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MAPPINGS OF $*$-DERIVATION-TYPE ON SUM OF PRODUCTS $ab - b \circ a^*$ ON $*$-ALGEBRAS

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Abstract Let $\mathcal{A}$ be a prime $*$-algebra. In this paper, we prove that every mapping $\delta : \mathcal{A} \to \mathcal{A}$ satisfying $\delta(ab - b \circ a^*) = \delta(a)b + a\delta(b) - \delta(b) \circ a^* - b \circ \delta(a)^*$ (where $\circ$ is the special Jordan product on $\mathcal{A}$), for all elements $a, b \in \mathcal{A}$, is an additive $*$-derivation.

1 Introduction

Let $\mathcal{A}$ be a $*$-algebra over the complex field $\mathbb{C}$. Denote by $a \circ b = \frac{1}{2}(ab + ba)$, for all elements $a, b \in \mathcal{A}$, the *special Jordan product* and by $[a, b]_* = ab - ba^*$, for all elements $a, b \in \mathcal{A}$, the $*$-*Lie product*. A mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called *additive* $*$-*derivation* if it is an additive derivation and satisfies $\delta(a^*) = \delta(a)^*$, for all element $a \in \mathcal{A}$, and it is called *nonlinear* $*$-*Lie derivation* if $\delta([a, b]_*) = [\delta(a), b]_* + [a, \delta(b)]_*$, for all elements $a, b \in \mathcal{A}$. A mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called $*$-*derivation-type on sum of products* $ab - b \circ a^*$ if

$$\delta(ab - b \circ a^*) = \delta(a)b + a\delta(b) - \delta(b) \circ a^* - b \circ \delta(a)^*, \tag{1.1}$$

for all elements $a, b \in \mathcal{A}$.

Recently, several authors have presented new results in the study of some classes of mappings on $*$-algebras. Within the scope of this paper some examples can be seen in [1], [2], [3], [4]. In that regard, Cui and Li [1] proved that every bijective map preserving $*$-Lie product on factor von Neumann algebras is a $*$-ring isomorphism, Yu and Zhang [4] proved that every $*$-Lie derivation on factor von Neumann algebras is an additive $*$-derivation and Ferreira e Marietto [2] proved that every mapping preserving sum of products $ab - b \circ a^*$ on factor von Neumann algebras is a $*$-ring isomorphism. Motivated by these results, the aim of the present paper is to prove that a mapping of $*$-derivation-type on sum of products $ab - b \circ a^*$ on prime $*$-algebras is an additive $*$-derivation.

Our main result reads as follows.

Theorem 1.1. *Let $\mathcal{A}$ be a prime $*$-algebra with $1_{\mathcal{A}}$ its identity and such that $\mathcal{A}$ has a non-trivial projection. Then every mapping $\delta : \mathcal{A} \to \mathcal{A}$ satisfying $\delta(ab - b \circ a^*) = \delta(a)b + a\delta(b) - \delta(b) \circ a^* - b \circ \delta(a)^*$, for all elements $a, b \in \mathcal{A}$, is an additive $*$-derivation.*

2 The proof of the Theorem 1.1

The proof of the Theorem 1.1 is made by proving several lemmas. We begin with the following lemma.

Lemma 2.1. *Let $\mathcal{A}$ be a $*$-algebra and a mapping $\delta : \mathcal{A} \to \mathcal{A}$ of $*$-derivation-type on sum of products $ab - b \circ a^*$. Then $\delta(0) = 0$.*

Proof. Indeed, $\delta(0) = \delta(00 - 0 \circ 0^*) = \delta(0)0 + 0\delta(0) - \delta(0) \circ 0^* - 0 \circ \delta(0)^* = 0.$ □

Lemma 2.2. *Let $\mathcal{A}$ be a prime $*$-algebra with $1_{\mathcal{A}}$ its identity and such that $\mathcal{A}$ has a nontrivial projection. Then every mapping $\delta : \mathcal{A} \to \mathcal{A}$ of $*$-derivation-type on sum of products $ab - b \circ a^*$ is additive.*

Based on the approach taken by Cui and Li [1], Ferreira and Marietto [2] and Yu and Zhang [4], we shall organize the proof of Lemma 2.2 in a series of properties. We begin, though, with a well-known result that will be used throughout this paper.

Let p_1 be an arbitrary non-trivial projection of $\mathcal{A}$ and write $p_2 = 1_{\mathcal{A}} - p_1$. Then $\mathcal{A}$ has a Peirce decomposition $\mathcal{A} = \mathcal{A}_{11} \oplus \mathcal{A}_{12} \oplus \mathcal{A}_{21} \oplus \mathcal{A}_{22}$, where $\mathcal{A}_{ij} = p_i \mathcal{A} p_j$ $(i, j = 1, 2)$, satisfying the following multiplicative relations: $\mathcal{A}_{ij}\mathcal{A}_{kl} \subseteq \delta_{jk}\mathcal{A}_{il}$, where δ_{jk} is the *Kronecker delta function*.

Property 2.3. For arbitrary elements $a_{11} \in \mathcal{A}_{11}$, $b_{12} \in \mathcal{A}_{12}$, $c_{21} \in \mathcal{A}_{21}$ and $d_{22} \in \mathcal{A}_{22}$, the following assertions hold: (i) $\delta(a_{11} + b_{12}) = \delta(a_{11}) + \delta(b_{12})$, (ii) $\delta(a_{11} + c_{21}) = \delta(a_{11}) + \delta(c_{21})$, (iii) $\delta(b_{12} + d_{22}) = \delta(b_{12}) + \delta(d_{22})$ and (iv) $\delta(c_{21} + d_{22}) = \delta(c_{21}) + \delta(d_{22})$.

