of the dual surface, we obtain

$$K^* = \frac{1}{K}, \text{ and } H^* = \frac{H}{K}.$$

3 Statement of the Problem

In the work by M.S.Lone, M.K.Karacan, and M.Aydin [16, 14], it is considered the problem of the existence of a surface when the total curvature K^* and the mean curvature H^* of a surface are constant. We consider this problem in the case when the total and mean curvatures of the surface are arbitrarily given functions. In particular, we consider the problem in the class of transfer surfaces. The considered problem in R_3^2 essentially depends on how the transfer surface is defined. Three variants are possible. Let's move on their presentation.

In the general case, the transfer surface given by the vector equation can be written as the sum of two curves in space,

$$\overrightarrow{r}(u,v) = \overrightarrow{\alpha}(u) + \overrightarrow{\beta}(v)$$

here $\overrightarrow{\alpha}(u)$, and $\overrightarrow{\beta}(v)$ are the vector equations of the curves.

Case 1. The surface is uniquely projected onto a plane that is not parallel to the axis Oz in an isotropic space:

$$\overrightarrow{r}(u,v) = u \cdot \overrightarrow{i} + v \cdot \overrightarrow{j} + (f(u) + g(v)) \cdot \overrightarrow{k} \tag{3.1}$$

where $\overrightarrow{\alpha}(u) = (\,u,\, 0,\, f(u)\,); \quad \overrightarrow{\beta}(v) = (0,\; v,\; g(v))$.
In this case, the coefficients of the first and second fundamental form
$E = 1,\; F = 0,\; G = 1,$ and $L = f''_{uu},\; M = 0,\; N = g''_{vv}$.
Taking this into account, one can get a formula for the total and mean curvature of the surface:

$$K = f''_{uu}(u) \cdot g''_{vv}(v), \quad \text{and} \quad H = f''_{uu}(u) + g''_{vv}(v).$$

Respectively, the total and mean curvatures of the dual surface

$$K^* = \frac{1}{f''_{uu}(u) \cdot g''_{vv}(v)} \tag{3.2}$$

$$H^* = \frac{f''_{uu}(u) + g''_{vv}(v)}{f''_{uu}(u)g''_{vv}(v)} \tag{3.3}$$

Case 2. The transfer surface is one-to-one projected onto the plane Oxz. Recall that this plane is the Galilean plane. Then the vector equation of the transfer surface has the form

$$\overrightarrow{r}(u,v) = u\cdot\overrightarrow{i} + (f(u)+g(v))\cdot\overrightarrow{j} + v\cdot\overrightarrow{k} \tag{3.4}$$

where $\alpha(u) = (u, f(u), 0)$ and $\beta(v) = (0, g(v), v)$.

The coefficients of the fundamental forms are the following:
$E = 1 + {f'_u}^2, \quad F = f'_u g'_v, \quad G = {g'_v}^2$ and
$L = -\dfrac{f''_{uu}}{g'_u}, \quad M = 0, \quad N = -\dfrac{g''_{uu}}{g'_u}.$

Respectively, the total and mean curvatures of the transfer surface

$$K = \frac{f''_{uu}(u)\cdot g''_{vv}(v)}{{g'_v}^4(v)}, \quad \text{and} \quad H = \frac{f''_{uu}(u)g'_v(v) + g''_{vv}(v)(1+{f'_u}^2(u))}{{g'_v}^3(v)},$$

and for the dual surface, they are the following:

$$K^* = \frac{{g'_v}^4(v)}{f''_{uu}(u)\cdot g''_{vv}(v)} \tag{3.5}$$

$$H^* = \frac{f''_{uu}(u){g'_v}^3(v) + g'_v(v)g''_{vv}(v)\left(1+{f'_u}^2(u\right)}{2f''_{uu}(u)g''_{vv}(v)} \tag{3.6}$$

Case 3. The surface is uniquely projected onto the plane Oyz, and the parameters are chosen in a special way.

$$\overrightarrow{r}(u,v) = \frac{1}{2}\left(f(u)+g(v), u-v+\pi, u+v\right) \tag{3.7}$$

where $\overrightarrow{\alpha}(u) = \frac{1}{2}\left(f(u),\, u+\frac{\pi}{2},\, u-\frac{\pi}{2}\right)$; $\quad \overrightarrow{\beta}(v) = \frac{1}{2}\left(g(v),\, \frac{\pi}{2}-v,\, \frac{\pi}{2}+v\right)$.

