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PRÜFER-CLOSED EXTENSIONS AND FCP λ-EXTENSIONS OF COMMUTATIVE RINGS

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Abstract If $A \subseteq B$ is a (unital) extension of (commutative) rings, we say that A is Prüfer-closed in B if A is the Prüfer hull of A in B. Fix a ring extension $R \subseteq S$, with $\overline{R}_S$ denoting the integral closure of R in S. If $R \subseteq S$ is a P-extension, then the set of rings contained between R and S is pinched at $\overline{R}_S$ if and only if J is Prüfer-closed in S for each ring J such that $R \subseteq J \subset \overline{R}_S$. Assume henceforth that $R \subseteq S$ satisfies FCP. Then R is Prüfer-closed in S if and only if either S is integral over R or $R \subset \overline{R}_S \subset S$ with $(R : \overline{R}_S) \subseteq N$ for each $N \in \text{MSupp}(S/\overline{R}_S)$. Applications include ring-theoretic generalizations of several domain-theoretic results of Ben Nasr and Zeidi, as well as a characterization of λ-extensions (also known as chained ring extensions). If S is integral over R, this characterization uses some results of M. S. Gilbert for the case where R is a field. If R is integrally closed in S, then $R \subseteq S$ is a λ-extension if and only if $\text{Supp}_R(S/R)$ $(:= \{P \in \text{Spec}(R) \mid R_P \subset S_P\})$ is linearly ordered by inclusion.

1 Introduction

All rings and algebras considered below are commutative and unital, all inclusions of rings are considered to be (unital) ring extensions, and all algebra/ring homomorphisms are unital. If R is a ring, then $\text{Spec}(R)$ (resp., $\text{Max}(R)$) denotes the set of prime (resp., maximal) ideals of R; $\sqrt{R}$ denotes the nilradical of R (in the sense of [31, page 16]); and $\text{Rad}_R(I)$ denotes the radical of an ideal I of R. Given a ring extension $R \subseteq S$, we will use the following notation: $\overline{R}_S$ denotes the integral closure of R in S; $[R, S]$ denotes the set of intermediate rings between R and S; $(R : S) := \{x \in R \mid xS \subseteq R\}$, the conductor of R in S; $\text{Supp}(S/R) := \text{Supp}_R(S/R) := \{P \in \text{Spec}(R) \mid R_P \neq S_P := S_{R\setminus P}\}$; $\text{MSupp}_R(S/R) := \text{Max}(R) \cap \text{Supp}_R(S/R)$; and $[R, S[= [R, S] \setminus \{R\}$.

This paragraph will give a very brief summary of this paper. The main purpose of Section 2 is to characterize the P-extensions $R \subseteq S$ such that $[R, S] = [R, \overline{R}_S] \cup [\overline{R}_S, S]$. That characterization involves the concept of a Prüfer-closed ring extension, which we introduce for arbitrary ring extensions and then give an MSupp-theoretic characterization of in the class of (necessarily P-extension) ring extensions satisfying the FCP property. This work in Section 2 leads to a generalization of a result from [3] and generalizations (with much shorter proofs) of most of the (domain-theoretic) results in [8] to the setting of arbitrary ring extensions. Section 3 characterizes the λ-extensions from [26] (which were termed "chained extensions" in [36]) which satisfy FCP by combining results from [26] (for the case where the base ring is a field) to treat the integral case (with an arbitrary base ring), developing a Supp-theoretic characterization for the integrally closed case, and applying the above-mentioned characterization from Section 2 of when $[R, S]$ is pinched at $\overline{R}_S$. The rest of the Introduction provides more details and some background.

Following [26], we say that a ring extension $R \subseteq S$ is a λ-*extension* if the set $[R, S]$ is linearly ordered by inclusion. Perhaps the two most familiar examples of a λ-extension $R \subseteq S$ are given by a valuation domain R with quotient field S (cf. [31, Theorems 64 and 65]) and by a minimal ring extension (in the sense of [25]). For an arbitrary ring extension $R \subseteq S$, it is clear that $R \subseteq S$ is a λ-extension if and only if each of the following three conditions hold:

$R \subseteq \overline{R}_S$ is a λ-extension; $\overline{R}_S \subseteq S$ is a λ-extension; and $[R, S]$ is pinched at $\overline{R}_S$ (in the sense that $[R, S] = [R, \overline{R}_S] \cup [\overline{R}_S, S]$). Going significantly beyond this triviality, G. Picavet and M. Picavet-L'Hermitte showed in [36, Proposition 5.3] that the second condition is equivalent to $\overline{R}_S \subseteq S$ being a Prüfer-Manis extension (in the sense of [32]). Our main purpose here is to deepen our understanding of λ-extensions $R \subseteq S$ by characterizing each of the above three conditions in case $R \subseteq S$ satisfies FCP. (Recall that a ring extension is said to satisfy FCP if, when $[R, S]$ is viewed as a poset under inclusion, each chain in $[R, S]$ is finite. To motivate the partitioning of this project in conjunction with the condition that $[R, S]$ is pinched at $\overline{R}_S$, recall the result [19, Theorem 3.13] that a ring extension $R \subseteq S$ satisfies FCP if and only if both $R \subseteq \overline{R}_S$ and $\overline{R}_S \subseteq S$ satisfy FCP.) The next three paragraphs summarize our work here. The first and second of those paragraphs concern the third condition that was mentioned above, how our work on that condition connects to the titular "Prüfer-closed" concept that is being introduced here, and how the two main theorems in Section 2 serve to generalize many of the results of Ben Nasr and Zeidi in [8] from the setting of integral domains to that of arbitrary (commutative) rings. The following paragraph (after those two paragraphs) summarizes our work in Section 3 on the first and second conditions, as well as where the reader can see how the individual pieces are put together to characterize the λ-extensions satisfying FCP.

Section 2 is devoted to studying the ring extensions $R \subseteq S$ such that $[R, S]$ is pinched at $\overline{R}_S$. (Earlier studies of these extensions appeared in [8] and [13]; we say more about how our present work relates to those papers in the next paragraph.) Recall from [28, Theorem 1] and [12] that an element $s \in S$ is called *primitive* over R if s is a root of some polynomial in $R[X]$ whose coefficients generate the unit ideal of R (equivalently, if s is a root of some polynomial in $R[X]$ at least one of whose coefficients is 1); and that if each element of S is primitive over R, then $R \subseteq S$ is said to be a P-*extension*. If R is an integral domain with quotient field K, then $R \subseteq K$ is a P-extension if and only if the integral closure of R (in K) is a Prüfer domain [28, Theorem 5]. In [32], Knebusch and Zhang introduced the following concept, which can be considered as a relativization of the concept of a Prüfer ring. A ring extension $R \subseteq S$ is called a *Prüfer extension* if $R \subseteq T$ is a flat epimorphism (in the category of commutative rings) for each $T \in [R, S]$. Any Prüfer extension is integrally closed. It is noteworthy that a ring extension $R \subseteq S$ is a P-extension if and only if $\overline{R}_S \subseteq S$ is a Prüfer extension: cf. [7, Corollary 1] or [37, Theorem 2.3]. (This is a far-reaching generalization of the above-mentioned result of Gilmer and Hoffmann [28, Theorem 5].) For an arbitrary ring extension $R \subseteq S$, it was proved in [32, Theorem 5.2, page 47] that a ring extension $R \subseteq S$ is a Prüfer extension if and only if (R, S) is a normal pair. (Recall from [10] that if $R \subseteq S$ are rings, then (R, S) is called a *normal pair* if the ring extension $T \subseteq S$ is integrally closed for each $T \in [R, S]$. Normal pairs have been extensively studied: cf. [4, 7, 10, 22, 23, 30]. The above "noteworthy" result has the important consequence that a ring extension $R \subseteq S$ is integrally closed and a P-extension if and only if (R, S) is a normal pair.) Recall also from [32] that any ring extension $R \subseteq S$ has a maximum Prüfer subextension $R \subseteq \widetilde{R}^S$, called the *Prüfer hull* of R in S. (Note that $\widetilde{R}^S$ was denoted by $P(R, S)$ in [32].) We say that a ring extension $R \subseteq S$ is *Prüfer-closed* (or that R is *Prüfer-closed in* S) if $R = \widetilde{R}^S$. It is clear that any Prüfer extension is integrally closed; and it is easy to see that any integral extension is Prüfer-closed (Proposition 2.1 (c)). However, even though there exist distinct rings $R \subset S$ such that R is both integrally closed in S and Prüfer-closed in S (Example 2.2), no such $R \subset S$ can satisfy FCP (Proposition 2.1 (d)). Given that FCP $\Rightarrow$ P-extension (cf. Proposition 2.4), one may reasonably expect that assuming FCP would lead to a deeper insight into Prüfer-closed extensions. In fact, Theorem 2.7 establishes a characterization of the Prüfer-closed extensions $R \subseteq S$ that makes use of the set $\mathrm{MSupp}(S/\overline{R}_S)$. Note that Theorem 2.7 generalizes a result of Ayache [3, Proposition 10]. The other main result in Section 2 is Theorem 2.5: if $R \subseteq S$ is a P-extension (which may or may not satisfy FCP), then $J \subset S$ is a Prüfer-closed extension for each ring J such that $R \subseteq J \subset \overline{R}_S$ if and only if $[R, S]$ is pinched at $\overline{R}_S$.

By combining Theorems 2.5 and 2.7, we obtain five corollaries (2.9-2.13) that generalize various results in [8] from integral domains to the ring-theoretic setting. In particular, note that Corollary 2.9 generalizes the main result in [8], [8, Theorem 2.7]. The fact that some results in [8] could be proved more generally by using other methods was already noted in [13, Theorem 2.1] and its applications. We focus here on generalizations of some of the results in [8] that were not addressed in [13]. For further specifics in this regard, see Remark 2.14.