Proof. At first, observe that $(ip_2)(a_{11} + b_{12}) - (a_{11} + b_{12}) \circ (ip_2)^* = (ip_2)b_{12} - b_{12} \circ (ip_2)^*$, since $(ip_2)a_{11} - a_{11} \circ (ip_2)^* = 0$. For this reason, by Lemma 2.1, we have

$$\begin{aligned}
&\delta(ip_2)(a_{11} + b_{12}) + (ip_2)\delta(a_{11} + b_{12})\\
&\quad - \delta(a_{11} + b_{12}) \circ (ip_2)^* - (a_{11} + b_{12}) \circ \delta(ip_2)^*\\
=&\delta((ip_2)(a_{11} + b_{12}) - (a_{11} + b_{12}) \circ (ip_2)^*)\\
=&\delta((ip_2)a_{11} - a_{11} \circ (ip_2)^*) + \delta((ip_2)b_{12} - b_{12} \circ (ip_2)^*)\\
=&\delta(ip_2)a_{11} + (ip_2)\delta(a_{11}) - \delta(a_{11}) \circ (ip_2)^* - a_{11} \circ \delta(ip_2)^*\\
&\quad + \delta(ip_2)b_{12} + (ip_2)\delta(b_{12}) - \delta(b_{12}) \circ (ip_2)^* - b_{12} \circ \delta(ip_2)^*\\
=&\delta(ip_2)(a_{11} + b_{12}) + (ip_2)(\delta(a_{11}) + \delta(b_{12}))\\
&\quad - (\delta(a_{11}) + \delta(b_{12})) \circ (ip_2)^* - (a_{11} + b_{12}) \circ \delta(ip_2)^*
\end{aligned}$$

which demonstrates that

$$(ip_2)(\delta(a_{11} + b_{12}) - \delta(a_{11}) - \delta(b_{12})) - (\delta(a_{11} + b_{12}) - \delta(a_{11}) - \delta(b_{12})) \circ (ip_2)^* = 0.$$

Defining $t = \delta(a_{11} + b_{12}) - \delta(a_{11}) - \delta(b_{12}) = t_{11} + t_{12} + t_{21} + t_{22}$, we can write the above equation as

$$(ip_2)(t_{11} + t_{12} + t_{21} + t_{22}) - (t_{11} + t_{12} + t_{21} + t_{22}) \circ (ip_2)^* = 0$$

which implies that $\frac{1}{2}t_{12} + \frac{3}{2}t_{21} + 2t_{22} = 0$. It results in $t_{12} = t_{21} = t_{22} = 0$. Now, observe that $(ip_1 - 3ip_2)(a_{11} + b_{12}) - (a_{11} + b_{12}) \circ (ip_1 - 3ip_2)^* = (ip_1 - 3ip_2)a_{11} - a_{11} \circ (ip_1 - 3ip_2)^*$, since $(ip_1 - 3ip_2)b_{12} - b_{12} \circ (ip_1 - 3ip_2)^* = 0$. It therefore follows that

$$\begin{aligned}
&\delta(ip_1 - 3ip_2)(a_{11} + b_{12}) + (ip_1 - 3ip_2)\delta(a_{11} + b_{12})\\
&\quad - \delta(a_{11} + b_{12}) \circ (ip_1 - 3ip_2)^* - (a_{11} + b_{12}) \circ \delta(ip_1 - 3ip_2)^*\\
=&\delta((ip_1 - 3ip_2)(a_{11} + b_{12}) - (a_{11} + b_{12}) \circ (ip_1 - 3ip_2)^*)\\
=&\delta((ip_1 - 3ip_2)a_{11} - a_{11} \circ (ip_1 - 3ip_2)^*)\\
&\quad + \delta((ip_1 - 3ip_2)b_{12} - b_{12} \circ (ip_1 - 3ip_2)^*)\\
=&\delta(ip_1 - 3ip_2)a_{11} + (ip_1 - 3ip_2)\delta(a_{11})\\
&\quad - \delta(a_{11}) \circ (ip_1 - 3ip_2)^* - a_{11} \circ \delta(ip_1 - 3ip_2)^*\\
&\quad + \delta(ip_1 - 3ip_2)b_{12} + (ip_1 - 3ip_2)\delta(b_{12})\\
&\quad - \delta(b_{12}) \circ (ip_1 - 3ip_2)^* - b_{12} \circ \delta(ip_1 - 3ip_2)^*\\
=&\delta(ip_1 - 3ip_2)(a_{11} + b_{12}) + (ip_1 - 3ip_2)(\delta(a_{11}) + \delta(b_{12}))\\
&\quad - (\delta(a_{11}) + \delta(b_{12})) \circ (ip_1 - 3ip_2)^* - (a_{11} + b_{12}) \circ \delta(ip_1 - 3ip_2)^*
\end{aligned}$$

which produces

$$(ip_1 - 3ip_2)(\delta(a_{11} + b_{12}) - \delta(a_{11}) - \delta(b_{12})) \\ - (\delta(a_{11} + b_{12}) - \delta(a_{11}) - \delta(b_{12})) \circ (ip_1 - 3ip_2)^* = 0.$$

This shows that

$$(ip_1 - 3ip_2)t_{11} - t_{11} \circ (ip_1 - 3ip_2)^* = 0$$

which leads to $t_{11} = 0$. As consequence, we obtain $\delta(a_{11} + b_{12}) = \delta(a_{11}) + \delta(b_{12})$.

