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THE FEKETE-SZEGÖ PROBLEM FOR A GENERALIZED CLASS OF ANALYTIC FUNCTIONS OF COMPLEX ORDER ASSOCIATED WITH $q-$CALCULUS

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Abstract In the present investigation, by using the concept of convolution and $q-$calculus, we define a certain $q-$derivative operator for analytic functions in the open unit disk. We obtain bounds for the Fekete-Szegö functional $|a_3 - \eta a_2^2|$ for new subclasses of analytic functions of complex order by using this operator. Relevant connections of the results are briefly indicated for these subclasses.

1 Introduction

Let $\mathscr{A}$ represent the class of functions f of the form

$$f(z) = z + \sum_{\kappa=2}^{\infty} a_\kappa z^\kappa, \tag{1.1}$$

which are analytic in the open unit disk $\Delta = \{z \ : \ z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$ and satisfy the normalization condition $f(0) = f'(0) - 1 = 0$. Further, we represent by $\mathscr{S}$ the subclass of $\mathscr{A}$ consisting of functions f of the form (1.1) which are also univalent in Δ.

A function $f \in \mathscr{A}$ is said to be in the class $\mathscr{S}(\varsigma,\ \lambda)$, if it satisfy the condition

$$\Re\left(1 + \frac{1}{\varsigma}\left(\frac{zf'(z)}{\lambda z f'(z) + (1-\lambda)f(z)} - 1\right)\right) > 0, \qquad 0 \le \lambda < 1,\ \varsigma \in \mathbb{C}\setminus\{0\},\ z \in \Delta.$$

Similarly, a function $f \in \mathscr{A}$ is said to be in the class $\mathscr{C}(\varsigma,\ \lambda)$, if it satisfy the condition

$$\Re\left(1 + \frac{1}{\varsigma}\left(\frac{(1-\lambda)zf''(z)}{f'(z) + \lambda f''(z)} - 1\right)\right) > 0, \qquad 0 \le \lambda < 1,\ \varsigma \in \mathbb{C}\setminus\{0\},\ z \in \Delta.$$

It is worthy to note that

(i) $\mathscr{S}(\varsigma,\ 0) = \mathscr{S}(\varsigma)$ studied by Nasar and Aouf [12]

(ii) $\mathscr{C}(\varsigma,\ 0) = \mathscr{C}(\varsigma)$ studied by Wiatrowski [21].

(iii) $\mathscr{S}(1,\ \lambda) = \mathscr{S}(\lambda)$ studied by Altintas and Owa [4].

(iv) $\mathscr{C}(1,\ \lambda) = \mathscr{C}(\lambda)$ studied by Altintas and Owa [4].

(v) $\mathscr{S}(1,\ 0) = \mathscr{S}^*$ studied by Robertson [17].

(vi) $\mathscr{C}(1,\ 0) = \mathscr{C}$ studied by Robertson and Silverman [19].

The convolution (or Hadamard product) of two functions $f(z)$ of the form (1.1) and $g(z) = z + \sum_{\kappa=2}^{\infty} b_\kappa z^\kappa$ is defined by

$$(f * g)(z) = z + \sum_{\kappa=2}^{\infty} a_\kappa b_\kappa z^\kappa = (g * f)(z)\,. \tag{1.2}$$

In 1981, Sălăgean [18] introduced the Sălăgean derivative operator $\mathscr{D}^n$ for functions f of the form (1.1) as

$$\mathscr{D}^0 f(z) = f(z), \qquad \mathscr{D} f(z) = zf'(z), \qquad \cdots \qquad \text{and}$$
$$\mathscr{D}^n f(z) = \mathscr{D}^{n-1}\left(\mathscr{D} f(z)\right), \qquad n \in \mathbb{N} = \{1, 2, 3, \cdots\}.$$

Also, we note that

$$\mathscr{D}^n f(z) := z + \sum_{\kappa=2}^{\infty} \kappa^n a_\kappa z^\kappa, \qquad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \cdots\}.$$

In 1986, Owa et al. [14] gave important some results related with certain subclass of analytic functions introduced by Sălăgean.