Then the coefficients of the fundamental forms are calculated using the following formulas:
$E = \dfrac{1}{4}(f'_u(u))^2 + \dfrac{1}{4}, \quad F = \dfrac{1}{4}f'_u(u)g'_v(v) - \dfrac{1}{4}, \quad G = \dfrac{1}{4}(g'_v(v))^2 + \dfrac{1}{4}$, and
$L = \dfrac{f''_{uu}(v)}{\frac{1}{2}(f'_u(u)+g'_v(v))}, \quad M = 0, \quad N = \dfrac{g''_{vv}(v)}{\frac{1}{2}(f'_u(u)+g'_v(v))}.$

Respectively, the total and mean curvatures of the transfer surface

$$K = \frac{16f''_{uu}(u)\cdot g''_{vv}(v)}{(f'_u(u)+g'_v(v))^4}, \qquad H = \frac{f''_{uu}(u)\left(1+{g'_v}^2(v)\right) + g''_{vv}(v)\left(1+{f'_u}^2(u)\right)}{(f'_u(u)+g'_v(v))^3}$$

and its dual space: $K^* = \frac{\left(f'_u(u)+g'_v(v)\right)^4}{16f''_{uu}(u)\cdot g''_{vv}(v)}$, and

$$H^* = \frac{(f'_u(u)+g'_v(v))\left(f''_{uu}(u)\left(1+{g'_v}^2(v)\right) + g''_{vv}(v)\left(1+{f'_u}^2(u)\right)\right)}{16f''_{uu}(u)\cdot g''_{vv}(v)}$$

given that $f'(u) + g'(u) \neq 0$, and $f'(u) \neq const$, $g'(u) \neq const$.

In the work by Mohamd Saleem Lone and Murat Kemal Karacan [15, 5], the problem of the existence of a transfer surface is solved when $K^* = const$.

We solve this problem in the case when it is given the total curvature K^* of its dual surface being a specific given function.

As we noted above, we will consider the above three cases separately.

4 The Solution of the Problem

Let

$$K^* = \varphi(u) \cdot \psi(v) \neq 0$$

be a function defined on the domain $D \subset R_2$, and $\varphi(u)$, $\psi(v)$ be nonzero, continuous functions.

In Case 1, the following assertion holds.

Lemma 4.1. *If it is given the function $K^* = \varphi(u)\psi(v)$, then there exists a surface*

$$\overrightarrow{r_\lambda}(u,v) = u \cdot \overrightarrow{i} + v \cdot \overrightarrow{j} + \\ + \left(\int \left[\int \frac{1}{\lambda \cdot \varphi(u)} du + C_1 \right] du + \int \left[\int \frac{\lambda}{\psi(v)} dv + C_2 \right] dv + C \right) \cdot \overrightarrow{k} \quad (4.1)$$

for which it is the total curvature of the dual map. Here C_1 , C_2 , C are constants, $\varphi(u)$, $\psi(v) \in C^2(D)$ and the parameter is $\lambda \neq 0$ and $\lambda \in \mathbb{R}$.

Proof. In Case 1, the total curvature of the dual surface is calculated by formula (3.2).