Using a similar reasoning as above, we prove the cases (ii), (iii) and (iv). □

Property 2.4. For arbitrary elements $b_{12} \in \mathcal{A}_{12}$ and $c_{21} \in \mathcal{A}_{21}$, the following assertion holds: $\delta(b_{12} + c_{21}) = \delta(b_{12}) + \delta(c_{21})$.

Proof. At first, observe that $(ip_1 - 3ip_2)(b_{12} + c_{21}) - (b_{12} + c_{21}) \circ (ip_1 - 3ip_2)^* = (ip_1 - 3ip_2)c_{21} - c_{21} \circ (ip_1 - 3ip_2)^*$, since $(ip_1 - 3ip_2)b_{12} - b_{12} \circ (ip_1 - 3ip_2)^* = 0$. On account of this, we have

$$\begin{aligned}
&\delta(ip_1 - 3ip_2)(b_{12} + c_{21}) + (ip_1 - 3ip_2)\delta(b_{12} + c_{21}) \\
&\quad - \delta(b_{12} + c_{21}) \circ (ip_1 - 3ip_2)^* - (b_{12} + c_{21}) \circ \delta(ip_1 - 3ip_2)^* \\
=&\delta((ip_1 - 3ip_2)(b_{12} + c_{21}) - (b_{12} + c_{21}) \circ (ip_1 - 3ip_2)^*) \\
=&\delta((ip_1 - 3ip_2)b_{12} - b_{12} \circ (ip_1 - 3ip_2)^*) \\
&\quad + \delta((ip_1 - 3ip_2)c_{21} - c_{21} \circ (ip_1 - 3ip_2)^*) \\
=&\delta(ip_1 - 3ip_2)b_{12} + (ip_1 - 3ip_2)\delta(b_{12}) \\
&\quad - \delta(b_{12}) \circ (ip_1 - 3ip_2)^* - b_{12} \circ \delta(ip_1 - 3ip_2)^* \\
&\quad + \delta(ip_1 - 3ip_2)c_{21} + (ip_1 - 3ip_2)\delta(c_{21}) \\
&\quad - \delta(c_{21}) \circ (ip_1 - 3ip_2)^* - c_{21} \circ \delta(ip_1 - 3ip_2)^* \\
=&\delta(ip_1 - 3ip_2)(b_{12} + c_{21}) + (ip_1 - 3ip_2)(\delta(b_{12}) + \delta(c_{21})) \\
&\quad - (\delta(b_{12}) + \delta(c_{21})) \circ (ip_1 - 3ip_2)^* - (b_{12} + c_{21}) \circ \delta(ip_1 - 3ip_2)^*
\end{aligned}$$

which gives

$$(ip_1 - 3ip_2)(\delta(b_{12} + c_{21}) - \delta(b_{12}) - \delta(c_{21})) \\ - (\delta(b_{12} + c_{21}) - \delta(b_{12}) - \delta(c_{21})) \circ (ip_1 - 3ip_2)^* = 0.$$

Defining $t = \delta(b_{12} + c_{21}) - \delta(b_{12}) - \delta(c_{21}) = t_{11} + t_{12} + t_{21} + t_{22}$, we can write the above equation as

$$(ip_1 - 3ip_2)(t_{11} + t_{12} + t_{21} + t_{22}) - (t_{11} + t_{12} + t_{21} + t_{22}) \circ (ip_1 - 3ip_2)^* = 0$$

through which we get $2t_{11} - 4t_{21} - 6t_{22} = 0$. This implies that $t_{11} = t_{21} = t_{22} = 0$. Now, observe that $(-3ip_1 + ip_2)(b_{12} + c_{21}) - (b_{12} + c_{21}) \circ (-3ip_1 + ip_2)^* = (-3ip_1 + ip_2)b_{12} - b_{12} \circ (-3ip_1 + ip_2)^*$, since $(-3ip_1 + ip_2)c_{21} - c_{21} \circ (-3ip_1 + ip_2)^* = 0$. On account of this, we have

$$\begin{aligned}
&\delta(-3ip_1 + ip_2)(b_{12} + c_{21}) + (-3ip_1 + ip_2)\delta(b_{12} + c_{21}) \\
&\quad - \delta(b_{12} + c_{21}) \circ (-3ip_1 + ip_2)^* - (b_{12} + c_{21}) \circ \delta(-3ip_1 + ip_2)^* \\
=&\delta((-3ip_1 + ip_2)(b_{12} + c_{21}) - (b_{12} + c_{21}) \circ (-3ip_1 + ip_2)^*) \\
=&\delta((-3ip_1 + ip_2)b_{12} - b_{12} \circ (-3ip_1 + ip_2)^*) \\
&\quad + \delta((-3ip_1 + ip_2)c_{21} - c_{21} \circ (-3ip_1 + ip_2)^*) \\
=&\delta(-3ip_1 + ip_2)b_{12} + (-3ip_1 + ip_2)\delta(b_{12}) \\
&\quad - \delta(b_{12}) \circ (-3ip_1 + ip_2)^* - b_{12} \circ \delta(-3ip_1 + ip_2)^*
\end{aligned}$$

$$
\begin{aligned}
&+\delta(-3ip_1+ip_2)c_{21}+(-3ip_1+ip_2)\delta(c_{21})\\
&-\delta(c_{21})\circ(-3ip_1+ip_2)^*-c_{21}\circ\delta(-3ip_1+ip_2)^*\\
=&\delta(-3ip_1+ip_2)(b_{12}+c_{21})+(-3ip_1+ip_2)(\delta(b_{12})+\delta(c_{21}))\\
&-(\delta(b_{12})+\delta(c_{21}))\circ(-3ip_1+ip_2)^*-(b_{12}+c_{21})\circ\delta(-3ip_1+ip_2)^*
\end{aligned}
$$

which gives

$$
\begin{aligned}
&(-3ip_1+ip_2)(\delta(b_{12}+c_{21})-\delta(b_{12})-\delta(c_{21}))\\
&\qquad-(\delta(b_{12}+c_{21})-\delta(b_{12})-\delta(c_{21}))\circ(-3ip_1+ip_2)^*=0.
\end{aligned}
$$