Now, let the function

$$f_\nu(z) = \int_0^z \left(\frac{1+r}{1-r}\right)^\nu \frac{1}{1-r^2} dr = z + \sum_{\kappa=2}^{\infty} \varphi_\kappa(\nu) z^\kappa, \quad \nu > 0, \quad z \in \Delta,$$

where

$$\varphi_2(\nu) = \nu \qquad \textit{and} \qquad \varphi_3(\nu) = \frac{1}{3}(2\nu^2 + 1).$$

It is worthy to note that for $\nu < 1$, the function $zf'_\nu(z)$ is starlike with two slits. Moreover, since $zf'_1(z)$ is the Koebe function, all functions f_ν for $0 < \nu \leq 1$ are univalent and convex. For detailed study of the function f_ν one may refer Trimble [20].

Now, we consider the function

$$\mathcal{F}(z) = f_\nu(z) * \mathscr{D}^n f(z) = z + \sum_{\kappa=2}^{\infty} \varphi_\kappa(\nu) \kappa^n a_\kappa z^\kappa, \quad \nu > 0, \ z \in \Delta. \tag{1.3}$$

Recently, it has come to know that the concept of $q-$calculus is widely used in geometric function theory. The concept of $q-$calculus were initially introduced by Jackson [8, 9] and Purohit and Raina [16]. The $q-$number for $\kappa \in \mathbb{N}$ defined by

$$[\kappa]_q = \frac{1-q^\kappa}{1-q}, \qquad 0 < q < 1,$$

$[\kappa]_q$ can also be represented as geometric series in the following way

$$[\kappa]_q = \sum_{l=0}^{\kappa-1} q^l, \qquad \lim_{\kappa\to\infty} : [\kappa]_q = \frac{1}{1-q} \qquad \text{and} \qquad \lim_{q\to 1} : [\kappa]_q = \kappa. \tag{1.4}$$

The q-derivative operator $\mathscr{D}_q$ of a function $f \in \mathscr{S}$ is defined as

$$\mathscr{D}_q f(z) = 1 + \sum_{\kappa=2}^{\infty} [\kappa]_q a_\kappa z^{\kappa-1}. \tag{1.5}$$

For a function $f \in \mathscr{S}$, it can be easily seen that

$$\mathscr{D}_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, \qquad q \neq 1, \ z \neq 0 \tag{1.6}$$

and $(\mathscr{D}_q f)(0) = f'(0)$. If we take the function $h(z) = z^\kappa$, then the $q-$derivative of $h(z)$ is defined as

$$\mathscr{D}_q h(z) = \mathscr{D}_q z^\kappa = \frac{1-q^\kappa}{1-q} z^{\kappa-1} = [\kappa]_q z^{\kappa-1}.$$

Then

$$\lim_{q\to 1} \mathscr{D}_q h(z) = \lim_{q\to 1} [\kappa]_q z^{\kappa-1} = \kappa z^{\kappa-1} = h'(z),$$

where h' is the ordinary derivative.

By using subordination, we define q-analogue of the subclasses $\mathscr{S}(\varsigma, \lambda)$ and $\mathscr{C}(\varsigma, \lambda)$.

Let χ be an analytic function with positive real part in Δ with $\chi(0) = 1$, $\chi'(0) > 1$.

Definition 1.1. Let $0 \leq \lambda < 1$, $\varsigma \in \mathbb{C} \setminus \{0\}$, $\nu > 0$ and $n \geq 0$. Also, let $f \in \mathscr{A}$. We say that f belongs to the class $\mathscr{S}_{n,\, q}^{\nu,\, \lambda}(\varsigma,\, \chi)$, if

$$1 + \frac{1}{\varsigma}\left(\frac{z\mathscr{D}_q\mathcal{F}(z)}{\lambda z\mathscr{D}_q\mathcal{F}(z) + (1-\lambda)\mathcal{F}(z)} - 1\right) \prec \chi(z), \tag{1.7}$$

where $\mathcal{F}(z)$ is defined by (1.3).

Definition 1.2. Let $0 \leq \lambda < 1$, $\varsigma \in \mathbb{C} \setminus \{0\}$, $\nu > 0$ and $n \geq 0$. Also, let $f \in \mathscr{A}$. We say that f belongs to the class $\mathscr{C}_{n,\, q}^{\nu,\, \lambda}(\varsigma,\, \chi)$, if

$$1 + \frac{1}{\varsigma}\left(\frac{(1-\lambda)\left(\mathscr{D}_q(z\mathcal{F}'(z)) - \mathcal{F}'(z)\right)}{\lambda\mathscr{D}_q(z\mathcal{F}'(z)) + (1-\lambda)\mathcal{F}'(z)}\right) \prec \chi(z), \tag{1.8}$$

where $\mathcal{F}(z)$ is defined by (1.3).