By the condition of Lemma, the given total curvature of the dual space has the form: $\frac{1}{f''_{uu}(u) \cdot g''_{vv}(v)} = \varphi(u) \cdot \psi(v)$. By transformation, we obtain the following equality

$$\frac{1}{f''_{uu}(u) \cdot \varphi(u)} = g''_{vv}(v) \cdot \psi(v). \quad (4.2)$$

The left-hand side of the last equality depends only on the variable u and cannot change when v changes. Therefore, if one fix u and change v, the left-hand side, and, hence, the right-hand one, will remain constant. Arguing similarly, we will establish that the right-hand side, and hence the left-hand side, cannot change when u changes. This will be true only if both sides of equality (4.2) do not depend at all on either v or u, i.e., when both ratios $\frac{1}{f''_{uu}(u) \cdot \varphi(u)}$ and $g''_{vv}(v) \cdot \psi(v)$ are constant values [9]:

$$\frac{1}{f''_{uu}(u) \cdot \varphi(u)} = \lambda = g''_{vv}(v) \cdot \psi(v),$$

what implies that functions f and g must satisfy the differential equations

$$\frac{1}{f''_{uu}(u) \cdot \varphi(u)} = \lambda, \quad \text{and} \quad g''_{vv}(v) \cdot \psi(v) = \lambda.$$

Integrating these equalities, we obtain

$$f_\lambda(u) = \int \left[\int \frac{1}{\lambda \cdot \varphi(u)} du + C_1 \right] du + C_1', \quad \text{and} \quad g_\lambda(v) = \int \left[\int \frac{\lambda}{\varphi(v)} dv + C_2 \right] dv + C_2'.$$

Substituting these functions $f_\lambda(u)$ and $g_\lambda(v)$ into the equation of the transfer surface, we obtain the formula for the surface given in Lemma 4.1 . This is a one-parameter set of surfaces $\vec{r} = \vec{r}_\lambda(u,v)$. □

Theorem 4.2. *If the surface is of case 1, and $K^* = C_0 = \text{const} \neq 0$, then the surface has the equation:*

$$\overrightarrow{r}(u,v) = u\overrightarrow{i} + v\overrightarrow{j} + \left(\frac{C_o}{2}u^2 + \frac{1}{2C_0}v^2 + C_1 u + C_2 v + C \right) \overrightarrow{k}.$$

If $K^* = \varphi(u)\psi(v)$, then the surface equation is given by formula (4.1). For arbitrary functions with non-separable variables $K^* = K^*(u,v) \neq \varphi(u)\psi(v)$, the problem has no solution.

Proof. We consider each of the above cases separately. When $K^* = C_0 = const$, the problem was solved in the work by M.S. Lone and M.K. Karaken [16].

If K^* is a function with separable variables, then the statement of Theorem 1 follows from Lemma 1.

Finally, when $K^* = K^*(u,v)$ is a function with non- separable variables, the surface with equation (3.1) and having the total curvature K^*- does not exist. Because in the case where the surface is given by equation (3.1), the total curvature $K^*(u,v) = \frac{1}{f_{uu}{}''}\frac{1}{g_{vv}{}''}$, that is, the solution must be a function with separable variables. This contradiction proves the statement of Theorem 4.1. □

Theorem 4.3. *If it is given the mean curvature H^* of the surface F^*, then the surface equation is of case 1.*

1. If $H^* = 0$, then the surface F has the equation
$\overrightarrow{r_\lambda}(u,v) = u\overrightarrow{i} + v\overrightarrow{j} + (\frac{\lambda}{2}u^2 - \frac{\lambda}{2}v^2 + C_1u + C_2v + C)\overrightarrow{k}.$

2. If $H^* = C_0$ $(C_0 \neq 0,\ \ C_0 = const)$, then the surface F has the equation
$\overrightarrow{r_\lambda}(u,v) = u\overrightarrow{i} + v\overrightarrow{j} + (\frac{u^2}{2\lambda} - \frac{v^2}{2(2C_0 - \lambda)} + C_1u + C_2v + C)\overrightarrow{k}.$