It follows from the above equation that

$$
(-3ip_1+ip_2)t_{12}-t_{12}\circ(-3ip_1+ip_2)^*=0
$$

which results in $t_{12}=0$. Consequently, $\delta(b_{12}+c_{21})=\delta(b_{12})+\delta(c_{21})$. □

Property 2.5. For arbitrary elements $a_{11}\in\mathcal{A}_{11}$, $b_{12}\in\mathcal{A}_{12}$, $c_{21}\in\mathcal{A}_{21}$ and $d_{22}\in\mathcal{A}_{22}$, the following assertions hold: (i) $\delta(a_{11}+b_{12}+c_{21})=\delta(a_{11})+\delta(b_{12})+\delta(c_{21})$ and (ii) $\delta(b_{12}+c_{21}+d_{22})=\delta(b_{12})+\delta(c_{21})+\delta(d_{22})$.

Proof. It follows from the Property 2.4 that

$$
\begin{aligned}
&\delta(ip_2)(a_{11}+b_{12}+c_{21})+(ip_2)\delta(a_{11}+b_{12}+c_{21})\\
&-\delta(a_{11}+b_{12}+c_{21})\circ(ip_2)^*-(a_{11}+b_{12}+c_{21})\circ\delta(ip_2)^*\\
=&\delta((ip_2)(a_{11}+b_{12}+c_{21})-(a_{11}+b_{12}+c_{21})\circ(ip_2)^*)\\
=&\delta((ip_2)a_{11}-a_{11}\circ(ip_2)^*)+\delta((ip_2)b_{12}-b_{12}\circ(ip_2)^*)\\
&+\delta((ip_2)c_{21}-c_{21}\circ(ip_2)^*)\\
=&\delta(ip_2)a_{11}+(ip_2)\delta(a_{11})-\delta(a_{11})\circ(ip_2)^*-a_{11}\circ\delta(ip_2)^*\\
&+\delta(ip_2)b_{12}+(ip_2)\delta(b_{12})-\delta(b_{12})\circ(ip_2)^*-b_{12}\circ\delta(ip_2)^*\\
&+\delta(ip_2)c_{21}+(ip_2)\delta(c_{21})-\delta(c_{21})\circ(ip_2)^*-c_{21}\circ\delta(ip_2)^*\\
=&\delta(ip_2)(a_{11}+b_{12}+c_{21})+(ip_2)(\delta(a_{11})+\delta(b_{12})+\delta(c_{21}))\\
&-(\delta(a_{11})+\delta(b_{12})+\delta(c_{21}))\circ(ip_2)^*-(a_{11}+b_{12}+c_{21})\circ\delta(ip_2)^*
\end{aligned}
$$

which produces the equation

$$
\begin{aligned}
&(ip_2)(\delta(a_{11}+b_{12}+c_{21})-\delta(a_{11})-\delta(b_{12})-\delta(c_{21}))\\
&\qquad-(\delta(a_{11}+b_{12}+c_{21})-\delta(a_{11})-\delta(b_{12})-\delta(c_{21}))\circ(ip_2)^*=0.
\end{aligned}
$$

Defining $t=\delta(a_{11}+b_{12}+c_{21})-\delta(a_{11})-\delta(b_{12})-\delta(c_{21})=t_{11}+t_{12}+t_{21}+t_{22}$, so we get

$$
(ip_2)(t_{11}+t_{12}+t_{21}+t_{22})-(t_{11}+t_{12}+t_{21}+t_{22})\circ(ip_2)^*=0
$$

which leads to $\frac{1}{2}t_{12}+\frac{3}{2}t_{21}+2t_{22}=0$. It therefore follows that $t_{12}=t_{21}=t_{22}=0$. Also, by Property 2.3(ii), we have

$$
\begin{aligned}
&\delta(ip_1-3ip_2)(a_{11}+b_{12}+c_{21})+(ip_1-3ip_2)\delta(a_{11}+b_{12}+c_{21})\\
&-\delta(a_{11}+b_{12}+c_{21})\circ(ip_1-3ip_2)^*-(a_{11}+b_{12}+c_{21})\circ\delta(ip_1-3ip_2)^*\\
=&\delta((ip_1-3ip_2)(a_{11}+b_{12}+c_{21})-(a_{11}+b_{12}+c_{21})\circ(ip_1-3ip_2)^*)\\
=&\delta((ip_1-3ip_2)a_{11}-a_{11}\circ(ip_1-3ip_2)^*)\\
&+\delta((ip_1-3ip_2)b_{12}-b_{12}\circ(ip_1-3ip_2)^*)\\
&+\delta((ip_1-3ip_2)c_{21}-c_{21}\circ(ip_1-3ip_2)^*)
\end{aligned}
$$