The first seven results in Section 3 concern the integral λ-extensions (such as $R \subseteq \overline{R}_S$ arising from an arbitrary λ-extension $R \subseteq S$). Several of these results produce characterizations, especially in case the given ring extension satisfies FCP. Perhaps the most trenchant of those characterizations is the one in Theorem 3.5 (d), for the following three reasons: it takes advantage of the fact that whenever $R \subset S$ is an integral λ-extension satisfying FCP, the set $\mathrm{Supp}(S/R)$ consists of a single element, which is necessarily some maximal ideal M of R; in it, there is a natural role for the condition that $[R, S]$ is pinched at $R + \mathrm{Rad}_S(MS)$; and (as is also true for several of the other characterizations in the integral case) it builds upon the work of Gilbert [26, Chapter 3] on the λ-extensions whose base ring is a field. Next, Section 3 turns to the integrally closed λ-extensions (such as $\overline{R}_S \subseteq S$ arising from an arbitrary λ-extension $R \subseteq S$). Recall that [36, Proposition 5.3] developed the characterization that an integrally closed extension $R \subseteq S$ is a λ-extension if and only if $R \subseteq S$ is a Prüfer-Manis extension. At the cost of assuming FCP, Theorem 3.9 gives the following (to our minds, much more tractable) characterization: if $R \subset S$ is an integrally closed ring extension that satisfies FCP, then $R \subset S$ is a λ-extension if and only if $\mathrm{Supp}_R(S/R)$ is linearly ordered by inclusion. Finally, note that the question of whether $[R, S]$ is pinched at $\overline{R}_S$ is only nontrivial if $R \subset \overline{R}_S \subset S$. For that setting, provided that one also assumes that $R \subseteq S$ satisfies FCP, Corollary 3.11 characterizes when $R \subseteq S$ is a λ-extension, by using Theorem 3.5 (d) to handle $R \subseteq \overline{R}_S$, Theorem 3.9 to handle $\overline{R}_S \subseteq S$, and Theorems 2.5 and 2.7 to handle pinchedness of $[R, S]$ at $\overline{R}_S$.

As usual, $\subset$ denotes proper inclusion. While reading Corollaries 2.9-2.13, the reader may find it useful to have a copy of [8] at hand. Any otherwise unexplained material or terminology is standard, as in [27] and [31].

2 When $[R, S]$ is pinched at $\overline{R}_S$

We begin with a result that collects some useful information.

Proposition 2.1. *Let $R \subseteq S$ be rings. Then:*

(*a*) $\widetilde{R}^S \cap \overline{R}_S = R$.

(*b*) *Assume, in addition, that $R \subseteq S$ is a* P*-extension. Then $\widetilde{R}^S$ is the maximum element in $\{T \in [R, S] \mid T \cap \overline{R}_S = R\}$, and so, for each $T \in [R, S]$ such that $T \cap \overline{R}_S = R$, one has $T \subseteq \widetilde{R}^S$.*

(*c*) *Assume, in addition, that S is integral over R. Then R is Prüfer-closed in S.*

(*d*) *Assume, in addition, that R is integrally closed in S, $R \subseteq S$ satisfies* FCP*, and $R \neq S$. Then R is not Prüfer-closed in S.*

(*e*) *Assume, in addition, that $R = D + M$ and $S = E + M$, where $D \subseteq E$ are integral domains with quotient field K and M is the maximal ideal of a valuation domain V of the form $V = K + M$. Then $\widetilde{R}^S = \widetilde{D}^E + M$. In particular, R is Prüfer-closed in S if and only if D is Prüfer-closed in E.*

Proof. (a) It is clear that $R \subseteq \widetilde{R}^S \cap \overline{R}_S$. For the reverse inclusion, it suffices to prove that if $t \in \widetilde{R}^S \cap \overline{R}_S$, then $t \in R$. Consider $T := R[t] \in [R, \widetilde{R}^S]$. As $(R, \widetilde{R}^S)$ is a normal pair, R is integrally closed in $\widetilde{R}^S$ and, *a fortiori*, integrally closed in T. However, since $T \in [R, \overline{R}_S]$, T is also integral over R. Hence $T = R$, and so $t \in R$, as desired.

(b) For each $T \in [R, S]$, one has that $T \cap \overline{R}_S$ is integrally closed in T. Suppose now that $T \in [R, S]$ satisfies $T \cap \overline{R}_S = R$. Then R is integrally closed in T and, since $R \subseteq T$ inherits the "P-extension" property from $R \subseteq S$, it now follows that (R, T) is a normal pair. Hence, it follows from the definition of $\widetilde{R}^S$ that $T \subseteq \widetilde{R}^S$. In view of (a), this completes the proof of (b).

(c) Our task is to prove that $\widetilde{R}^S = R$. This, in turn, can be obtained as a consequence of (a), since the assumption that S is integral over R can be restated as $\overline{R}_S = S$. As any integral extension is a P-extension, an alternate proof of (c) is available by using the first assertion in (b).

(d) By [19, Theorem 6.3 (b)], the first two additional hypotheses in (c) ensure that (R, S) is a normal pair. Hence $\widetilde{R}^S = S$, by the definition of $\widetilde{R}^S$. As $R \neq S$, we get $\widetilde{R}^S \neq R$, as desired.

(e) By a well known fact about the classical $(D+M)$-construction [6, Theorem 3.1], $[R, S] = \{H + M \mid H \in [D, E]\}$. For any $H \in [D, E]$, it follows from a result of Rhodes (cf. [32, Proposition 5.8, page 52]) that $(R, H+M)$ is a normal pair if and only if $(R/M, (H+M)/M)$ is a normal pair; that is, if and only if (D, H) is a normal pair. The maximum $H+M$ (resp., maximal H) satisfying these conditions is $\widetilde{R}^S$ (resp., $\widetilde{D}^E$). The first assertion now follows immediately.

The "In particular" assertion is then a consequence, since R is Prüfer-closed in S (resp., D is Prüfer-closed in E) if and only if $\widetilde{R}^S = R$ (resp., $\widetilde{D}^E = D$). The proof is complete. □

The result in Proposition 2.1 (e) about the $(D+M)$-construction can be established for some more general pullbacks. However, as Example 2.2 will illustrate, it will suffice to use Proposition 2.1 (e) in order to show the necessity of assuming (something with the flavor of) the FCP condition in many of the results in this paper.

Example 2.2. The assertion in Proposition 2.1 (d) would become false if one were to delete the hypothesis that $R \subseteq S$ satisfies FCP. In other words, there exist (distinct) rings $R \subset S$ such that R is integrally closed in S and R is Prüfer-closed in S. One way to produce such data is the following. Take $R := k + M$ and $S := k[X] + M$, where k is a field, X is an indeterminate over k, and (V, M) is a valuation domain of the form $V = k(X) + M$. Moreover, if $0 \leq d < \infty$, one can arrange that V has (Krull) dimension d, so that the integral domain R also has dimension d.

Proof. Let $0 \leq d < \infty$. It is well known that we can find a d-dimensional valuation domain of the form $V = k(X) + M$ (cf. [27, Corollary 18.5 and page 199]) and (since k is zero-dimensional) that this forces $R := k + M$ to also be d-dimensional (cf. [27, Exercise 12 (4), page 203]). If A is any integral domain and X is an indeterminate over A, an easy degree argument shows that A is integrally closed in $A[X]$. In particular, k is integrally closed in $k[X]$. Hence (cf. [27, Example 11 (2), page 202]), R is integrally closed in $S := k[X] + M$. Of course, $R \subset S$. It remains only to prove that R is Prüfer-closed in S. Hence, by Proposition 2.1 (e), it suffices to prove that k is Prüfer-closed in $k[X]$. Our task then is to prove that if (k, T) is a normal pair for some ring $T \in [k, k[X]]$, then $T = k$. For any such T, the "normal pair" condition ensures that $k \subseteq T$ is a P-extension. As k is a field, it follows that the extension $k \subseteq T$ is algebraic and, hence, integral. Therefore, since k is integrally closed in $k[X]$, we get $T = k$. The proof is complete. □

The next result begins our examination of connections between the "Prüfer-closed extension" concept and the property that $[R, S]$ is pinched at $\overline{R}_S$.

Proposition 2.3. *Suppose that $[R, S] = [R, \overline{R}_S] \cup [\overline{R}_S, S]$ for some given rings $R \subseteq S$. Then $J \subset S$ is a Prüfer-closed extension for each ring $J \in [R, \overline{R}_S[$.*

Proof. Let $J \in [R, \overline{R}_S[$. Our task is to show that $J = \widetilde{J}^S$. If $\overline{R}_S \subseteq \widetilde{J}^S$, then $(J, \overline{R}_S)$ would inherit the "normal pair" property from $(J, \widetilde{J}^S)$, so that the extension $J \subseteq \overline{R}_S$ would be both integrally closed and integral, whence $J = \overline{R}_S$, contrary to hypothesis. Therefore, since $\widetilde{J}^S$ and $\overline{R}_S$ are assumed to be comparable under inclusion, we have $(J \subseteq)\ \widetilde{J}^S \subset \overline{R}_S$. As the extension $J \subseteq \widetilde{J}^S$ is then both integrally closed and integral, $J = \widetilde{J}^S$. □

Theorem 2.5 will present a partial converse of Proposition 2.3. First, we isolate a useful fact that deserves to be more widely known.

Proposition 2.4. *Let $R \subseteq S$ be rings such that there exists a finite maximal chain $R = R_0 \subset \cdots \subset R_n = S$ in $[R, S]$. (For instance, let $R \subseteq S$ be a ring extension that satisfies* FCP.) *Then $R \subseteq S$ is a* P*-extension.*

Proof. It was proven in [7, Theorem 2] (and can also be seen by combining [37, Corollary 3.3 and Theorem 2.3] with [12, Theorem]) that if both $A \subseteq B$ and $B \subseteq C$ are P-extensions, then $A \subseteq C$ is also a P-extension. Therefore, it will suffice to show that the minimal extension $R_{i-1} \subset R_i$ satisfies FCP for each $i = 1, \ldots, n$. Fix i. As all integral extensions are P-extensions, we may assume, without loss of generality, that $R_{i-1} \subset R_i$ is an integrally closed extension. Then (R_{i-1}, R_i) is a normal pair, and so $R_{i-1} \subset R_i$ is a P-extension. □

Theorem 2.5. *Let $R \subseteq S$ be a* P*-extension. Then the following conditions are equivalent:*

(*1*) $[R, S] = [R, \overline{R}_S] \cup [\overline{R}_S, S]$;

(*2*) *$J \subset S$ is a Prüfer-closed extension for each ring $J \in [R, \overline{R}_S[$.*

Proof. (1) ⇒ (2): Apply Proposition 2.3.

