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Study of Commutativity Theorems in rings with involution

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Abstract The purpose of this paper is to study the commutativity of ring R with involution $*$ which admits pair of derivations satisfying certain algebraic identities. In fact certain well known problems like Herstein problem and strong commutativity preserving problem have been studied in the setting of pair of derivations in rings with involution. Finally, we give two examples to prove that various restrictions imposed in the hypotheses of our results are not superfluous.

1 Introduction

Throughout this article R will represent an associative ring with center $Z(R)$. We denote $[x, y] = xy - yx$, the commutator of x and y and $x \circ y = xy + yx$, the anti-commutator of x and y. A ring is said to 2-torsion free if $2x = 0$ (where $x \in R$) implies $x = 0$. A ring R is said to be prime if $aRb = (0)$ (where $a, b \in R$) implies either $a = 0$ or $b = 0$ and is called semiprime ring if $aRa = (0)$ (where $a \in R$) implies $a = 0$. An additive mapping $* : R \to R$ is called an involution if $*$ is an anti-automorphism of order 2; that is, $(x^*)^* = x$ for all $x \in R$. An element x in a ring with involution is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of R will be denoted by $H(R)$ and $S(R)$, respectively. A ring equipped with an involution is known as ring with involution or $*$-ring. The involution is said to be of the first kind if $Z(R) \subseteq H(R)$, otherwise it is said to be of the second kind. In the later case, $S(R) \cap Z(R) \neq (0)$. If R is 2-torsion free then every $x \in R$ can be uniquely represented in the form $2x = h + k$, where $h \in H(R)$ and $k \in S(R)$. Note that in this case x is normal i.e., $xx^* = x^*x$, if and only if h and k commute. If all elements in R are normal, then R is called a normal ring. An example is the ring of quaternions. A description of such rings can be found in [14], where further references can be found.

A derivation on R is an additive mapping $d : R \to R$ such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. A derivation d is said to be inner if there exists $a \in R$ such that $d(x) = ax - xa$ for all $x \in R$. Over the last 30 years, several authors have investigated the relationship between commutativity of the ring R and certain special types of maps on R. The first result in this direction is due to Divinsky [13], who proved that a simple artinian ring is commutative if it has a commuting non-trivial automorphism. Two years later, Posner [23] proved that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. Over the last few decades, several authors have refined and extended these results in various directions (see [4, 5, 6, 7, 10] where further references can be found).

In [15], Herstein proved that a prime ring R of characteristic not two with a nonzero derivation d satisfying $d(x)d(y) = d(y)d(x)$ for all $x, y \in R$, must be commutative. Further, Daif [11] showed that a 2-torsion free semiprime ring R admitting a derivation d such that $d(x)d(y) = d(y)d(x)$ for all $x, y \in I$, where I is a nonzero ideal of R and d is nonzero on I, then R contains a nonzero central ideal. Further this result was extended by many authors (viz [4, 16], where further references can be found). The first aim of this paper is to continue this line of study in the setting of rings with involution involving pair of derivations.

We say that a map $f : R \to R$ preserves commutativity if $[f(x), f(y)] = 0$ whenever $[x, y] = 0$ for all $x, y \in R$. The study of commutativity preserving mappings has been an active research area in matrix theory, operator theory and ring theory (see [9, 24] for references). According to [8], let S be a subset of R, a map $f : R \to R$ is said to be strong commutativity preserving (SCP) on S if $[f(x), f(y)] = [x, y]$ for all $x, y \in S$. In [6], Bell and Daif investigated the commutativity in rings admitting a derivation which is SCP on a nonzero right ideal. Precisely, they proved that if a semiprime ring R admits a derivation d satisfying $[d(x), d(y)] = [x, y]$ for all x, y in a right ideal I of R, then $I \subseteq Z(R)$. In particular, R is commutative if $I = R$. Later, Deng and Ashraf [12] proved that if there exists a derivation d of a semiprime ring R and a map $f : I \to R$ defined on a nonzero ideal I of R such that $[f(x), d(y)] = [x, y]$ for all $x, y \in I$, then R contains a nonzero central ideal. In particular, they showed that R is commutative if $I = R$. Recently, this result was extended to Lie ideals and symmetric elements of prime rings by Lin and Liu in [19] and [20], respectively. Many related generalizations of these results can be found in the literature (see for instance [10, 17, 18, 21, 22]).