$$\begin{aligned}
=&\delta(ip_1-3ip_2)a_{11}+(ip_1-3ip_2)\delta(a_{11})\\
&-\delta(a_{11})\circ(ip_1-3ip_2)^*-a_{11}\circ\delta(ip_1-3ip_2)^*\\
&+\delta(ip_1-3ip_2)b_{12}+(ip_1-3ip_2)\delta(b_{12})\\
&-\delta(b_{12})\circ(ip_1-3ip_2)^*-b_{12}\circ\delta(ip_1-3ip_2)^*\\
&+\delta(ip_1-3ip_2)c_{21}+(ip_1-3ip_2)\delta(c_{21})\\
&-\delta(c_{21})\circ(ip_1-3ip_2)^*-c_{21}\circ\delta(ip_1-3ip_2)^*\\
=&\delta(ip_1-3ip_2)(a_{11}+b_{12}+c_{21})+(ip_1-3ip_2)(\delta(a_{11})+\delta(b_{12})+\delta(c_{21}))\\
&-(\delta(a_{11})+\delta(b_{12})+\delta(c_{21}))\circ(ip_1-3ip_2)^*\\
&-(a_{11}+b_{12}+c_{21})\circ\delta(ip_1-3ip_2)^*
\end{aligned}$$

which demonstrates that

$$\begin{aligned}
&(ip_1-3ip_2)(\delta(a_{11}+b_{12}+c_{21})-\delta(a_{11})-\delta(b_{12})-\delta(c_{21}))\\
&\qquad-(\delta(a_{11}+b_{12}+c_{21})-\delta(a_{11})-\delta(b_{12})-\delta(c_{21}))\circ(ip_1-3ip_2)^*=0.
\end{aligned}$$

It results in

$$(ip_1-3ip_2)t_{11}-t_{11}\circ(ip_1-3ip_2)^*=0$$

which leads to $t_{11}=0$. Therefore, $\delta(a_{11}+b_{12}+c_{21})=\delta(a_{11})+\delta(b_{12})+\delta(c_{21})$.

Using a similar reasoning as above, we prove the case (ii). □

Property 2.6. For arbitrary elements $a_{11}\in\mathcal{A}_{11}$, $b_{12}\in\mathcal{A}_{12}$, $c_{21}\in\mathcal{A}_{21}$ and $d_{22}\in\mathcal{A}_{22}$, the following assertion holds: $\delta(a_{11}+b_{12}+c_{21}+d_{22})=\delta(a_{11})+\delta(b_{12})+\delta(c_{21})+\delta(d_{22})$.

Proof. From Property 2.5(i), it follows that

$$\begin{aligned}
&\delta(ip_1)(a_{11}+b_{12}+c_{21}+d_{22})+(ip_1)\delta(a_{11}+b_{12}+c_{21}+d_{22})\\
&-\delta(a_{11}+b_{12}+c_{21}+d_{22})\circ(ip_1)^*-(a_{11}+b_{12}+c_{21}+d_{22})\circ\delta(ip_1)^*\\
=&\delta((ip_1)(a_{11}+b_{12}+c_{21}+d_{22})-(a_{11}+b_{12}+c_{21}+d_{22})\circ(ip_1)^*)\\
=&\delta((ip_1)a_{11}-a_{11}\circ(ip_1)^*)+\delta((ip_1)b_{12}-b_{12}\circ(ip_1)^*)\\
&+\delta((ip_1)c_{21}-c_{21}\circ(ip_1)^*)+\delta((ip_1)d_{22}-d_{22}\circ(ip_1)^*)\\
=&\delta(ip_1)a_{11}+(ip_1)\delta(a_{11})-\delta(a_{11})\circ(ip_1)^*-a_{11}\circ\delta(ip_1)^*\\
&+\delta(ip_1)b_{12}+(ip_1)\delta(b_{12})-\delta(b_{12})\circ(ip_1)^*-b_{12}\circ\delta(ip_1)^*\\
&+\delta(ip_1)c_{21}+(ip_1)\delta(c_{21})-\delta(c_{21})\circ(ip_1)^*-c_{21}\circ\delta(ip_1)^*\\
&+\delta(ip_1)d_{22}+(ip_1)\delta(d_{22})-\delta(d_{22})\circ(ip_1)^*-d_{22}\circ\delta(ip_1)^*\\
=&\delta(ip_1)(a_{11}+b_{12}+c_{21}+d_{22})+(ip_1)(\delta(a_{11})+\delta(b_{21})+\delta(c_{21})+\delta(d_{22}))\\
&-(\delta(a_{11})+\delta(b_{21})+\delta(c_{21})+\delta(d_{22}))\circ(ip_1)^*-(a_{11}+b_{12}+c_{21}+d_{22})\circ\delta(ip_1)^*
\end{aligned}$$

which results in the equation

$$\begin{aligned}
&(ip_1)(\delta(a_{11}+b_{12}+c_{21}+d_{22})-\delta(a_{11})-\delta(b_{12})-\delta(c_{21})-\delta(d_{22}))\\
&\qquad-(\delta(a_{11}+b_{12}+c_{21}+d_{22})-\delta(a_{11})-\delta(b_{12})-\delta(c_{21})-\delta(d_{22}))\circ(ip_1)^*=0.
\end{aligned}$$

Defining $t=\delta(a_{11}+b_{12}+c_{21}+d_{22})-\delta(a_{11})-\delta(b_{12})-\delta(c_{21})-\delta(d_{22})=t_{11}+t_{12}+t_{21}+t_{22}$, we obtain

$$(ip_1)(t_{11}+t_{12}+t_{21}+t_{22})-(t_{11}+t_{12}+t_{21}+t_{22})\circ(ip_1)^*=0$$

which implies that $2t_{11}+\frac{3}{2}t_{12}+\frac{1}{2}t_{21}=0$. Therefore, $t_{11}=t_{12}=t_{21}=0$.