(2) ⇒ (1): Assume (2). We must show that if $T \in [R, S]$, then T is comparable with $\overline{R}_S$ under inclusion. Consider $J := T \cap \overline{R}_S \in [R, \overline{R}_S]$. If $J = \overline{R}_S$, then $\overline{R}_S \subseteq T$. Thus, without loss of generality, $J \neq \overline{R}_S$. Then, by (2), $J \subset S$ is a Prüfer-closed extension; that is, $J = \widetilde{J}^S$. Also, since $\overline{J}_S = \overline{R}_S$, we have $T \cap \overline{J}_S = J$, and so $J \subseteq T$ is an integrally closed extension. Moreover, $J \subseteq T$ inherits the P-extension property from $R \subset S$. Hence, (J, T) is a normal pair. Therefore, by the definition of $\widetilde{J}^S$, we have $T \subseteq \widetilde{J}^S$ $(= J \subseteq \overline{R}_S)$, and so $T \subseteq \overline{R}_S$. This completes the proof. □

Remark 2.6. For each integral domain R (with quotient field S) belonging to the class of rings studied in [18, Theorem 3.4], we have that $R \subseteq S$ is a P-extension (by a theorem of Davis, cf. [27, Theorem 26.2 (a) ⇔ (d)], since the integral closure of R is a Prüfer domain), but neither condition (1) nor condition (2) from Theorem 2.5 is satisfied by $R \subseteq S$. Indeed, for any such $R \subseteq S$, there is only one ring W in $[R, S]$ which is not comparable to $\overline{R}_S$ under inclusion. Moreover, R is the only element of $[R, \overline{R}_S[$. So, as Theorem 2.5 requires, $R \subset S$ is not a Prüfer-closed extension. One could verify this last fact directly because (R, W) is a normal pair for any such data (the point being that $R \subset W$ is a minimal extension such that $W \cap \overline{R}_S = R$). This concludes the remark.

We pause to recall the definition of, and some background concerning, a concept that was introduced in [25] by Ferrand and Olivier. A ring extension $R \subset S$ is said to be *minimal* if ($R \subset S$ and) $[R, S] = \{R, S\}$. Any minimal (ring) extension must be either integrally closed or integral. If $R \subset S$ is a minimal extension, it follows from [25, Théorème 2.2 (i) and Lemme 1.3] that there exists a (necessarily unique) maximal ideal M of R, called the *crucial maximal ideal* of $R \subset S$, such that the canonical injective ring homomorphism $R_M \to S_M$ $(:= S_{R\setminus M})$ can be viewed as a minimal extension, while the canonical ring homomorphism $R_P \to S_P$ $(:= S_{R\setminus P})$ is an isomorphism for all prime ideals P of R except M (cf. also [33, p. 37], [11]). If a minimal ring extension $R \subset S$ is integral, then M is precisely the conductor $(R : S) := \{x \in R \mid xS \subseteq R\}$ (by [25, Théorème 2.2 (ii)]). A minimal ring extension $R \subset S$ is integrally closed if and only if $R \hookrightarrow S$ is a flat epimorphism (in the category of commutative rings); also, if and only if $R \subset S$ is a Prüfer extension, in the sense of [32], that is, if and only if (R, S) is a normal pair.

It will be useful to note that if $R \subset S$ has FCP, then any maximal (necessarily finite) chain of R-subalgebras of S, $R = R_0 \subset R_1 \subset ... \subset R_{n-1} \subset R_n = S$, results from juxtaposing n minimal extensions $R_i \subset R_{i+1}$ (with $0 \le i \le n-1$).

The next theorem characterizes the Prüfer-closed extensions that satisfy FCP. This result generalizes [3, Proposition 10].

Theorem 2.7. *Let $R \subseteq S$ be a ring extension that satisfies* FCP. *Then the following conditions are equivalent:*

(1) $R \subseteq S$ is a Prüfer-closed extension;

(2) Either (i) $R \subseteq S$ is an integral extension
or (ii) $R \subset \overline{R}_S \subset S$ and $(R : \overline{R}_S) \subseteq N$ for each $N \in$ MSupp$(S/\overline{R}_S)$.

Proof. Both (1) and (2) hold trivially if $R = S$, and so we can assume henceforth that $R \subset S$.

(1) ⇒ (2): Assume (1). By parts (c) and (d) of Proposition 2.1, we can assume, without loss of generality, that S is not integral over R (that is, $\overline{R}_S \subset S$) and R is not integrally closed in S (that is, $R \subset \overline{R}_S$). It remains to show that $(R : \overline{R}_S) \subseteq N$ for each $N \in$ MSupp$(S/\overline{R}_S)$.

Let $N \in$ MSupp$(S/\overline{R}_S)$. As $\overline{R}_S \subset S$ satisfies FCP, it follows from [38, Lemma 1.8] that there exists a ring $T_0 \in [\overline{R}_S, S]$ such that $\overline{R}_S \subset T_0$ is a minimal extension with crucial maximal ideal N. Moreover, this (minimal) extension is integrally closed, since $(\overline{R}_S, S)$ is a normal pair (by virtue of [19, Theorem 6.3 (b)]). On the other hand, since $R \subset \overline{R}_S$ satisfies FCP, there exists a (finite) maximal chain $R = R_0 \subset R_1 \subset ... \subset R_n = \overline{R}_S$ of rings, for some positive integer n. For each $i \in \{1, 2, ..., n\}$, $R_{i-1} \subset R_i$ is an integral minimal extension and so, by [25, Théorème 2.2], its crucial maximal ideal is $Q_i := (R_{i-1} : R_i)$. It will be useful to note that $(R : \overline{R}_S) \subseteq Q_i$ for each i.

By integrality, $N \cap R_{n-1}$ is a maximal ideal of R_{n-1}. So, if $N \cap R_{n-1} \subseteq Q_n$, then $N \cap R_{n-1} = Q_n$, whence $Q_n \subseteq N$ and $(R : \overline{R}_S) \subseteq N$, as desired. In the remaining case, $N \cap R_{n-1} \nsubseteq Q_n$.

Then the conditions of the crosswise exchange lemma [19, Lemma 2.7] apply, thus producing a ring $T_1 \in [R_{n-1}, T_0]$ such that $R_{n-1} \subset T_1$ is an integrally closed minimal extension with crucial maximal ideal $N \cap R_{n-1}$. In the subcase $(N \cap R_{n-1}) \cap R_{n-2}$ $(= N \cap R_{n-2}) \subseteq Q_{n-1}$, we get $N \cap R_{n-2} = Q_{n-1}$ (since integrality ensures that $N \cap R_{n-2}$ is a maximal ideal of R_{n-2}), whence $Q_{n-1} \subseteq N$ and $(R : \overline{R}_S) \subseteq N$, as desired. Thus, we have reduced to the subcase where $N \cap R_{n-2} \nsubseteq Q_{n-1}$. By repeating the above reasoning, another application of [19, Lemma 2.7] and [19, Theorem 6.3 (b)] produces a ring $T_2 \in [R_{n-2}, T_1]$ such that $R_{n-2} \subset T_2$ is an integrally closed minimal extension with crucial maximal ideal $N \cap R_{n-2}$.

If repeating the above "downward" process of argumentation has not led to a subcase with the desired conclusion before the focus falls naturally on the extension $R_0 \subset R_1$, iteration does eventually reduce to the subcase where $N \cap R_1 \nsubseteq Q_1$. Then [19, Lemma 2.7] and [19, Theorem 6.3 (b)] combine one final time, producing a ring $T_n \in [R, T_{n-1}]$ such that $R \subset T_n$ is an integrally closed minimal extension. In particular, (R, T_n) is a normal pair, which contradicts the hypothesis that $R \subset S$ is a Prüfer-closed extension. Thus, some (earlier) point of the above iterative reasoning produced an index j such that $N \cap R_j = Q_j$, with the consequence that $(R : \overline{R}_S) \subseteq (Q_j \subseteq)\, N$.

(2) $\Rightarrow$ (1): By Proposition 2.1 (c), we may assume that the condition (2) (ii) holds and our task is to prove (1), that is, that $R = \widetilde{R}^S$. Suppose that the assertion fails. Then, since $R \subset \widetilde{R}^S$ satisfies FCP, [19, Corollary 3.2] guarantees that $\mathrm{Supp}(\widetilde{R}^S/R)$ is a finite nonempty set. As $\mathrm{Supp}_A(E)$ is stable under generalization for any module E over any ring A, $\mathrm{MSupp}(\widetilde{R}^S/R)$ is also a finite nonempty set. Let M be one of its elements. We will show (two paragraphs hence) that M does not contain $(R : \overline{R}_S)$.