Our purpose here is to continue this line of investigation by studying commutativity criteria for rings with involution admitting pair of derivations satisfying certain algebraic identities and some more identities have also been studied.

2 Main Results

In [1], it is proved that if R is a prime ring with involution of the second kind with a derivation d such that $[d(x), d(x^*)] = 0$ for all $x \in R$, then R is commutative. In the present paper our aim is to establish the generalized version of this result by considering pair of derivations.

Theorem 2.1. *Let R be a prime ring with involution $*$ of the second kind such that char$(R) \neq 2$. Let d_1 and d_2 be non-zero derivations of R such that $[d_1(x), d_2(x^*)] = 0$ for all $x \in R$. Then R is commutative.*

Proof. By the given assumption, we have

$$[d_1(x), d_2(x^*)] = 0 \tag{2.1}$$

for all $x \in R$. A linearization of (2.1) yields that

$$[d_1(x), d_2(y^*)] + [d_1(y), d_2(x^*)] = 0 \tag{2.2}$$

for all $x, y \in R$. Replacing y by $h'y$ in (2.2), where $y \in R$ and $h' \in H(R) \cap Z(R)$, We get

$$h'([d_1(x), d_2(y^*)] + [d_1(y), d_2(x^*)]) + d_2(h')[d_1(x), y^*] + d_1(h')[y, d_2(x^*)] = 0.$$

for all $x, y \in R$ and $h\prime \in H(R) \cap Z(R)$. Using (2.2), we get

$$d_2(h')[d_1(x), y^*] + d_1(h')[y, d_2(x^*)] = 0 \tag{2.3}$$

for all $x, y \in R$ and $h' \in H(R) \cap Z(R)$. Replacing y by $k'y$ in (2.3), where $y \in R$ and $k' \in S(R) \cap Z(R)$, we have

$$-d_2(h')[d_1(x), y^*k'] + d_1(h')[k'y, d_2(x^*)] = 0.$$

for all $x, y \in R$ and $h' \in H(R) \cap Z(R)$. This further implies that

$$-d_2(h')k'[d_1(x), y^*] + d_1(h')k'[y, d_2(x^*)] = 0. \tag{2.4}$$

for all $x, y \in R$. Multiplying (2.3) by k' and comparing with (2.4), we obtain

$$2d_1(h')k'[y, d_2(x^*)] = 0 \text{ for all } x, y \in R.$$

Since char $(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$, we have

$$d_1(h')[y, d_2(x^*)] = 0 \tag{2.5}$$

for all $x, y \in R$ and $h' \in H(R) \cap Z(R)$. Using the primeness of R, we get $d_1(h') = 0$ for all $h' \in H(R) \cap Z(R)$ or $[y, d_2(x^*)] = 0$ for all $x, y \in R$. Suppose $[y, d_2(x^*)] = 0$ for all $x, y \in R$. Replacing x by x^* we get $[y, d_2(x)] = 0$ for all $x, y \in R$. Thus in view Posner's Result [23], R is commutative. Now suppose $d_1(h') = 0$ for all $h' \in H(R) \cap Z(R)$. This further implies that $0 = d_1((k')^2) = 2d_1(k')k'$. Since char$(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$, we have $d_1(k') = 0$

for all $k' \in S(R) \cap Z(R)$. Since every $z \in Z(R)$ can be represented as $2z = h' + k'$ where $h' \in H(R) \cap Z(R)$ and $k' \in S(R) \cap Z(R)$, we get $d_1(Z(R)) = (0)$. Now in view of (2.4), we have