Using a similar reasoning as above, we prove that $t_{22}=0$. As consequence we have $\delta(a_{11}+b_{12}+c_{21}+d_{22})=\delta(a_{11})+\delta(b_{12})+\delta(c_{21})+\delta(d_{22})$. □

Property 2.7. For arbitrary elements $a_{12}, b_{12} \in \mathcal{A}_{12}$ and $c_{21}, d_{21} \in \mathcal{A}_{21}$, the following assertions hold: (i) $\delta(a_{12}+b_{12}) = \delta(a_{12}) + \delta(b_{12})$ and (ii) $\delta(c_{21}+d_{21}) = \delta(c_{21}) + \delta(d_{21})$.

Proof. First, we note that the following identity holds

$$(p_1+a_{12})(p_2+b_{12}) - (p_2+b_{12}) \circ (p_1+a_{12})^* = a_{12} + \tfrac{1}{2}b_{12} - \tfrac{1}{2}a_{12}^* - \tfrac{1}{2}b_{12}a_{12}^* - \tfrac{1}{2}a_{12}^*b_{12}.$$

Then, by Property 2.6, we have

$$\begin{aligned}
&\delta(a_{12}+\tfrac{1}{2}b_{12}) + \delta(-\tfrac{1}{2}a_{12}^*) + \delta(-\tfrac{1}{2}b_{12}a_{12}^*) + \delta(-\tfrac{1}{2}a_{12}^*b_{12})\\
=&\delta(a_{12}+\tfrac{1}{2}b_{12} - \tfrac{1}{2}a_{12}^* - \tfrac{1}{2}b_{12}a_{12}^* - \tfrac{1}{2}a_{12}^*b_{12})\\
=&\delta((p_1+a_{12})(p_2+b_{12}) - (p_2+b_{12}) \circ (p_1+a_{12})^*)\\
=&\delta(p_1+a_{12})(p_2+b_{12}) + (p_1+a_{12})\delta(p_2+b_{12})\\
&-\delta(p_2+b_{12}) \circ (p_1+a_{12})^* - (p_2+b_{12}) \circ \delta(p_1+a_{12})^*\\
=&(\delta(p_1)+\delta(a_{12}))(p_2+b_{12}) + (p_1+a_{12})(\delta(p_2)+\delta(b_{12}))\\
&-(\delta(p_2)+\delta(b_{12})) \circ (p_1+a_{12})^* - (p_2+b_{12}) \circ (\delta(p_1)+\delta(a_{12}))^*\\
=&\delta(p_1)p_2 + p_1\delta(p_2) - \delta(p_2) \circ p_1^* - p_2 \circ \delta(p_1)^*\\
&+\delta(p_1)b_{12} + p_1\delta(b_{12}) - \delta(b_{12}) \circ p_1^* - b_{12} \circ \delta(p_1)^*\\
&+\delta(a_{12})p_2 + a_{12}\delta(p_2) - \delta(p_2) \circ a_{12}^* - p_2 \circ \delta(a_{12})^*\\
&+\delta(a_{12})b_{12} + a_{12}\delta(b_{12}) - \delta(b_{12}) \circ a_{12}^* - b_{12} \circ \delta(a_{12})^*\\
=&\delta(p_1p_2 - p_2 \circ p_1^*) + \delta(p_1b_{12} - b_{12} \circ p_1^*)\\
&+\delta(a_{12}p_2 - p_2 \circ a_{12}^*) + \delta(a_{12}b_{12} - b_{12} \circ a_{12}^*)\\
=&\delta(\tfrac{1}{2}b_{12}) + \delta(a_{12} - \tfrac{1}{2}a_{12}^*) + \delta(-\tfrac{1}{2}b_{12}a_{12}^* - \tfrac{1}{2}a_{12}^*b_{12})\\
=&\delta(a_{12}) + \delta(\tfrac{1}{2}b_{12}) + \delta(-\tfrac{1}{2}a_{12}^*) + \delta(-\tfrac{1}{2}b_{12}a_{12}^*) + \delta(-\tfrac{1}{2}a_{12}^*b_{12})
\end{aligned}$$

which shows that $\delta(a_{12}+\frac{1}{2}b_{12}) = \delta(a_{12}) + \delta(\frac{1}{2}b_{12})$. Due to this result, we conclude that $\delta(a_{12}+b_{12}) = \delta(a_{12}) + \delta(b_{12})$.

Similarly, we prove the case (ii) using the identity

$$(p_2+c_{21})(p_1+d_{21}) - (p_1+d_{21}) \circ (p_2+c_{21})^* = c_{21} + \tfrac{1}{2}d_{21} - \tfrac{1}{2}c_{21}^* - \tfrac{1}{2}d_{21}c_{21}^* - \tfrac{1}{2}c_{21}^*d_{21}.$$

□

Property 2.8. For arbitrary elements $a_{11}, b_{11} \in \mathcal{A}_{11}$ and $c_{22}, d_{22} \in \mathcal{A}_{22}$, the following assertions hold: (i) $\delta(a_{11}+b_{11}) = \delta(a_{11}) + \delta(b_{11})$ and (ii) $\delta(c_{22}+d_{22}) = \delta(c_{22}) + \delta(d_{22})$.