Since $R \subset \overline{R}_S$ is an integral extension that satisfies FCP, [19, Theorem 4.2 (a)] guarantees that the nonzero ring $R/(R : \overline{R}_S)$ is Artinian. As this ring is also nonzero, it has Krull dimension 0 and its prime spectrum is finite and nonempty (cf. [2, Propositions 8.1 and 8.3]). So, for some positive integer n, the set consisting of the prime (that is, the maximal) ideals of R that contain $(R : \overline{R}_S)$ can be denoted by $\{M_1, M_2, ..., M_n\}$. We claim that $R_{M_i} = (\widetilde{R}^S)_{M_i}$ for each $i \in \{1, 2, ..., n\}$. Suppose that this claim fails for some (temporarily fixed) i. Note that $R_{M_i} \subset (\widetilde{R}^S)_{M_i}$ satisfies FCP and inherits the "integrally closed extension" property from $R \subset \widetilde{R}^S$. Thus, $(R_{M_i}, (\widetilde{R}^S)_{M_i})$ is a normal pair by [19, Theorem 6.3 (b)]. It follows that there exists a ring A_i in $[R_{M_i}, (\widetilde{R}^S)_{M_i}]$ such that $R_{M_i} \subset A_i$ is an integrally closed minimal extension. On the other hand, the (integral) ring extension $R_{M_i} \subset (\overline{R}_S)_{M_i}$ also satisfies FCP. Thus, there exists a ring B_i in $[R_{M_i}, (\overline{R}_S)_{M_i}]$ such that $R_{M_i} \subset B_i$ is an integral minimal extension. However, since R_{M_i} is quasi-local, it cannot be the base ring of both an integral minimal extension and an integrally closed minimal extension that have the same crucial maximal ideal and are subrings of some common "universal" ring extension (in the present instance, S_{M_i} can play the role of such a "universal" ring), by [38, Lemma 1.5]. This (desired) contradiction completes the proof of the above claim.

For each $i \in \{1, 2, ..., n\}$, we have shown that $R_{M_i} = (\widetilde{R}^S)_{M_i}$, and so $M_i \notin \mathrm{Supp}(\widetilde{R}^S/R)$. Hence, $M \notin \{M_1, M_2, ..., M_n\}$. It follows that M does not contain $(R : \overline{R}_S)$. (Two paragraphs ago, we promised to prove this fact.) As $R \subset \overline{R}_S$ is an integral extension, the Lying-over Theorem (cf. [27, Theorem 11.5], [31, Theorem 44]) supplies a maximal ideal M' of $\overline{R}_S$ that lies over M. It is worth noticing that, because $R \subset \widetilde{R}^S$ is an integrally closed extension satisfying FCP, [19, Remark 6.14] ensures that $\mathrm{Supp}(\widetilde{R}^S/R) = \{P \in \mathrm{Spec}(R) \mid R_P \neq (\widetilde{R}^S)_P\} = \{P \in \mathrm{Spec}(R) \mid P\widetilde{R}^S = \widetilde{R}^S\}$. (Although [19, Remark 6.14] included a hypothesis that the ambient rings were integral domains, a sympathetic reading of its proof shows that the argument goes through without the domain-theoretic hypothesis.) Hence, $M\widetilde{R}^S = \widetilde{R}^S$. It follows that $MS = S$ and, consequently, $M'S = S$. Once again using [19, Remark 6.14] and the fact that $\overline{R}_S \subset S$ is an integrally closed extension satisfying FCP, we infer that $M' \in \mathrm{Supp}(S/\overline{R}_S)$. Hence, by (2) (ii), $(R : \overline{R}_S) \subseteq M'$. Therefore,

$$(R : \overline{R}_S) = (R : \overline{R}_S) \cap R \subseteq M' \cap R = M,$$

the desired contradiction. The proof is complete. $\square$

Remark 2.8. (a) The implication (1) $\Rightarrow$ (2) in Theorem 2.7 would become false if one were to delete the hypothesis that $R \subseteq S$ satisfies FCP. The easiest way to see this is to take $R \subseteq S$ to be the ring extension $k + M \subset k[X] + M$ that was studied in Example 2.2. Consequently, by Theorem 2.7, $R \subseteq S$ does not satisfy FCP. This last fact can also be seen directly by using the infinite strictly decreasing chain of rings $\{k[X^{2^n}] + M \mid n \geq 0\}$.

(b) The above proof of Theorem 2.7 included a proof of the following fact: if $R \subset S$ is a Prüfer-closed extension that satisfies FCP and is not integral, then $R \subset \overline{R}_S$. In case R is quasi-local, this fact can also be gleaned from [35, Proposition 3.3]. We became aware of [35] after the research for the present paper had been completed. This concludes the remark.

As a consequence of Theorems 2.5 and 2.7, we next generalize [8, Theorem 2.7].

Corollary 2.9. *Let $R \subset S$ be a ring extension that satisfies* FCP *and $R \subset \overline{R}_S \subset S$. Then the following conditions are equivalent:*

(1) $[R, S] = [R, \overline{R}_S] \cup [\overline{R}_S, S]$;

(2) For each ring $J \in [R, \overline{R}_S[$ and each $N \in$ MSupp$(S/\overline{R}_S)$*, one has $(J : \overline{R}_S) \subseteq N$;*

(3) For each ring $J \in [R, \overline{R}_S[$ such that $J \subset \overline{R}_S$ is a minimal extension and for each $N \in$ MSupp$(S/\overline{R}_S)$*, one has $(J : \overline{R}_S) \subseteq N$.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows by combining Proposition 2.4 with Theorems 2.5 and 2.7, while the implication (2) $\Rightarrow$ (3) is trivial. It remains to prove that (3) $\Rightarrow$ (2). Assume (3) and let $J \in [R, \overline{R}_S[$. As $J \subset \overline{R}_S$ inherits FCP from $R \subset S$, there exists $H \in [J, \overline{R}_S[$ such that $H \subset \overline{R}_S$ is a minimal extension. Since $(H : \overline{R}_S)$ is contained in each maximal element of Supp$(S/\overline{R}_S)$ by (3), it follows from $(J : \overline{R}_S) \subseteq (H : \overline{R}_S)$ that $(J : \overline{R}_S)$ is also contained in each maximal element of Supp$(S/\overline{R}_S)$. □

The next result generalizes [8, Corollary 2.9].

Corollary 2.10. *Let $R \subset S$ be rings such that $R \subset \overline{R}_S \subset S$ and $R \subset S$ satisfies* FCP. *Then the following conditions are equivalent:*

(1) $[R, S] \setminus \{R\} = [\overline{R}_S, S]$; that is, $\overline{R}_S$ is the (unique) minimum element of $[R, S] \setminus \{R\}$;

(2) $R \subset \overline{R}_S$ is a minimal extension and $(R : \overline{R}_S) \subseteq N$ for each $N \in$ MSupp$(S/\overline{R}_S)$*.*

Proof. (1) $\Rightarrow$ (2): Assume (1). Then $[R, S] = \{R\} \cup [\overline{R}_S, S]$. Thus $R \subset \overline{R}_S$ is a minimal extension and $[R, S] = [R, \overline{R}_S] \cup [\overline{R}_S, S]$. Then for each $N \in$ MSupp$(S/\overline{R}_S)$, one gets $(R : \overline{R}_S) \subseteq N$ by applying Corollary 2.9 (with $J := R$).

(2) $\Rightarrow$ (1): This implication also follows from Corollary 2.9 since (2) ensures that $[R, \overline{R}_S[= \{R\}$. □

The next two corollaries generalize [8, Proposition 2.2] and [8, Corollary 2.3], respectively. These two results were proved by Ben Nasr and Zeidi for integral domains using [8, Theorem 2.1], a result that determined the structure of the intermediate rings between R and S. The statement of [8, Theorem 2.1] was markedly domain-theoretic. The same could be said of its proof, which cannot be adapted to the more general ring-theoretic context. Next, we give a proof that is much shorter and is valid for arbitrary rings.

Corollary 2.11. *Let $R \subseteq S$ be a ring extension satisfying* FCP *such that each ring in $[R, \overline{R}_S[$ is quasi-local. Then $[R, S] = [R, \overline{R}_S] \cup [\overline{R}_S, S]$.*

Proof. Without loss of generality, we can assume that $R \subset \overline{R}_S \subset S$. By Corollary 2.9, it suffices to show that for each $J \in [R, \overline{R}_S[$ such that $J \subset \overline{R}_S$ is minimal, the conductor $(J : \overline{R}_S)$ is contained in each $N \in$ MSupp$(S/\overline{R}_S)$. For any such N, integrality ensures that $N \cap J \in$ Max(J). As J is assumed to be quasi-local, its maximal ideal must be $(J : \overline{R}_S)$, as that is the crucial maximal ideal of $J \subset \overline{R}_S$, by [25, Théorème 2.2]. Thus, $(J : \overline{R}_S) = N \cap J \subseteq N$. □

Corollary 2.12. *Let $R \subseteq S$ be a ring extension satisfying* FCP *such that $\overline{R}_S$ is quasi-local. Then* $[R, S] = [R, \overline{R}_S] \cup [\overline{R}_S, S]$.

Proof. It follows from integrality (cf. [2, Theorem 5.10 and Corollary 5.8]) that each ring in $[R, \overline{R}_S[$ inherits the" quasi-local" property from $\overline{R}_S$. Hence the assertion is a special case of Corollary 2.11. □

We next generalize [8, Proposition 2.5]. While [8, Proposition 2.5] played the role of being a basic result that led to the proof of [8, Theorem 2.7] in [8], we can now see that [8, Proposition 2.5] is, like [8, Theorem 2.7], just another easy consequence of Corollary 2.9.

Corollary 2.13. *Let $R \subset S$ be rings such that $R \subset S$ satisfies* FCP*, $R \subset \overline{R}_S \subset S$, $[R, S] = [R, \overline{R}_S] \cup [\overline{R}_S, S]$ and $\overline{R}_S \subset S$ is a minimal extension. Let N denote the crucial maximal ideal of $\overline{R}_S \subset S$. Then $(R : \overline{R}_S) \subseteq N \cap R$.*

Proof. As it is clear that $\mathrm{MSupp}(S/\overline{R}_S) = \{N\}$, the assertion follows from the implication (1) $\Rightarrow$ (2) in Corollary 2.9 (with $J = R$). □

Remark 2.14. Corollary 2.13 has the following corollary. If $R \subset S$ are rings such that both $R \subset \overline{R}_S$ and $\overline{R}_S \subset S$ are minimal extensions, with respective crucial maximal ideals M and N, and if $[R, S] = [R, \overline{R}_S] \cup [\overline{R}_S, S]$, then $M = N \cap R$. (Indeed, since [25, Théorème 2.2] ensures that $(R : \overline{R}_S) = M$, it follows from Corollary 2.13 that $M \subseteq N \cap R$ and then equality holds because $M \in \mathrm{Max}(R)$.) More is already known. In fact, [13, Theorem 2.1] can be restated as follows. If $R \subset S$ are rings such that both $R \subset \overline{R}_S$ and $\overline{R}_S \subset S$ are minimal extensions, with respective crucial maximal ideals M and N, then: $[R, S] = [R, \overline{R}_S] \cup [\overline{R}_S, S]$ if and only if $M = N \cap R$. Another equivalent condition that could be added to this result is that $[R, S]$ is linearly ordered by inclusion. This observation gives additional motivation for the study of λ-extensions in the next section and, for the context of ring extensions that satisfy FCP, it will be generalized in the final result of this paper.