3. If
$$H^* = \varphi(u) + \psi(v), \tag{4.3}$$
then the surface F has the equation
$$\begin{aligned}\overrightarrow{r_\lambda}(u,v) = u\overrightarrow{i} + v\overrightarrow{j} + \\ + \left(\int\left[\int\frac{du}{\lambda + 2\varphi(u)}\right]du + \int\left[\int\frac{dv}{2\psi(v) - \lambda}\right]dv + C_1u + C_2v + C\right)\overrightarrow{k}.\end{aligned} \tag{4.4}$$
Here $C_i = const$ and $\lambda \in \mathbb{R}\backslash\{0\}$

Proof. The first and second cases of the statement were proved in [5, 16]. We prove the third case. According to (3.3) and (4.3) we have
$$2\left(\varphi(u) + \psi(v)\right) = \frac{f''(u) + g''(v)}{f''(u)g''(v)}, \qquad 2\varphi(u) + 2\psi(v) = \frac{1}{f''(u)} + \frac{1}{g''(v)},$$
$$\frac{1}{f''(u)} - 2\varphi(u) = -\frac{1}{g''(v)} + 2\psi(v).$$

We write the last equality in the form of two differential equations as follows:
$$\frac{1}{f''(u)} - 2\varphi(u) = \lambda = -\frac{1}{g''(v)} + 2\psi(v),$$
$$\frac{1}{f''(u)} - 2\varphi(u) = \lambda, \quad \text{and} \quad -\frac{1}{g''(v)} + 2\psi(v) = \lambda.$$

These equations have the following solutions:
$$f(u) = \int\left[\int\frac{du}{\lambda + 2\varphi(u)}\right]du + C_1u + C_1{}', \quad g(v) = \int\left[\int\frac{dv}{2\psi(v) - \lambda}\right]dv + C_2v + C_2{}'.$$

Substituting these functions into equality (3.1), we obtain equation (4.4) given in the theorem. Thus, Theorem is proved for this case as well. □

In Case 2, we obtain the following

Theorem 4.4. *If it is given a function $K^* = \varphi(u)\psi(v)$, then there exists a surface*
$$\overrightarrow{r_\lambda}(u,v) = u\cdot\overrightarrow{i} + \left(\int\left[\int\lambda\varphi(u)du + C_1\right]du + \int\left[-\frac{1}{\frac{3}{\lambda}\int\frac{1}{\psi(v)}dv + C_2}\right]dv + C\right)\cdot\overrightarrow{j} + v\cdot\overrightarrow{k}, \tag{4.5}$$
for which it is the total curvature of the dual image. Here $C_1,\ C_2,\ C$ are arbitrary constants and $\varphi(u),\ \ \psi(v) \in C^2(D)$, $\lambda \in \mathbb{R}\backslash\{0\}$.

Proof. The total curvature of the surface, corresponding to Case 2, is calculated by the formula

$$K^* = \frac{{g'_v}^4(v)}{f''_{uu}(u) \cdot g''_{vv}(v)}.$$

Therefore, $\dfrac{{g'_v}^4(v)}{f''_{uu}(u) \cdot g''_{vv}(v)} = \varphi(u)\psi(v)$. Separating the variables, we obtain $\dfrac{f''_{uu}(u)}{\varphi(u)} = \dfrac{{g'_v}^4(v)}{\psi(v) \cdot g''_{vv}(v)}$, what implies

$$\frac{f''_{uu}(u)}{\varphi(u)} = \lambda, \qquad \frac{{g'_v}^4(v)}{\psi(v) \cdot g''_{vv}(v)} = \lambda, \qquad \lambda \in R.$$

By direct integration, we obtain

$$f_\lambda(u) = \int \left[\int \lambda\varphi(u)du + C_1 \right] du + C_1{}', \quad g_\lambda(v) = \int \left[-\frac{1}{3\int \frac{1}{\lambda\psi(v)}dv + C_2} \right] dv + C_2{}'.$$

Substituting functions $f_\lambda(u)$ and $g_\lambda(v)$, we get (3.4), which is a solution to the problem indicated in Theorem 4.4. □

The problem of the existence of a surface by a given mean curvature of the dual surface in the second case, that is, when the surface belongs to the class of surfaces given by equation (3.4), has a solution only if $H^* = 0$ and $H^* = C$ $(C = const)$ [13].