Proof. At first, observe that $(ip_2)(a_{11}+b_{11}) - (a_{11}+b_{11}) \circ (ip_2)^* = 0$, since $(ip_2)a_{11} - a_{11} \circ (ip_2)^* = (ip_2)b_{11} - b_{11} \circ (ip_2)^* = 0$. For this reason, by Lemma 2.1, we have

$$\begin{aligned}
&\delta(ip_2)(a_{11}+b_{11}) + (ip_2)\delta(a_{11}+b_{11})\\
&-\delta(a_{11}+b_{11}) \circ (ip_2)^* - (a_{11}+b_{11}) \circ \delta(ip_2)^*\\
=\;&\delta((ip_2)(a_{11}+b_{11}) - (a_{11}+b_{11}) \circ (ip_2)^*)\\
=\;&\delta((ip_2)a_{11} - a_{11} \circ (ip_2)^*) + \delta((ip_2)b_{11} - b_{11} \circ (ip_2)^*)\\
=\;&\delta(ip_2)a_{11} + (ip_2)\delta(a_{11}) - \delta(a_{11}) \circ (ip_2)^* - a_{11} \circ \delta(ip_2)^*\\
&+\delta(ip_2)b_{11} + (ip_2)\delta(b_{11}) - \delta(b_{11}) \circ (ip_2)^* - b_{11} \circ \delta(ip_2)^*\\
=\;&\delta(ip_2)(a_{11}+b_{11}) + (ip_2)(\delta(a_{11})+\delta(b_{11}))\\
&-(\delta(a_{11})+\delta(b_{11})) \circ (ip_2)^* - (a_{11}+b_{11}) \circ \delta(ip_2)^*
\end{aligned}$$

which demonstrates that

$$(ip_2)(\delta(a_{11}+b_{11}) - \delta(a_{11}) - \delta(b_{11})) - (\delta(a_{11}+b_{11}) - \delta(a_{11}) - \delta(b_{11})) \circ (ip_2)^* = 0.$$

Defining $t = \delta(a_{11}+b_{11}) - \delta(a_{11}) - \delta(b_{11}) = t_{11}+t_{12}+t_{21}+t_{22}$, we can write the above equation as

$$(ip_2)(t_{11}+t_{12}+t_{21}+t_{22}) - (t_{11}+t_{12}+t_{21}+t_{22}) \circ (ip_2)^* = 0$$

which yields in $\frac{1}{2}t_{12} + \frac{3}{2}t_{21} + 2t_{22} = 0$. It therefore follows that $t_{12} = t_{21} = t_{22} = 0$. Next, for an arbitrary element $c_{12} \in \mathcal{A}_{12}$ we have, by Property 2.7(i),

$$\begin{aligned}
& \delta(a_{11}+b_{11})c_{12} + (a_{11}+b_{11})\delta(c_{12}) \\
& -\delta(c_{12}) \circ (a_{11}+b_{11})^* - c_{12} \circ \delta(a_{11}+b_{11})^* \\
= \; & \delta((a_{11}+b_{11})c_{12} - c_{12} \circ (a_{11}+b_{11})^*) \\
= \; & \delta(a_{11}c_{12} - c_{12} \circ a_{11}^*) + \delta(b_{11}c_{12} - c_{12} \circ b_{11}^*) \\
= \; & \delta(a_{11})c_{12} + a_{11}\delta(c_{12}) - \delta(c_{12}) \circ a_{11}^* - c_{12} \circ \delta(a_{11})^* \\
& +\delta(b_{11})c_{12} + b_{11}\delta(c_{12}) - \delta(c_{12}) \circ b_{11}^* - c_{12} \circ \delta(b_{11})^* \\
= \; & (\delta(a_{11}) + \delta(b_{11}))c_{12} + (a_{11}+b_{11})\delta(c_{12}) \\
& -\delta(c_{12}) \circ (a_{11}+b_{11})^* - c_{12} \circ (\delta(a_{11}) + \delta(b_{11}))^*.
\end{aligned}$$

This produces the equation

$$(\delta(a_{11}+b_{11}) - \delta(a_{11}) - \delta(b_{11}))c_{12} - c_{12} \circ (\delta(a_{11}+b_{11}) - \delta(a_{11}) - \delta(b_{11}))^* = 0$$

which leads to

$$t_{11}c_{12} - c_{12} \circ t_{11}^* = 0.$$

Due to this result, we conclude that $t_{11} = 0$. Consequently, we obtain $\delta(a_{11}+b_{11}) = \delta(a_{11}) + \delta(b_{11})$.

Similarly, we prove the case (ii). □

Property 2.9. δ is an additive mapping.

Proof. The result is a direct consequence of Properties 2.6, 2.7 and 2.8. □

From now on, we assume that all lemmas satisfy the conditions of the Theorem 1.1.

Lemma 2.10. *(i)* $\delta(1_{\mathcal{A}}) = 0$ *and* $\delta(i1_{\mathcal{A}}) = 0$, *(ii)* $\delta(a^*) = \delta(a)^*$, *for all element* $a \in \mathcal{A}$, *and (iii)* $\delta(ia) = i\delta(a)$, *for all element* $a \in \mathcal{A}$.