$$d_2(h')k'[d_1(x), y^*] = 0$$

for all $x, y \in R, h' \in H(R) \cap Z(R)$ and $k' \in S(R) \cap Z(R)$. Using primeness, we get either $d_2(h') = 0$ for all $h' \in H(R) \cap Z(R)$ or $[d_1(x), y^*] = 0$ for all $x, y \in R$. Replacing y by y^*, we get $[d_1(x), y]$ for all $x, y \in R$. Again using Posner's result [23], we get R is commutative. Suppose $d_2(h') = 0$ for all $h' \in H(R) \cap Z(R)$. This intern implies that $d_2(Z(R)) = (0)$. Replacing y by $-k'y$ in (2.2), we have

$$k'([d_1(x), d_2(y^*)] - [d_1(y), d_2(x^*)] = 0.$$

for all $x, y \in R$ and $k' \in S(R) \cap Z(R)$. Since $S(R) \cap Z(R) \neq (0)$, this further implies that by the primeness of R

$$[d_1(x), d_2(y^*)] - [d_1(y), d_2(x^*)] = 0 \tag{2.6}$$

for all $x, y \in R$. On Comparing (2.6) with (2.2), We get $2[d_1(x), d_2(y^*)] = 0$ for all $x, y \in R$. Hence $[d_1(x), d_2(y)] = 0$ for all $x, y \in R$. Using [3, Theorem 3.1] in the setting of $m = n = 1$, we get R is commutative. □

Corollary 2.2. *[1, Theorem 3.1] Let R be a prime ring with involution $*$ of the second kind such that char$(R) \neq 2$. Let d be a nonzero derivation of R such that $[d(x), d(x^*)] = 0$ for all $x \in R$. Then R is commutative.*

In view of our Theorem 2.1, we get a version of Herstein's result for rings with involution.

Corollary 2.3. *Let R be a prime ring with involution $*$ of the second kind such that char$(R) \neq 2$. Let d be a nonzero derivation of R such that $[d(x), d(y)] = 0$ for all $x, y \in R$. Then R is commutative.*

Theorem 2.4. *Let R be a prime ring with involution $*$ of the second kind such that char$R \neq 2$. Let d_1 and d_2 be non-zero derivations of R such that*

$$d_1(x) \circ d_2(x^*) = 0 \quad \textit{for all} \quad x \in R.$$

Then R is commutative.

Proof. By the given assumption, we have

$$d_1(x) \circ d_2(x^*) = 0 \quad \text{for all} \quad x \in R.$$

This can be further written as

$$d_1(x)d_2(x^*) + d_2(x^*)d_1(x) = 0 \text{ for all } x \in R. \tag{2.7}$$

Replacing x by $x + y$, we get

$$d_1(x)d_2(x^*) + d_1(x)d_2(y^*) + d_1(y)d_2(x^*) + d_1(y)d_2(y^*) + d_2(x^*)d_1(x)$$
$$+d_2(x^*)d_1(y) + d_2(y^*)d_1(x) + d_2(y^*)d_1(y) = 0 \text{ for all } x, y \in R.$$

Using (2.7), we arrive at

$$d_1(x)d_2(y^*) + d_1(y)d_2(x^*) + d_2(x^*)d_1(y) + d_2(y^*)d_1(x) = 0 \;\; for \;\; all \;\; x, y \in R. \tag{2.8}$$

Replace x by h' in (2.7) where $h' \in H(R) \cap Z(R)$. We get $d_1(h')d_2(h') + d_2(h')d_1(h') = 0$. This implies $2d_1(h')d_2(h') = 0$. Since char$(R) \neq 2$, we have $d_1(h')d_2(h') = 0$. Using Primeness of R, either $d_1(h') = 0$ or $d_2(h') = 0$ for all $h' \in H(R) \cap Z(R)$. First consider the case $d_1(h') = 0$ for all $h' \in H(R) \cap Z(R)$. This intern implies that $d_1(Z(R)) = (0)$. Taking $y = h'$ in (2.8), where $h' \in H(R) \cap Z(R)$. We get $2d_2(h')d_1(x) = 0$ for all $x \in R$ and $h' \in H(R) \cap Z(R)$. Since char$(R) \neq 2$, we get $d_2(h')d_1(x) = 0$. Using the primeness, we get either $d_2(h') = 0$