Lemma 4.5. *If $H^* = \psi(v)$, then the equation of the surface F has the form:*

$$\vec{r_\lambda}(u,v) = u \cdot \vec{i} + \left(-\lambda \ln \left| \cos\left(\frac{u}{\lambda} + C_1\right) \right| + \int \mu(v)dv + C_2 \right) \cdot \vec{j} + v \cdot \vec{k}, \tag{4.6}$$

where $\mu(v)$ is a solution of the equation $\mu'(v)\,(2\psi(v) - \lambda\mu(v)) - \mu^3(v) = 0$. Here $\lambda \in \mathbb{R}\backslash\{0\}$.

Proof. The mean curvature H^* is calculated by the formula (3.6). Substituting its value, we obtain

$$\psi(v) = \frac{f''_{uu}(u){g'_v}^3(v) + g'_v(v)g''_{vv}(v)\left(1 + {f'_u}^2(u)\right)}{2f''_{uu}(u)g''_{vv}(v)}.$$

Reduce this equation to the following form

$$2\psi(v) = \frac{{g'_v}^3(v)}{g''_{vv}(v)} + \frac{g'_v(v)\left(1 + {f'_u}^2(u)\right)}{f''_{uu}(u)}.$$

After not difficult transformations,

$$\frac{2\psi(v)}{g'_v(v)} - \frac{{g'_v}^3(v)}{g''_{vv}(v)} = \frac{1 + {f'_u}^2(u)}{f''_{uu}(u)}, \quad \text{and} \quad \frac{2\psi(v)}{g'_v(v)} - \frac{{g'_v}^3(v)}{g''_{vv}(v)} = \lambda = \frac{1 + {f'_u}^2(u)}{f''_{uu}(u)}$$

it is reduced to an equation with separated variables.

Rewriting the equation in the form of two differential equations, we obtain

$a)\, \dfrac{1 + {f'_u}^2(u)}{f''_{uu}(u)} = \lambda, \quad f(u) = C_2 - \lambda \ln \left| \cos\left(\cos\left(\dfrac{u}{\lambda} + C_1 \right) \right) \right|;$

$b)\, \dfrac{2\psi(v)}{g'_v(v)} - \dfrac{{g'_v}^2(v)}{g''_{vv}(v)} = \lambda.$

Introducing the replacement $g'(v) = \mu(v)$, we have $\mu'(v)\,(2\psi(v) - \lambda\mu(v)) - \mu^3(v) = 0$. Since the obtained equation is an equation with separable variables, one can find the function $\mu(v)$. Hence, $g(v) = \int \mu(v)dv$ also can be found. Substituting $f(u)$ and $g(v)$ into equation (3.6), we obtain function (4.6), which is a solution to the problem. Lemma 4.5 is proved. □

Theorem 4.6. *If it is given a function $K^* = \varphi(u)\psi(v)$ then there exists a surface*

$$\overrightarrow{r_\lambda}(u,v) = \frac{1}{2}\left(\int\left[\int \frac{16\lambda^4}{\varphi(u)}du + C_1\right]du + \int\left[\int \frac{16\lambda^4}{\psi(v)}dv + C_2\right]dv + C\right)\cdot\overrightarrow{i} +$$

$$+\frac{1}{2}(u - v + \pi)\cdot\overrightarrow{j} + \frac{1}{2}(u+v)\cdot\overrightarrow{k}, \tag{4.7}$$

for which it is the total curvature of the dual image. Here $C_1,\ C_2,\ C$ are arbitrary constants and $\varphi(u),\ \ \psi(v) \in C^2(D)$, $\lambda \in \mathbb{R}\backslash\{0\}$.