3 Applications to the λ-extensions satisfying FCP

The definition of a λ-extension in [26] was motivated by the following two observations: any minimal (ring) extension is a λ-extension, and any λ-extension is a Δ-extension (in the sense of [29]). As we mentioned in the Introduction, λ-extensions were characterized in [36, Proposition 5.3] and our purpose in Section 3 is to give a deeper characterization of them for the case of ring extensions that satisfy FCP.

Once again, we begin a section with a result that collects some useful information.

Proposition 3.1. *Let $R \subseteq S$ be a λ-extension. Then:*

(a) *$R \subseteq S$ is a* P*-extension. Consequently, no element of S is transcendental over R.*

(b) *The following conditions are equivalent:*

(1) *There exists a finite maximal chain $R = R_0 \subset \cdots \subset R_n = S$ in $[R, S]$;*

(2) *$R \subseteq S$ satisfies* FCP.

Moreover, if the above (hypothesis and) equivalent conditions hold, then $[R, S] = \{R_i \mid 0 \leq i \leq n\}$ and $S = R[u]$ for some $u \in S$.

(c) *Suppose, in addition, that $R = K$ is a field (such that $K \subseteq S$ is a λ-extension). Then S is algebraic over K. Moreover, the above conditions (1) and (2) in (b) are also equivalent to the following conditions:*

(3) *S is a finitely generated as a K-module;*

(4) $\dim_K(S) < \infty$.

Proof. (a) One way to prove the first assertion is to argue as in [26, page 1 and Lemma 1.1] by noting that each λ-extension is a Δ-extension and then use the proof of [29, Lemma 3] to show that any Δ-extension is a P-extension. (For an alternate (and simpler) proof that does not refer to Δ-extensions, use the "λ-extension" hypothesis to show that if $u \in S$, then either $u^2 \in R[u^3]$ or $u^3 \in R[u^2]$, whence u is primitive over R.) The "'Consequently" assertion follows because any element that is primitive over R cannot be transcendental over R.

(b) Regardless of whether $R \subseteq S$ is a λ-extension, (2) $\Rightarrow$ (1). Next, given that $R \subseteq S$ is a λ-extension and assuming (1), we see that if $T \in [R, S]$, then $T \in \mathcal{C} := \{R_i \mid 0 \leq i \leq n\}$, by virtue of the maximality of $\mathcal{C}$. This observation establishes the first of the "Moreover" assertions and also completes the proof that (1) $\Rightarrow$ (2), since any subset of $\mathcal{C}$ is finite. To prove the second of the "Moreover" assertions, we may adapt the proof of [26, Theorem 3.2, (b) $\Rightarrow$ (c)]. In detail, one may assume, without loss of generality, that $R \subset S$, so that $n \geq 1$; then any $u \in S \setminus R_{n-1}$ satisfies $R[u] = S$, since $R[u]$ and R_{n-1} are comparable under inclusion and $R_{n-1} \subset S$ is a minimal extension.

(c) Since $R = K$ is assumed to be a field, the first assertion follows from either of the conclusions in (a). Hence S is integral over K. Thus (cf. [31, Theorem 17]), (3) is equivalent to (3)$'$: S is finitely generated as a K-algebra. As we saw in (b) that (1) and (2) imply that $S = R[u]$ for some $u \in S$, it follows that (1) and (2) imply (3)$'$ (and hence that they imply (3)). Of course, (3) $\Leftrightarrow$ (4) by the theory of finite-dimensional vector spaces. Thus, it remains only to show that (4) $\Rightarrow$ (2). This, in turn, also follows from the theory of finite-dimensional vector spaces. Indeed, if $d := \dim_K(S) < \infty$ and $A_0 \subset \ldots \subset A_m$ is a finite chain in $[R, S]$, then $\{\dim_K(A_j) \mid 0 \leq j \leq m\}$ is a strictly increasing (finite) sequence of non-negative integers that are each less than $d + 1$, so that $m + 1 \leq d + 1$ and $m \leq d$. The proof is complete. □

It seems natural to explore the extent to which the equivalences in Proposition 3.1 (c) may have valid analogues in case the base ring R is not a field. However, since one of our purposes here is to reduce the theory of λ-extensions that satisfy FCP to, insofar as possible, contexts having a field as a base ring, we move at once to that context. Fortunately, Corollary 3.2 will provide a considerable amount of information along those lines, thanks to copious citations from [26]. First, recall that if S is a (commutative) algebra over a ring R, then S is said to be *decomposable* as an R-algebra if S is R-algebra isomorphic to a direct product $\prod_{i \in I} A_i$ where each A_i is a nonzero R-algebra and the index set I has cardinality at least 2; of course, if an R-algebra S is not decomposable as an R-algebra, S is said to be *indecomposable* as an R-algebra. Also, recall that a ring is said to be *reduced* if it has no nonzero nilpotent elements.

Corollary 3.2. *Let K be a field and S a nonzero K-algebra. View $K \subseteq S$ via the (injective, unique) K-algebra homomorphism $K \to S$. Then:*

(a) If S is a field, then $K \subseteq S$ is a λ-extension if (and only if) the set of fields contained between K and S is linearly ordered by inclusion.

(b) S is decomposable as a K-algebra and $K \subseteq S$ is a λ-extension if and only if S is K-algebra isomorphic to $K \times L$ for some field extension L of K such that the set of fields contained between K and L is linearly ordered by inclusion.

(c) Suppose that S is (nonzero and) indecomposable as a K-algebra and let $J := \sqrt{S}$. Then $K \subseteq S$ is a λ-extension if and only if $K \subseteq K + J$ is a λ-extension, $K + J$ is comparable under inclusion with each ring in $[K, S]$, $L := S/J$ is a field, and (when we view $K \subseteq L$) the set of fields contained between K and L is linearly ordered by inclusion. Moreover, if the above (hypotheses and) equivalent conditions hold, then each ring in $[K, S]$ has a unique prime ideal, each ring in $[K + J, S]$ has J as its unique prime ideal, J is a principal nilpotent ideal of S of nilpotency index r for some $r \in \{1, 2, 3\}$ and $K + J$ is K-algebra isomorphic to $K[X]/(X^r)$ with X an indeterminate over K. Finally, if the above (hypotheses and) equivalent conditions hold and S is not a field, the field extension $K \subseteq L$ is purely inseparable.

(d) Suppose that (K is a field and) $K \subseteq S$ satisfies FCP *and S is a reduced ring. Then $K \subseteq S$ is a λ-extension if and only if there exists a field extension L of K such that $[L : K] < \infty$, $K \subseteq L$ is a λ-extension, and S is K-algebra isomorphic to either L or $K \times L$.*

Proof. (a) Since any λ-extension is a Δ-extension, the assertion follows from [29, Theorem 1]. An alternate proof of the assertion, which does not make use of the "Δ-extension" concept, can be found in [26, Proposition 3.17 (2)].

(b) It suffices to combine [26, Corollary 2.14 (b)] and (a).

(c) Since $(K + J)/J \cong K$, the assignment $E \mapsto E/J$ induces an order isomorphism $[K + J, S] \to [K, L]$. Thus, $K + J \subseteq S$ is a λ-extension if and only if $K \subseteq L$ is a λ-extension. It follows that $K \subseteq S$ is a λ-extension if and only if $K \subseteq K + J$ is a λ-extension, $K + J$ is comparable under inclusion with each ring in $[K, S]$ and $K \subseteq L$ is a λ-extension.

Suppose first that $K \subseteq S$ is a λ-extension. Then, by [26, Theorem 3.8 (2)], J is the unique prime ideal of each ring in $[K + J, S]$. It follows that L is a field. In view of the preceding paragraph, this completes the proof of the "only if" assertion.

To prove the "if" assertion, it suffices to combine (a) with the first paragraph of this proof.

All parts of the "Moreover" and "Finally" assertions now follow from what was established in [26, Theorem 3.8].

(d) The "if" assertion follows by combining (a) and (b). For the converse, suppose that $K \subseteq S$ is a λ-extension. Since $K \subseteq S$ satisfies FCP, Proposition 3.1 (c) gives $\dim_K(S) < \infty$. Thus, by once again combining (a) and (b), we may assume, without loss of generality, that S is an indecomposable K-algebra. Since S is assumed to be reduced, $J := \sqrt{S} = 0$. Hence, by yet another combination of (a) and (b), S is a field (identified with $L := S/J$). Finally, $[L : K] = \dim_K(S) < \infty$. The proof is complete. □

Remark 3.3. (a) One cannot remove the hypothesis in the "Finally" assertion in Theorem 3.2 (c) that S is not a field. In other words, a field extension which is also a λ-extension need not be purely inseparable. Examples of non-minimal field extensions that are also λ-extensions and illustrate this fact are easy to find in any prime characteristic $p > 0$: cf. $\mathbb{F}_p \subset \mathbb{F}_{p^{(q^2)}}$ for any prime number q. With some more effort, one can verify that $\mathbb{Q} \subset \mathbb{Q}(2^{1/4})$ provides such an example in characteristic 0. We will meet these field extensions again in Remark 3.7 (b).