for all $h' \in H(R) \cap Z(R)$ or $d_1(x) = 0$ for all $x \in R$. But $d_1(x) \neq 0$ by our assumption, hence $d_2(h') = 0$ for all $h' \in H(R) \cap Z(R)$. Hence $d_2(Z(R)) = (0)$. Similarly on considering the case $d_2(h') = 0$ for all $h' \in H(R) \cap Z(R)$, we get $d_2(Z(R)) = (0)$ and $d_1(Z(R)) = (0)$. Thus in both cases we have $d_1(Z(R)) = (0)$ and $d_2(Z(R)) = (0)$. Replacing y by $k'y$ where $k' \in S(R) \cap Z(R)$ in (2.8)

$$k'(-d_1(x)d_2(y)^* + d_1(y)d_2(x^*) + d_2(x^*)d_1(y) - d_2(y)^*d_1(x)) = 0 \text{ for all } x, y \in R.$$

Using the primeness of R and the fact that $S(R) \cap Z(R) \neq (0)$, we arrive at

$$-d_1(x)d_2(y)^* + d_1(y)d_2(x^*) + d_2(x^*)d_1(y) - d_2(y)^*d_1(x) = 0 \text{ for all } x, y \in R. \tag{2.9}$$

On comparing (2.8) and (2.9), we get

$$2(d_1(y)d_2(x^*) + d_2(x^*)d_1(y)) = 0 \quad \text{for all } x, y \in R.$$

Since the char$(R) \neq 2$, we obtain

$$d_1(y)d_2(x^*) + d_2(x^*)d_1(y) = 0 \quad \text{for all } x, y \in R.$$

Replacing x by x^*, we get

$$d_1(y)d_2(x) + d_2(x)d_1(y) = 0 \quad \text{for all } x, y \in R.$$

That is,

$$d_1(y) \circ d_2(x) = 0 \quad \text{for all } x, y \in R.$$

Hence in view of [3, Theorem 3.5], R is commutative. □

Corollary 2.5. *[1, Theorem 3.3] Let R be a prime ring with involution $*$ of the second kind such that char$(R) \neq 2$. Let d be a nonzero derivation of R such that $d(x) \circ d(x^*) = 0$ for all $x \in R$. Then R is commutative.*

Corollary 2.6. *Let R be a prime ring with involution $*$ of the second kind such that char$(R) \neq 2$. Let d be a nonzero derivation of R such that $d(x) \circ d(y) = 0$ for all $x, y \in R$. Then R is commutative.*

Motivated by the notion of strong commutativity preserving derivation, in [2] author studied the more general concept by considering the identity $[d(x), d(x^*)] = [x, x^*]$. In fact they proved that if R is a prime ring involution of the second kind and admits a nonzero derivation d such that $[d(x), d(x^*)] = [x, x^*]$ for all $x \in R$, then R must be commutative. In the following theorem we study the more general case by considering pair of derivation.

Theorem 2.7. *Let R be a prime ring with involution $*$ of the second kind and with char$(R) \neq 2$. Let d_1 and d_2 be two nonzero derivations of R, such that $[d_1(x), d_2(x^*)] = \pm[x, x^*]$ for all $x \in R$. Then R is commutative.*

Proof. First consider the case

$$[d_1(x), d_2(x^*)] = [x, x^*] \text{ for all } x \in R. \tag{2.10}$$

A lineralization of (2.10) yields that

$$[d_1(x), d_2(y^*)] + [d_1(y), d_2(x^*)] = [x, y^*] + [y, x^*] \text{ for all } x, y \in R. \tag{2.11}$$