Proof. From the formula for the total curvature of the surface related to Case 3, we have

$$\frac{\left(f'_u(u) + g'_v(v)\right)^4}{16 f''_{uu}(u)\cdot g''_{vv}(v)} = \varphi(u)\psi(v) \tag{4.8}$$

what can be reduced to the following form

$$f'_u(u) + g'_v(v) = 2\sqrt[4]{\varphi(u)}\sqrt[4]{\psi(v)}\cdot\left(f''_{uu}(u)\cdot g''_{vv}(v)\right)^{\frac{1}{4}}.$$

Differentiating the equality with respect to u, we obtain

$$f''_u(u) = 2\cdot\frac{1}{4}(\varphi(u))^{-\frac{3}{4}}\varphi'_u(u)\sqrt[4]{\psi(v)}\cdot\left(f''_{uu}(u)\cdot g''_{vv}(v)\right)^{\frac{1}{4}} +$$

$$+2\cdot\frac{1}{4}\sqrt[4]{\varphi(u)\psi(v)}\cdot\left(g''_{vv}(v)\right)^{\frac{1}{4}}\left(f''_{uu}(u)\right)^{\frac{-3}{4}} f'''_{uuu}(u)\cdot$$

Transforming this equality, we have

$$\frac{\left(f''_u(u)\right)^{\frac{7}{4}}}{\left[\frac{\varphi'_u(u)}{\sqrt{\varphi^3(u)}} f''_{uu}(u) + \sqrt[4]{\varphi(u)}\cdot f'''_{uuu}(u)\right]} = \frac{1}{2}\sqrt[4]{\psi(v) g''_{vv}(v)}.$$

Since the equality holds for arbitrary values of the independent variables, we get

$$\frac{\left(f''_u(u)\right)^{\frac{7}{4}}}{\left[\frac{\varphi'_u(u)}{\sqrt{\varphi^3(u)}} f''_{uu}(u) + \sqrt[4]{\varphi(u)}\cdot f'''_{uuu}(u)\right]} = \lambda = \frac{1}{2}\sqrt[4]{\psi(v) g''_{vv}(v)}.$$

Therefore, we consider two differential equations:

$$\frac{\left(f''_u(u)\right)^{\frac{7}{4}}}{\left[\frac{\varphi'_u(u)}{\sqrt{\varphi^3(u)}} f''_{uu}(u) + \sqrt[4]{\varphi(u)}\cdot f'''_{uuu}(u)\right]} = \lambda, \qquad \frac{1}{2}\sqrt[4]{\psi(v) g''_{vv}(v)} = \lambda.$$

Solving the second of these equations: $g''_{vv}(v) = \dfrac{16\lambda^4}{\psi(v)}$ we obtain

$$g_\lambda(v) = \int\left[\int \frac{16\lambda^4}{\psi(v)}dv + C_2\right]dv.$$

But the functions $f_\lambda(u)$ and $g_\lambda(v)$ can be interchanged in equality (4.8), that is, they are equivalent. Reversing the functions, we get

$$f_\lambda(u) = \int\left[\int \frac{16\lambda^4}{\varphi(u)}du + C_1\right]du + C.$$

Substituting the values of $f_\lambda(u)$, and $g_\lambda(v)$, we obtain the function stated in Theorem 4.6.

□

In the third case, when the surface is considered in the class of functions defined by (3.7), the problem of the existence of a surface by a given mean curvature of the dual surface has a solution when $H^* = (\eta(u) + \mu(v))\,(\xi(u) + \rho(v))$.