Proof. Observe that the calculations

$$\delta(1_{\mathcal{A}})1_{\mathcal{A}} + 1_{\mathcal{A}}\delta(1_{\mathcal{A}}) - \delta(1_{\mathcal{A}}) \circ 1_{\mathcal{A}}^* - 1_{\mathcal{A}} \circ \delta(1_{\mathcal{A}})^* = \delta(1_{\mathcal{A}}1_{\mathcal{A}} - 1_{\mathcal{A}} \circ 1_{\mathcal{A}}^*) = 0$$

and

$$\begin{aligned}
\delta(i1_{\mathcal{A}})(i1_{\mathcal{A}}) + (i1_{\mathcal{A}})\delta(i1_{\mathcal{A}}) - \delta(i1_{\mathcal{A}}) \circ (i1_{\mathcal{A}})^* - (i1_{\mathcal{A}}) \circ \delta(i1_{\mathcal{A}})^* \\
= \delta((i1_{\mathcal{A}})(i1_{\mathcal{A}}) - (i1_{\mathcal{A}}) \circ (i1_{\mathcal{A}})^*) = -2\delta(1_{\mathcal{A}})
\end{aligned}$$

result in $\delta(1_{\mathcal{A}}) - \delta(1_{\mathcal{A}})^* = 0$ and $-2\delta(1_{\mathcal{A}}) = 3i\delta(i1_{\mathcal{A}}) - i\delta(i1_{\mathcal{A}})^*$, respectively. As $-2\delta(1_{\mathcal{A}})^* = -3i\delta(i1_{\mathcal{A}})^* + i\delta(i1_{\mathcal{A}})$, then we conclude that $\delta(i1_{\mathcal{A}})^* = -\delta(i1_{\mathcal{A}})$. As consequence of the last result, we have

$$\begin{aligned}
2\delta(i1_{\mathcal{A}}) =& \delta((i1_{\mathcal{A}})1_{\mathcal{A}} - 1_{\mathcal{A}} \circ (i1_{\mathcal{A}})^*) \\
=& \delta(i1_{\mathcal{A}})1_{\mathcal{A}} + (i1_{\mathcal{A}})\delta(1_{\mathcal{A}}) - \delta(1_{\mathcal{A}}) \circ (i1_{\mathcal{A}})^* - 1_{\mathcal{A}} \circ \delta(i1_{\mathcal{A}})^* \\
=& 2\delta(i1_{\mathcal{A}}) + 2i\delta(1_{\mathcal{A}})
\end{aligned}$$

which shows that $\delta(1_{\mathcal{A}}) = 0$ and this leads to conclusion that $\delta(i1_{\mathcal{A}}) = 0$. Thus, for an arbitrary element $a \in \mathcal{A}$, we have

$$\begin{aligned}\delta(a)-\delta(a)^* = \delta(a)1_{\mathcal{A}} + a\delta(1_{\mathcal{A}}) - \delta(1_{\mathcal{A}})\circ a^* - 1_{\mathcal{A}}\circ\delta(a)^* \\ = \delta(a1_{\mathcal{A}} - 1_{\mathcal{A}}\circ a^*) = \delta(a)-\delta(a^*)\end{aligned}$$

and

$$\begin{aligned}2\delta(ia) &= \delta((i1_{\mathcal{A}})a - a\circ(i1_{\mathcal{A}})^*) \\ &= \delta(i1_{\mathcal{A}})a + (i1_{\mathcal{A}})\delta(a) - \delta(a)\circ(i1_{\mathcal{A}})^* - a\circ\delta(i1_{\mathcal{A}})^* \\ &= 2i\delta(a).\end{aligned}$$

Consequently, $\delta(a^*) = \delta(a)^*$ and $\delta(ia) = i\delta(a)$, for all element $a \in \mathcal{A}$. □

Lemma 2.11. *The mapping δ is a derivation.*

Proof. For arbitrary elements $a, b \in \mathcal{A}$, we have

$$\begin{aligned}\delta(ab) - \delta(b\circ a^*) &= \delta(ab - b\circ a^*) \\ &= \delta(a)b + a\delta(b) - \delta(b)\circ a^* - b\circ\delta(a)^*, \end{aligned} \tag{2.1}$$

by Property 2.9. Replacing a by ia in the equality (2.1), we obtain

$$\begin{aligned}\delta((ia)b) - \delta(b\circ(ia)^*) &= \delta((ia)b - b\circ(ia)^*) \\ &= \delta(ia)b + (ia)\delta(b) - \delta(b)\circ(ia)^* - b\circ\delta(ia)^*\end{aligned}$$

which leads to

$$\delta(ab) + \delta(b\circ a^*) = \delta(a)b + a\delta(b) + \delta(b)\circ a^* + b\circ\delta(a)^*, \tag{2.2}$$

by Lemma 2.10. Adding (2.1) and (2.2), we have

$$\delta(ab) = \delta(a)b + a\delta(b).$$

□

Ending the demonstration of the Theorem 1.1, we conclude that δ is an additive $*$-derivation, by Property 2.9 and the Lemmas 2.10(ii) and 2.11.

The result that follows is a direct consequence of the Theorem 1.1.

Corollary 2.12. *Let $\mathscr{H}$ be an infinite dimensional complex Hilbert space, $\mathscr{B}(\mathscr{H})$ the algebra of all bounded linear operators on $\mathscr{H}$ and $\delta : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ be a mapping of $*$-derivation-type on sum of products $ab - b\circ a^*$. Then there exists an element $t \in \mathscr{B}(\mathscr{H})$ satisfying $t + t^* = 0$ and such that $\delta(a) = at - ta$, for all element $a \in \mathscr{B}(\mathscr{H})$.*

Proof. The prove is the same as in [4, Corollary 2.1]. □

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