Replace y by $h'y$ in (2.11), where $h' \in H(R) \cap Z(R)$, we get

$$[d_1(x), d_2((h'y)^*)] + [d_1(h'y), d_2(x^*)] = [x, (h'y)^*] + [h'y, x^*] \tag{2.12}$$

for all $x, y \in R$ and $h' \in H(R) \cap Z(R)$. On solving, we have

$$[d_1(x), y^*]d_2(h') + [y, d_2(x^*)]d_1(h')+ \tag{2.13}$$

$$h'([d_1(x), d_2(y^*)] + [d_1(y), d_2(x^*)]) = ([x, y^*] + [y, x^*])h'$$

for all $x, y \in R$ and $h' \in H(R) \cap Z(R)$. Multiplying (2.11) by h' and adding with (2.13), we arrive at

$$[d_1(x), y^*]d_2(h') + [y, d_2(x^*)]d_1(h') = 0 \tag{2.14}$$

for all $x, y \in R$ and $h' \in H(R) \cap Z(R)$. Now (2.14) is same as (2.3) and thus following the same technique we get $d_1(Z(R) = (0)$ and $d_2(Z(R) = (0)$. Now on replacing y by $k'y$ in (2.11), where $k' \in S(R) \cap Z(R)$, we get

$$[d_1(x), d_2((k'y)^*)] + [d_1(k'y), d_2(x^*)] = [x, (k'y)^*] + [k'y, x^*] \tag{2.15}$$

for all $x, y \in R$ and $k' \in S(R) \cap Z(R)$. On solving, we have

$$-[d_1(x), d_2(y^*)]k' + k'[d_1(y), d_2(x^*)] = -[x, y^*]k' + k'[y, x^*] \tag{2.16}$$

for all $x, y \in R$ and $k' \in S(R) \cap Z(R)$. Multiplying (2.11) by k' and adding with (2.16), we obtain $2k'[d_1(y), d_2(x^*)] = 2[y, x^*]$. Since char$(R) \neq 2$ and using primeness of R we get $[d_1(y), d_2(x^*)] = [y, x^*]$ for all $x, y \in R$. Hence $[d_1(y), d_2(x)] = [y, x]$ for all $x, y \in R$. Thus in view of [12, Theorem 1], R is commutative.
Similarly we can prove the second case with some necessary variations. □

Corollary 2.8. *[2, Theorem 1] Let R be a prime ring with involution $*$ of the second kind such that char$(R) \neq 2$. Let d be a nonzero derivation of R such that $[d(x), d(x^*)] - [x, x^*] = 0$ for all $x \in R$. Then R is commutative.*

Corollary 2.9. *Let R be a prime ring with involution $*$ of the second kind such that char$(R) \neq 2$. Let d be a nonzero derivation of R such that $d(x)d(x^*) - xx^* = 0$ for all $x \in R$. Then R is commutative.*

Corollary 2.10. *Let R be a prime ring with involution $*$ of the second kind such that char$(R) \neq 2$. Let d be a nonzero derivation of R such that $[d(x), d(y)] - [x, y] = 0$ for all $x, y \in R$. Then R is commutative.*

Theorem 2.11. *Let R be a prime ring with involution $*$ of the second kind such that char$(R) \neq 2$. Let d_1 and d_2 be nonzero derivations of R, such that $d_1(x) \circ d_2(x^*) = \pm x \circ x^*$ for all $x \in R$. Then R is commutative.*

Proof. First we consider the case

$$d_1(x) \circ d_2(x^*) = x \circ x^* \text{ for all } x \in R.$$

This can be further written as

$$d_1(x)d_2(x^*) + d_2(x)d_1(x^*) = xx^* + x^*x \text{ for all } x \in R. \tag{2.17}$$