Theorem 4.7. *In the case of $H^* = (\eta(u) + \mu(v))\,(\xi(u) + \rho(v))$, if functions $\eta(u), \mu(v), \xi(u), \rho(v)$ satisfy the equations*

$$\frac{1+\eta(u)}{\eta'(u)} = 16\xi(u), \ \ \textit{and} \ \ \frac{1+\mu(v)}{\mu'(v)} = 16\rho(v),$$

there exists a surface defined by an equation of the type (3.7) and its equations are as follows:

$$\overrightarrow{r}(u,v) = \frac{1}{2}\left(\int \eta(u)du + \int \mu(v)dv + C\right)\cdot \overrightarrow{i} + \frac{1}{2}\left(u - v + \pi\right)\cdot \overrightarrow{j} + \frac{1}{2}(u+v)\cdot \overrightarrow{k} \quad (4.9)$$

Proof. The mean curvature H^* for surfaces given by Equation (3.7) is calculated using the formula

$$H^* = \frac{(f'_u(u) + g'_v(v))\left(f''_{uu}(u)\left(1 + {g'_v}^2(v)\right) + g''_{vv}(v)\left(1 + {f'_u}^2(u)\right)\right)}{16 f''_{uu}(u)\cdot g''_{vv}(v)}.$$

Reduce it to the following form:

$$H^* = (f'_u(u) + g'_v(v))\left(\frac{1 + {f'_u}^2(u)}{16 f''_{uu}(u)} + \frac{1 + {g'_v}^2(v)}{16 g''_{vv}(v)}\right).$$

Substituting the value of H^*, the equation is reduced to the form

$$(f'_u(u) + g'_v(v))\left(\frac{1 + {f'_u}^2(u)}{16 f''_{uu}(u)} + \frac{1 + {g'_v}^2(v)}{16 g''_{vv}(v)}\right) = (\eta(u) + \mu(v))\,(\xi(u) + \rho(v))\,.$$

Separating functions with the same arguments, we obtain the system of differential equations

$$\begin{cases} f'_u(u) = \eta(u) \\ \dfrac{1 + {f'_u}^2(u)}{16 f''_{uu}(u)} = \xi(u) \end{cases}, \qquad \begin{cases} g'_v(v) = \mu(v) \\ \dfrac{1 + {g'_v}^2(v)}{16 g''_{vv}(v)} = \rho(v) \end{cases}.$$

It follows from the compatibility condition for differential equations that it suffices to solve the equations $f'_u(u) = \eta(u)$, $g'_v(v) = \mu(v)$, what will give solutions. Substituting the values of the functions, we obtain the equation of a surface that has the equations (4.9) indicated in the Theorem 4.7. □

Example 4.8. Let us give a simple example corresponding to Case 1. If $K^* = \dfrac{e^u}{\cos v}$, then

$$f_\lambda(u) = \frac{e^{-u}}{\lambda} + C_1 u + C'_1, \quad g_\lambda(v) = -\lambda\cos v + C_2 v + C_2{}'.$$

Hence, the equation of the corresponding transfer surface (Fig. 2) is

$$\overrightarrow{r_\lambda}(u,v) = u\cdot\overrightarrow{i} + v\cdot\overrightarrow{j} + \left(\frac{e^{-u}}{\lambda} - \lambda\cos v + C_1 u + C_2 v + C\right)\cdot\overrightarrow{k}.$$

Here $\lambda \in \mathbb{R}\backslash\{0\}$.

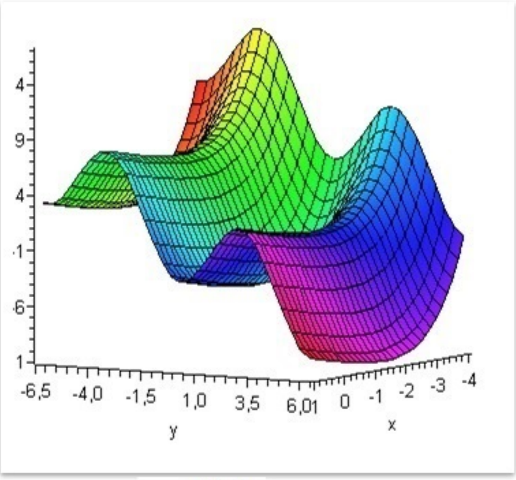

Fig.2

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