(b) As one may surmise from Corollary 3.2 (c), it remains an open question, for a field K, to classify (up to K-algebra isomorphism) the K-algebras S that are indecomposable, but not reduced, such that $K \subseteq S$ is a λ-extension. As noted in Corollary 3.2 (c), much is known about such rings S, including the fact that $\sqrt{S}$ is the only prime ideal of S and $\sqrt{S}$ is a principal nilpotent ideal of S whose nilpotency index is either 2 or 3. Moreover, if K is a field and $K \subseteq T$ is a λ-extension, then each nilpotent element of T has nilpotency index at most 3 [26, Corollary 3.6]. The restriction in Corollary 3.2 (c) to reduced extension rings was made in order to obtain the classification result there. In fact, if X is an indeterminate over a field K and $n \geq 1$, then $K \subseteq K[X]/(X^n)$ is a λ-extension if and only if $n \leq 3$ [26, Proposition 3.5]. This completes the remark.

We turn next to characterizing the integral λ-extensions satisfying FCP in the general case, that is, where the base ring may not be a field. The general procedure is summarized in Proposition 3.4 (which does not even need the "FCP" hypothesis).

Proposition 3.4. *Let $R \subseteq S$ be an integral ring extension, let $M \in \mathrm{Max}(R)$ and let J be an ideal of S such that $MS \subseteq J \subset S$. Put $A := R + J$. Then $R \subseteq S$ is a λ-extension if and only if the following three conditions hold:*

(i) $[R, S] = [R, A] \cup [A, S]$;

(ii) $R \subseteq A$ is a λ-extension;

(iii) $R/M \subseteq S/J$ is a λ-extension.

Proof. It will suffice to show that (iii) is equivalent to the condition that $A \subseteq S$ is a λ-extension. This, in turn, can be shown by reasoning as in the first paragraph of the proof of Corollary 3.2 (c). (In detail, combine the order-isomorphism $[A, S] \to [A/J, S/J]$ with the ring isomorphism $A/J \cong R/M$.) □

One may think that the most natural choice for J in Proposition 3.4 would be $J = MS$. However, Theorem 3.5 (c) will indicate how Proposition 3.4 can be sharpened when one adds the "FCP" hypothesis. We will show why a different choice of J (and a particular choice of M) may be appropriate when one is given S as the result of juxtaposing a finite number of minimal extensions whose smallest base ring is R.

Theorem 3.5. *Let $R \subset S$ be an integral ring extension with a finite maximal chain $R = R_0 \subset \cdots \subset R_n = S$ in $[R, S]$. Then:*

(a) If $1 \leq i \leq n$, put $Q_i := (R_{i-1} : R_i)$ and $M_i := Q_i \cap R$. Then

$$\mathrm{Supp}_R(S/R) = \mathrm{MSupp}_R(S/R) = \{M_i \mid 1 \leq i \leq n\}$$

and

$$\mathrm{Rad}_R(\prod_{i=1}^{n} M_i) = \mathrm{Rad}_R((R:S)) = \cap_{i=1}^n M_i.$$

(*b*) *Assume that the* (*integral*) *extension* $R \subset S$ *is a* λ*-extension and* $M \in \mathrm{Max}(R)$. *Then there are at most two distinct prime* (*in fact, maximal*) *ideals of* S *that lie over* M.

(*c*) *Assume that the* (*integral*) *extension* $R \subset S$ *is a* λ*-extension that satisfies* FCP. *Then* $\mathrm{Supp}_R(S/R) = \mathrm{MSupp}_R(S/R)$ *is the singleton set whose unique element is* $\mathrm{Rad}_R((R:S))$.

(*d*) *Assume that the* (*integral*) *extension* $R \subset S$ *satisfies* FCP. *Let* $M \in \mathrm{Max}(R)$ *and put* $A := R + \mathrm{Rad}_S(MS)$. *Then* $R \subset S$ *is a* λ*-extension if and only if the following three conditions hold:*

(*i*) $[R,S] = [R,A] \cup [A,S]$*;*

(*ii*) $R \subseteq A$ *is a* λ*-extension;*

(*iii*) *If* $M \in \mathrm{Supp}_R(S/R)$, *there are at most two prime* (*maximal*) *ideals* N *of* S *that lie over* M, *for at most one such* N *is it the case that* $R + N \subset S$, *and if there exists* $\mathcal{N} \in \mathrm{Max}(S)$ *such that* $R + \mathcal{N} \subset S$, *then* $R/M \subset S/\mathcal{N}$ *is a* (*necessarily finite-dimensional*) λ*-extension of fields.*

Proof. (a) For each i, it follows from [25, Théorème 2.2 (ii)] that Q_i is the crucial maximal ideal of the minimal extension $R_{i-1} \subset R_i$. Hence, by integrality, M_i is a maximal ideal of R. By [19, Corollary 3.2], $\mathrm{Supp}_R(S/R) = \{M_i \mid 1 \le i \le n\}$. This proves the first assertion of (a). Also, each M_i contains $(R:S)$, since $(R:S) \subseteq (R_{i-1}:R_i)$. On the other hand, since $Q_i R_i \subseteq R_{i-1}$, we have

$$(\prod_{1\le i\le n} M_i)S \subseteq (\prod_{1\le i\le n} Q_i)S \subseteq (\prod_{1\le i\le n-1} Q_i)R_{n-1} \subseteq$$
$$(\prod_{1\le i\le n-2} Q_i)R_{n-2} \subseteq (\prod_{1\le i\le n-3} Q_i)R_{n-3} \subseteq \ldots \subseteq R.$$

Thus, $\prod_{1\le i\le n} M_i \subseteq (R:S) \subseteq \bigcap_{1\le i\le n} M_i$. We can use this fact and [2, Proposition 1.11 (ii)] (which applies since each M_i is a prime ideal of R) to get the final assertion of (a).

(b) The parenthetical assertion is a standard consequence of integrality (cf. [2, Corollary 5.8]). If $M \notin \mathrm{Supp}_R(S/R)$, then $R_M = S_M$ and so there exists a unique prime (maximal) ideal of S lying over M. Thus, without loss of generality, we may assume that $M \in \mathrm{Supp}_R(S/R)$ (that is, $M \in \{M_i \mid 1 \le i \le n\}$). Since S is finitely generated as an R-module, there are only finitely many prime (maximal) ideals of S that lie over M, say, $N_1, \ldots, N_k$ for some integer $k \ge 1$ (cf. [9, Proposition 3, page 40], [2, Theorem 5.10]). Consider the ideal $J := \mathrm{Rad}_S(MS) = \cap_{j=1}^k N_j$ and the ring $A := R + J \in [R,S]$. We have canonical R-algebra isomorphisms $R/M \to A/J$ and $S/J \to \prod_{j=1}^k S/N_j$ (with the former holding because $J \cap R = M$ and the latter coming via the Chinese Remainder Theorem). By the order-isomorphism $[A,S] \to [A/J, S/J]$, $A/J \subseteq S/J$ inherits the "λ-extension" property from $A \subseteq S$, and so we may view the injective canonical R-algebra homomorphism $R/M \to \prod_{j=1}^k S/N_j$ as a λ-extension $R/M \subseteq \prod_{j=1}^k S/N_j$. Hence, as R/M is a field, [26, Corollary 2.14 (a)] ensures that $k \le 2$.

(c) Since $R \subset S$ is a λ-extension that satisfies FCP, it follows easily, by combining (a) with [19, Corollary 3.2] and [38, Lemma 1.7], that $\mathrm{Supp}_R(S/R) = \mathrm{MSupp}_R(S/R)$ is a singleton set. Let M denote its unique element. It remains only to prove that $M = \mathrm{Rad}_R((R:S))$. For $1 \le i \le n$, let M_i be as in (a). Then $M_i = M$ for all i. Hence, by (a), $\mathrm{Rad}_R((R:S)) = \cap_{i=1}^n M_i = M$, as desired.

(d) We must show that the present condition (iii) is equivalent to what was called condition (iii) in Proposition 3.4 (with $J = \mathrm{Rad}_S(MS)$). Recall that condition (iii) of Proposition 3.4 can be reformulated as stating that $R/M \subseteq S/\mathrm{Rad}_S(MS)$ is a λ-extension.

Suppose first that $M \notin \mathrm{Supp}_R(S/R)$, that is, $R_M = S_M$. Then the unique prime (maximal) ideal N of S lying over M is such that the residue fields S/N and R/M are isomorphic as R-algebras. In fact, if we view the canonical R-algebra homomorphism $R/M \to S/\mathrm{Rad}_S(MS) = S/N$ as an inclusion, it is the identity map on R/M, in which case $R/M \subseteq S/\mathrm{Rad}_S(MS)$ is trivially a λ-extension.

Suppose next that $M \in \mathrm{Supp}_R(S/R)$. There are two subcases. In the first subcase, there exists a unique prime (maximal) ideal N of S lying over M. In this subcase, $S/\mathrm{Rad}_S(MS) = S/N$.

By (b), there is only one other subcase, namely, where there are two distinct prime (maximal) ideals, say N_1 and N_2, of S lying over M. Then, by the Chinese Remainder Theorem, we have an R-algebra isomorphism

$$S/\text{Rad}_S(MS) = S/(N_1 \cap N_2) \cong S/N_1 \times S/N_2.$$

It remains only to explain why $R/M \subseteq S/N_1 \times S/N_2$ being a λ-extension is (after possibly relabeling N_1 and N_2) equivalent to having $R/M = S/N_1$ and $R/M \subseteq S/N_2$ a finite-dimensional λ-extension of fields. Thus, since $R/M \subseteq S/\text{Rad}_S(MS)$ inherits the "FCP" property from $R \subseteq S$, an appeal to Proposition 3.1 (c) and Corollary 3.2 (b) completes the proof. □

Remark 3.6. It is clear from the proofs of Corollary 3.2 (c), Proposition 3.4 and Theorem 3.5 (d) that the following conclusion holds. Let $R \subseteq S$ be rings, J an ideal of S, and $I := J \cap R$; then $R \subseteq S$ is a λ-extension if and only if $R \subseteq R+J$ is a λ-extension, $[R, S] = [R, R+J] \cup [R+J, S]$ and $R/I \subseteq S/J$ is a λ-extension. It is natural to ask why we chose to focus in Theorem 3.5 (d) on an intermediate ring of the form $R+\text{Rad}_S(MS)$, that is, on an ideal of the form $J = \text{Rad}_S(MS)$ for some $M \in \text{Max}(R)$. The answer is that this focus permitted a characterization in Theorem 3.5 (d) which made effective use of what is known about such problems when the base ring is a field. When we begin to move to the context of non-integral ring extensions in Theorem 3.9, an entirely different kind of analysis will be needed. But the methodology discussed above will return in Corollary 3.11, where it will be advantageous to use the natural intermediate ring in $[R, S]$, namely, $\overline{R}_S$. This completes the remark.