Replace x by $x + y$ in (2.17), where $x, y \in R$. we get

$$d_1(x)d_2(y^*) + d_1(y)d_2(x^*) + d_2(x^*)d_1(y) + d_2(y^*)d_1(x) = \tag{2.18}$$

$$xy^* + yx^* + x^*y + y^*x \text{ for all } x, y \in R.$$

Replacing y by $h'y$ (where $y \in R$ and $h' \in H(R) \cap Z(R)$) in (2.18) and using it, we have

$$(d_1(x)y^* + y^*d_1(x))d_2(h') + d_1(h')(yd_2(x^*) + d_2(x^*)y) = 0 \tag{2.19}$$

for all $x, y \in R$ and $h' \in H(R) \cap Z(R)$. Replacing y by $k'y$ in (2.19) where $y \in R$ and $k' \in S(R) \cap Z(R)$, we get

$$-d_1(x)y^*k' - y^*k'd_1(x)d_2(h') + d_1(h')(k'yd_2(x^*) + d_2(x^*)k'y) = 0 \tag{2.20}$$

for all $x, y \in R$, $h' \in H(R) \cap Z(R)$ and $k' \in S(R) \cap Z(R)$. Multiplying (2.19) by k' and adding with (2.20), we obtain

$$2d_1(h')k'(yd_2(x^*) + d_2(x^*)y) = 0$$

for all $x, y \in R$ and $k' \in S(R) \cap Z(R)$ and $h' \in H(R) \cap Z(R)$. Since char$(R) \neq 2$ this implies that $d_1(h')k'(yd_2(x^*) + d_2(x^*)y) = 0$ for all $x, y \in R$, $h' \in H(R) \cap Z(R)$ and $k' \in S(R) \cap Z(R)$. That is, $d_1(h')k'(y \circ d_2(x^*)) = 0$. Replacing x by x^*, we get $d_1(h')k'(y \circ d_2(x)) = 0$. By using the primeness and the fact that $S(R) \cap Z(R) \neq (0)$, we get either $d_1(h') = 0$ or $y \circ d_2(x) = 0$. Consider $y \circ d_2(x) = 0$ for all $x, y \in R$. Replace y by yu, where $u \in R$, we obtain $y(u \circ d_2(x)) - [y, d_2(x)]u = 0$ for all $x, y, u \in R$, implies that $[y, d_2(x)]u = 0$ for all $x, y, u \in R$. Taking $u \neq 0 \in Z(R)$ and applying the primeness of R, since $S(R) \cap Z(R) \neq (0)$, implies that $[y, d_2(x)] = 0$ for all $x, y \in R$. Hence in view of Posner's [23], we get R is commutative. Now consider $d_1(h') = 0$ for all $h' \in H(R) \cap Z(R)$. Hence

$$d_1(z) = 0 \text{ for all } z \in Z(R). \tag{2.21}$$

Using (2.21) in (2.19), we get $(d_1(x) \circ y^*)d_2(h') = 0$ for all $x, y \in R$ and $h' \in H(R) \cap Z(R)$. Replacing y by y^* we arrive at $(d_1(x) \circ y)d_2(h') = 0$. Now using the primeness we have either $(d_1(x) \circ y) = 0$ or $d_2(h') = 0$. Again $d_1(x) \circ y = 0$. Implies that either R is commutative or $d_2(h') = 0$ for all $h' \in H(R) \cap Z(R)$. Which intern implies that

$$d_2(z) = 0 \text{ for all } z \in Z(R). \tag{2.22}$$

Substituting $k'y$ for y in (2.18), where $y \in R$ and $k' \in S(R) \cap Z(R)$, we obtain

$$k'(-d_1(x)d_2(y^*) + d_1(y)d_2(x^*) + d_2(x^*)d_1(y) - d_2(y^*)d_1(x)) = \tag{2.23}$$
$$(-xy^* + yx^* + x^*y - y^*x)k' \text{ for all } x, y \in R \text{ and } k' \in S(R) \cap Z(R).$$

Multiplying (2.18)by k' and comparing with (2.23), we arrive at

$$k'(d_1(y)d_2(x^*) + d_2(x^*)d_1(y) - (yx^* + x^*y)) = 0 \tag{2.24}$$

for all $x, y \in R$ and $k' \in S(R) \cap Z(R)$. This can be further written as

$$k'(d_1(y) \circ d_2(x^*) - y \circ x^*) = 0 \tag{2.25}$$

for all $x, y \in R$ and $k' \in S(R) \cap Z(R)$. Replace x by x^* and interchange x by y, we have

$$k'(d_1(x) \circ d_2(y) - (x \circ y)) = 0 \tag{2.26}$$

for all $x, y \in R$ and $k' \in S(R) \cap Z(R)$. Since $S(R) \cap Z(R) \neq (0)$, we obtain

$$d_1(x) \circ d_2(y) = (x \circ y) \text{ for all } x, y \in R \tag{2.27}$$

Taking $y \neq 0 \in Z(R)$, we have $xy = 0$ for all $x \in R$ and $y \in Z(R)$, and thus $x = 0$ for all $x \in R$, which is a contradiction. Hence, we conclude that R is commutative.
Similarly we can prove the second case with some necessary variations. □

Corollary 2.12. *[2, Theorem 2] Let R be a prime ring with involution $*$ of the second kind such that char$(R) \neq 2$. Let d be a nonzero derivation of R such that $d(x) \circ d(x^*) - x \circ x^* = 0$ for all $x \in R$. Then R is commutative.*

Corollary 2.13. *Let R be a prime ring with involution $*$ of the second kind such that char$(R) \neq 2$. Let d be a nonzero derivation of R such that $d(x) \circ d(y) - x \circ y = 0$ for all $x, y \in R$. Then R is commutative.*

Remark 2.14. Following the proofs of our main results we see that if R is assumed to be a non commutative ring, then $d_1(Z(R)) = (0)$ and $d_2(Z(R)) = (0)$.

At the end, let us write an example which shows that the restriction of the second kind involution in Theorem 2.1 and Theorem 2.7 is not superfluous.

Example 2.15. Let $R = \left\{ \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \middle| \, a_1, a_2, a_3, a_4 \in \mathbb{Z} \right\}$. Of course R with matrix addition and matrix multiplication is a non commutative prime ring. Define mappings $d_1, d_2 : R \longrightarrow R$ and $* : R \longrightarrow R$ such that $d_1 \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} 0 & -a_2 \\ a_3 & 0 \end{pmatrix}$, $d_2 \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} 0 & a_2 \\ -a_3 & 0 \end{pmatrix}$, $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}^* = \begin{pmatrix} a_4 & -a_2 \\ -a_3 & a_1 \end{pmatrix}$. Obviously, $Z(R) = \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_1 \end{pmatrix} \middle| \, a_1 \in \mathbb{Z} \right\}$. Then $x^* = x$ for all $x \in Z(R)$, and hence $Z(R) \subseteq H(R)$, which shows that the involution $*$ is of the first kind. Moreover, d_1, d_2 are nonzero derivations and the following conditions hold $[d_1(x), d_2(x^*)] = 0$ and $[d_1(x), d_2(x^*)] = \pm[x, x^*]$ for all $x \in R$. However, R is not commutative. Hence, the hypothesis of second kind involution is crucial in Theorem 2.1 and Theorem 2.7.

We conclude the manuscript with the following example which reveals that Theorem 2.1 and Theorem 2.7 cannot be extended to semiprime rings.

Example 2.16. Let $S = R \times \mathbb{C}$, where R is same as in Example 2.15 with involution $*$ and derivations d_1 and d_2 same as in Example 2.15, $\mathbb{C}$ is the ring of complex numbers with conjugate involution τ. Thus, S is a non commutative semiprime ring with char$(R) \neq 2$. Now define an involution α on S, as $(x, y)^\alpha = (x^*, y^\tau)$. Clearly, α is an involution of the second kind. Further, we define the mappings d_1 and d_2 from S to S such that $D_1(x, y) = (d_1(x), 0)$ and $D_2(x, y) = (d_2(x), 0)$ for all $(x, y) \in S$. It can be easily checked that D_1 and D_2 are derivations on S and satisfying the identities of the Theorem 2.1 and Theorem 2.7 but S is not commutative. Hence, in our theorems, the hypothesis of primeness is essential.

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