Remark 3.7 will complete our contributions to the study of the integral case for the λ-extensions that satisfy FCP.

Remark 3.7. (a) This remark assumes familiarity with the inert/decomposed/ramified trichotomy for the integral minimal extensions of a given base ring R (cf. [34, Theorem 3.3]). The point of view in Theorem 3.5 leads naturally to the following question. If a ring extension $R \subset S$ admits a finite maximal chain $R = R_0 \subset \cdots \subset R_n = S$ in $[R, S]$, under what circumstances is $R \subset S$ a λ-extension? This question has been studied extensively in case $n = 2$, where the question is equivalent to asking when $[R, S] = \{R, R_1, S\}$. As the focus thus far in Section 3 has been on certain integral extensions, let us summarize here in case $n = 2$, supposing that one knows only that $E_1 : R \subset T$ and $E_2 : T \subset S$ are each integral minimal extensions with respective crucial maximal ideals M and N. It was shown in [24, Proposition 3.1 (d)] that a necessary condition for $R \subset T$ to be a λ-extension is that $N \cap R = M$. With this necessary condition also being assumed, it turns out that there are only two contexts for which one can be certain that $R \subset T$ is a λ-extension: E_2 is inert and E_1 is either decomposed or ramified. This fact is included in [14, Theorem 2.9 (a)].

Unfortunately, an error in [14, Theorem 2.8] led to condition (ix) of [24, Theorem 4.1] being mishandled in [14, Theorem 2.9]. Condition (ix) is the stipulation that E_1 is ramified, E_2 is decomposed and $N \cap R = M$. Part (b) of [14, Theorem 2.9] summarized the contexts for which $R \subset T$ is never a λ-extension; part (c) of [14, Theorem 2.9] summarized the contexts for which some instances have $R \subset T$ being a λ-extension and other instances have $R \subset T$ not being a λ-extension; and [14, Theorem 2.8] mistakenly classified condition (ix) into part (b) instead of part (c). Thanks to an example provided by Picavet and Picavet-L'Hermitte, this error was rectified in [15, Example], and results that classify up to isomorphism have been completed that are strong enough to cover the situation where R is the prime subring of S (cf. [16, Corollaries 2.21 (c) and 3.6 (f)], [20, Corollaries 2.15 and 2.18 (c)], [21, Corollary 2.4 and 2.5 (f)], [17, Lemma 2.2 (b)]).

The flavor of some of the underlying work may be seen by considering the situation where R is a field, say K, with E_1 and E_2 each being ramified. Up to isomorphism, it follows from [25, Lemme 1.2] and[17, Lemma 2.2 (b)] that we can take $T = K \times K$ (with K viewed inside T via the diagonal embedding, $a \mapsto (a, a)$) and $S = K \times K \times K$ (with T viewed inside S by taking the product of a diagonal map and the identity map on one of the copies of K). Then $K \subset S$ meets the stipulation of condition (xiii) of [24, Theorem 4.1] and, according to [14, Theorem 2.9] (as tempered by the above comments), condition (xiii) admits some examples that are λ-extensions and other examples that are not λ-extensions. So, even in an example as apparently simple as this, the theory covering the general case instructs us to analyze further. Fortunately, it is not difficult to give a direct argument showing that $K \subset S$ is not a λ-extension. Indeed, $[K, S] \setminus \{K, S\}$ contains at least (exactly, if $K = \mathbb{F}_2$) three pairwise distinct elements. These are, respectively, the collection of all $(a, b, c) \in S$ such that $a = b$ (resp., such that $a = c$; resp., such that $b = c$).

(b) By taking $p = 2 = q$ in an example from Remark 3.3 (a), we see that $\mathbb{F}_2 \subset \mathbb{F}_{16}$ is a non-minimal field extension which is also a λ-extension. Its only "properly intermediate" ring is $\mathbb{F}_4$. One can describe this situation as in the first paragraph of (a), by taking E_1 to be $\mathbb{F}_2 \subset \mathbb{F}_4$ and E_2 to be $\mathbb{F}_4 \subset \mathbb{F}_{16}$. The characteristic 0 example from Remark 3.3 (a) can be described similarly, by taking E_1 to be $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2})$ and E_2 to be $\mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(2^{1/4})$. Notice that in both examples, E_1 and E_2 are inert (and, of course, their crucial maximal ideals satisfy $N \cap R = M$, since N and M are each 0). According to [14, Theorem 2.9 (c)](as tempered by the above comments), there exist inert ring extensions $A \subset B$ and $B \subset C$ such that their crucial maximal ideals satisfy $\mathcal{N} \cap A = \mathcal{M}$ and $A \subset C$ is not a λ-extension. Indeed, field-theoretic examples that illustrate this fact are easy to find: consider the chains $\mathbb{F}_2 \subset \mathbb{F}_4 \subset \mathbb{F}_{64}$ and $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

(c) As noted in (a), it was proved in [14] that if E_2 is inert and E_1 is either decomposed or ramified, then the juxtaposition of E_2 atop E_1 produces a λ-extension. One such example, where $K \subset L$ is any minimal field extension, is given by the chain $K \subset K \times K \subset K \times L$. The reader is invited to similarly build many other examples by using some known inert, decomposed or ramified extensions in conjunction with [14, Theorem 2.9]. This completes the remark.

Theorem 3.9 identifies the integrally closed λ-extensions that satisfy FCP. This result generalizes [5, Corollary 2.7] and, in our opinion, improves upon [36, Proposition 5.2 (1)]. First, for the sake of completeness, Lemma 3.8 isolates some facts about support that will be needed in the proof of Theorem 3.9. Recall that if E is a (unital) module over a ring A, then $\text{Supp}_A(E) := \{P \in \text{Spec}(A) \mid E_P \neq 0\}$.

Lemma 3.8. *(a) Let R be a ring, let $P \subseteq M$ be prime ideals of R, and let E be an R-module. Then $P \in \text{Supp}_R(E)$ if and only if $PR_M \in \text{Supp}_{R_M}(E_M)$.*

(b) Let $A \subseteq B$ be rings and let $P \subseteq M$ be prime ideals of A. Then $P \in \text{Supp}_A(B/A)$ if and only if $PA_M \in \text{Supp}_{A_M}(B_M/A_M)$.

Proof. (a) We must show that $PR_M \in \text{Supp}_{R_M}(E_M)$ if and only if $E_P \neq 0$. This, in turn, follows since

$$(E_M)_{PR_M} = E_M \otimes_{R_M} (R_M)_{PR_M} \cong (E \otimes_R R_M) \otimes_{R_M} R_P \cong$$

$$E \otimes_R R_P \cong E_{R \setminus P} =: E_P.$$

(b) As $B_M/A_M \cong (B/A)_M$, an application of (a) (with $R := A$ and $E := B/A$) completes the proof. □

Theorem 3.9. *Let $R \subseteq S$ be an integrally closed ring extension that satisfies* FCP. *Then the following conditions are equivalent:*
(1) $\text{Supp}_R(S/R)$ *is linearly ordered by inclusion;*
(2) $R \subseteq S$ is a λ-extension.

Proof. If $R = S$, then both (1) and (2) hold trivially. Thus, we may henceforth assume, without loss of generality, that $R \subset S$.

(1) $\Rightarrow$ (2): Suppose that the implication fails. Then (1) holds, but there exist $T, U \in [R, S]$ such that $T \not\subseteq U$ and $U \not\subseteq T$. As $T \not\subseteq U$, it follows via globalization that $T_M \not\subseteq U_M$ for some $M \in \text{Max}(R)$. (As usual, if $A \subseteq B$ are rings, with $C \in [A, B]$ and $P \in \text{Spec}(A)$, then $C_P := C_{A \setminus P}$ and C_P is identified with its image under the canonical injection $C_P \longrightarrow B_P$.) Hence $T_M \neq R_M$, and so $R_M \subset T_M$; *a fortiori*, $R_M \subset S_M$, that is, $M \in \text{Supp}_R(S/R)$.

Consider $R_M \subset S_M$. On general principles, $[R_M, S_M] = \{C_M \mid C \in [R, S]\}$. (In detail, one inclusion is clear. For the reverse inclusion, if $D \in [R_M, S_M]$ and $j : S \longrightarrow S_M$ is the canonical injection, then $E := j^{-1}(D) \in [R, S]$ and $E_M = D$.) Thus, $R_M \subseteq S_M$ inherits the FCP (and FIP) properties from $R \subseteq S$; and, of course, R_M is integrally closed in S_M. Therefore, by

[19, Corollary 6.4], $[R_M, S_M]$ is a finite set that is linearly ordered by inclusion. Consequently, $U_M \subset T_M$. As $U \nsubseteq T$, it follows via globalization that $U_N \nsubseteq T_N$ for some $N \in \text{Max}(R)$.

Repeat the above reasoning, with $R_N \subseteq S_N$ replacing $R_M \subseteq S_M$. The upshot is that $[R_N, S_N]$ is a finite set that is linearly ordered by inclusion. Hence, $T_N \subset U_N$ (since $U_N \nsubseteq T_N$). Thus, $R_N \subset U_N \subseteq S_N$, and so $N \in \text{Supp}_R(S/R)$. As M and N are each maximal ideals of R that are in $\text{Supp}_R(S/R)$, it follows from (1) that $M = N$. Since $U_M \subset T_M$ and $U_N \nsubseteq T_N$, we have the (desired) contradiction, thus completing the (indirect) proof that (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1): Assume (2). We will derive a contradiction from the condition that there exist $P, Q \in \text{Supp}_R(S/R)$ such that $P \nsubseteq Q$ and $Q \nsubseteq P$. As $R \subset S$, we can use the proof of [36, Proposition 5.2 (2)] to conclude that $\text{MSupp}_R(S/R)$ consists of a single maximal ideal of R. Let M denote the unique element of $\text{MSupp}_R(S/R)$.

We claim that we can replace $R \subseteq S$ with $R_M \subseteq S_M$. To see this, note first that $R_M \subseteq S_M$ inherits the "λ-extension" property from $R \subset S$. Next, it follows from Lemma 3.8 (a) that $PR_M, QR_M \in \text{Supp}_{R_M}(S_M/R_M)$. Of course, PR_M and QR_M are not comparable since P and Q are not comparable. Next, it follows from Lemma 3.8 (b) and globalization that if $\text{Supp}_{R_M}(S_M/R_M)$ were linearly ordered by inclusion, then $\text{Supp}_R(S/R)$ would also be linearly ordered by inclusion. This proves the above claim and so, by *abus de langage*, we can assume henceforth that (R, M) is quasi-local.

As $R \subset S$ satisfies FCP, we can pick a finite maximal chain $R = R_0 \subset ... \subset R_k \subset ... \subset R_n = S$ in $[R, S]$. Since [19, Theorem 6.3 (b)] ensures that (R, S) is a normal pair, $R_{k-1} \subset R_k$ is an integrally closed minimal extension, for each $k \in \{1, ..., n\}$. For each such k, let C_k denote the crucial maximal ideal of the minimal extension $R_{k-1} \subset R_k$. As (R_{k-1}, R_k) is a normal pair and R_0 $(= R)$ is quasi-local, it follows from [19, Theorem 6.8] (by induction on k) that each R_k is quasi-local; $R_k = (R_{k-1})_{h_k}$ for some prime ideal h_k of R_{k-1} such that $h_kR_k = h_k$; and R_{k-1}/h_k is a valuation domain (necessarily with quotient field R_k/h_k). By [25, Théorème 2.2], C_k "blows up" in R_k (in the sense that $C_kR_k = R_k$) and *every other* prime ideal of R_{k-1} is lain over by exactly one prime ideal of R_k. So, for each k, there exists at most one $\mathfrak{p}_k$ (resp., at most one $\mathfrak{q}_k$) $\in \text{Spec}(R_k)$ such that $\mathfrak{p}_k \cap R = P$ (resp., such that $\mathfrak{q}_k \cap R = Q$). To show that $P \subseteq Q$ (resp., $Q \subseteq P$), it will be enough to find k such that $\mathfrak{p}_k \subseteq \mathfrak{q}_k$ (resp., $\mathfrak{q}_k \subseteq \mathfrak{p}_k$), as one would then intersect with R to get the desired assertion, that is, the desired contradiction.

By [19, Corollary 3.2], P can be expressed as the intersection of R with some appropriate C_k. Also, since $P \in \text{Supp}_R(S/R)$, it follows from [19, Remark 6.14] that P blows up in $S = R_n$, and so there exists a unique index k_1 such that $1 \leq k_1 \leq n$ and $\mathfrak{p}_{k_1-1}$ blows up in R_{k_1}. Similarly, there exists a unique index k_2 such that $1 \leq k_2 \leq n$ and $\mathfrak{q}_{k_2-1}$ blows up in R_{k_2}. For each index k, as noted above, only one prime ideal of R_{k-1} blows up in R_k. If $k_1 = k_2$, then $\mathfrak{p}_{k_1-1} = \mathfrak{q}_{k-1}$ and intersection with R would give $P = Q$, the desired contradiction. Therefore $k_1 \neq k_2$.

Without loss of generality, $k_1 < k_2$. Put $k := k_1$. Then $\mathfrak{p}_{k-1} = C_k \neq \mathfrak{q}_{k-1}$ in $\text{Spec}(R_{k-1})$. Recall that $R_k = (R_{k-1})_{h_k}$ and $h_kR_k = h_k$. It follows that h_k is the unique maximal ideal of R_k. We have that $\mathfrak{p}_{k-1}$ blows up in $R_k = (R_{k-1})_{h_k}$ and $\mathfrak{q}_{k-1}$ does not blow up in R_k. Thus, $\mathfrak{p}_{k-1} \nsubseteq h_k$ and $\mathfrak{q}_{k-1} \subseteq h_k$.

We claim that h_k is comparable under inclusion with each prime ideal $\mathfrak{p}$ of R_{k-1}. We will show that if $\mathfrak{p} \nsubseteq h_k$, then $h_k \subseteq \mathfrak{p}$. Pick $a \in \mathfrak{p} \setminus h_k$. Then, working in $R_k = (R_{k-1})_{h_k}$, we see that for each $u \in h_k$,

$$u = (\frac{u}{a})a = (\frac{u}{1})(\frac{1}{a})a \in h_k(R_{k-1})_{h_k}a = h_ka \subseteq R_{k-1}a \subseteq \mathfrak{p}.$$

This proves the above claim.

Apply the above claim to $\mathfrak{p} := \mathfrak{p}_{k-1}$. As $\mathfrak{p}_{k-1} \nsubseteq h_k$, we get $h_k \subseteq \mathfrak{p}_{k-1}$. Since $\mathfrak{q}_{k-1} \subseteq h_k$, it follows that $\mathfrak{q}_{k-1} \subseteq \mathfrak{p}_{k-1}$. As explained above, this leads to $Q \subseteq P$, the desired contradiction. The proof is complete. □

Remark 3.10. One cannot delete the "FCP" hypothesis in Theorem 3.9. Perhaps the easiest way to see this is to let R be any quasi-local treed integral domain (that is, an integral domain such that $\text{Spec}(R)$ is linearly ordered by inclusion; for instance, a valuation domain) and let S be the polynomial ring $R[X]$ (for an indeterminate X over R). As noted in the proof of Example 2.2, R is integrally closed in S. It is also clear that $\text{Supp}_R(S/R)$ is linearly ordered by inclusion, since it is a subset of Spec(R). However, $[R, S]$ is not linearly ordered by inclusion: notice, for instance, that $R[X^2]$ and $R[X^3]$ are incomparable under inclusion since $X^2 \notin R[X^3]$ and $X^3 \notin R[X^2]$. Of course, one can verify directly that the ring extension $R \subset S$ does not satisfy FCP, in view of infinite decreasing chains such as $\{R[X^{(2^n)}] \mid n \geq 0\}$. In fact, $R \subset S$ is not even a P-extension. This completes the remark.

In view of the earlier results in this section, our goal of characterizing the λ-extensions satisfying FCP will be accomplished if we do so for the FCP extensions that are neither integral nor integrally closed. This is done in our final result.

Corollary 3.11. *Let $R \subset S$ be rings such that $R \subset \overline{R}_S \subset S$ and $R \subset S$ satisfies* FCP. *Then the following conditions are equivalent:*

(1) $R \subset S$ is a λ-extension;

(2) $R \subset \overline{R}_S$ is a λ-extension, $[R, S] = [R, \overline{R}_S] \cup [\overline{R}_S, S]$ and $\overline{R}_S \subset S$ is a λ-extension;

(3) Let $M \in \mathrm{Max}(R)$ and put $E := R + \mathrm{Rad}_{\overline{R_S}}(M\overline{R_S})$. The following five conditions hold:

(i) $[R, \overline{R_S}] = [R, E] \cup [E, \overline{R_S}]$;

(ii) $R \subseteq E$ is a λ-extension;

(iii) If $M \in \mathrm{Supp}_R(\overline{R_S}/R)$, there are at most two prime (maximal) ideals N of $\overline{R_S}$ that lie over M, for at most one such N is it the case that $R + N \subset \overline{R_S}$, and if there exists $\mathcal{N} \in \mathrm{Max}(\overline{R_S})$ such that $R + \mathcal{N} \subset \overline{R_S}$, then $R/M \subset \overline{R_S}/\mathcal{N}$ is a (necessarily finite-dimensional) λ-extension of fields;

(iv) $J \subset S$ is a Prüfer-closed extension for each ring $J \in [R, \overline{R}_S[$;

(v) $\mathrm{Supp}_{\overline{R}_S}(S/\overline{R}_S)$ is linearly ordered by inclusion.

(4) Let $M \in \mathrm{Max}(R)$ and put $E := R + \mathrm{Rad}_{\overline{R_S}}(M\overline{R_S})$. The following five conditions hold: conditions (i), (ii), (iii) and (v) from the above statement of (3), as well as the following condition:

(iv)′ For each $J \in [R, \overline{R}_S[$ and for each $N \in \mathrm{MSupp}_{\overline{R}_S}(S/\overline{R}_S)$, one has $(J : \overline{R}_S) \subseteq N$.

Proof. It is easy to see that (1) ⇔ (2). Then (2) ⇔ (3) follows by combining Theorems 3.5 (d), 2.5 and 3.9; and (3) ⇔ (4) by Theorem 2.7. □

In closing, we point out four aspects of the characterization of $R \subset S$ being a λ-extension (in case $R \subset \overline{R}_S \subset S$ and $R \subset S$ satisfies FCP) that is provided by condition (4) in Corollary 3.11. First, three of the five conditions that comprise condition (4) are Supp-theoretic (namely, (iii), (iv)′ and (v)). Second, apart from its Supp-theoretic predication, (iii) is a condition that belongs to field theory, specifically, the (as yet unfinished) characterization of the finite-dimensional field extensions that are λ-extensions. Third, (ii) is a condition that shows the importance of understanding the integral case of λ-extensions and moreover, (ii) is stated in terms of an integral extension E of R that is often "smaller" than $\overline{R}_S$. Fourth, in view of the presence of condition (i), it seems (to us) that any result in the spirit of Corollary 3.11 will need to include a stipulation that (is equivalent to requiring that) the set of intermediate rings of some related ring extension is pinched somewhere.

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