If we take

$$\chi(z) = \frac{1+z}{1-z},$$

in Definitions 1.1 and 1.2, we have

$$\begin{aligned}\mathscr{S}_{n,\, q}^{\nu,\, \lambda}(\varsigma) &:= \mathscr{S}_{n,\, q}^{\nu,\, \lambda}\left(\varsigma,\, \frac{1+z}{1-z}\right) \\ &= \left\{f \in \mathscr{A} \;:\; \Re\left(1 + \frac{1}{\varsigma}\left(\frac{z\mathscr{D}_q\mathcal{F}(z)}{\lambda z\mathscr{D}_q\mathcal{F}(z) + (1-\lambda)\mathcal{F}(z)} - 1\right)\right) > 0\right\}\end{aligned} \tag{1.9}$$

and

$$\begin{aligned}\mathscr{C}_{n,\, q}^{\nu,\, \lambda}(\varsigma) &:= \mathscr{C}_{n,\, q}^{\nu,\, \lambda}\left(\varsigma,\, \frac{1+z}{1-z}\right) \\ &= \left\{f \in \mathscr{A} \;:\; \Re\left(1 + \frac{1}{\varsigma}\left(\frac{(1-\lambda)\left(\mathscr{D}_q(z\mathcal{F}'(z)) - \mathcal{F}'(z)\right)}{\lambda\mathscr{D}_q(z\mathcal{F}'(z)) + (1-\lambda)\mathcal{F}'(z)}\right)\right) > 0\right\}.\end{aligned} \tag{1.10}$$

It is worth mentioning that for $\lambda = 0$ and $q \to 1$, the classes $\mathscr{S}_{n,\, q}^{\nu,\, \lambda}(\varsigma, \chi)$, $\mathscr{C}_{n,\, q}^{\nu,\, \lambda}(\varsigma, \chi)$, $\mathscr{S}_{n,\, q}^{\nu,\, \lambda}(\varsigma)$ and $\mathscr{C}_{n,\, q}^{\nu,\, \lambda}(\varsigma)$ were studied by first author with Raducanu [13].

In 1933, Fekete and Szegö [6] found the maximum value of the coefficient functional

$$\Phi_\eta(f) := |a_3 - \eta a_2^2|$$

over the class $\mathscr{S}$ of univalent functions in Δ given by (1.1). By applying the Loewner method, they proved that

$$\max_{f\in\mathscr{S}} \Phi_\eta(f) = \begin{cases} 1 + 2\exp\left(\frac{-2\eta}{1-\eta}\right), & 0 \leq \eta < 1 \\ 1, & \eta = 1. \end{cases}$$

The inequality is sharp for each $\eta \in [0, 1]$. The problem of finding the maximum of $\Phi_\eta(f)$ for various subclasses of analytic functions $f \in \mathcal{A}$ with complex or real parameter η, is known as the Fekete-Szegö problem. Noteworthy contribution in this direction may be found in [1, 2, 3, 5, 7, 10, 13, 15].

In this paper, motivated with the above mentioned work, we consider the Fekete-Szegö problem for the classes $\mathscr{S}_{n,\, q}^{\nu,\, \lambda}(\varsigma,\, \chi)$, $\mathscr{C}_{n,\, q}^{\nu,\, \lambda}(\varsigma,\, \chi)$, $\mathscr{S}_{n,\, q}^{\nu,\, \lambda}(\varsigma)$ and $\mathscr{C}_{n,\, q}^{\nu,\, \lambda}(\varsigma)$.

2 The Fekete-Szegö Results

Let $\mathbb{B}$ denote the class of all analytic functions $w(z)$ in Δ with $w(0) = 0$ and $|w(z)| < 1$, $z \in \Delta$. A function f is said to be subordinate to a function g, denoted by $f \prec g$, if there exists $w \in \mathbb{B}$ such that $f(z) = g(w(z))$, $z \in \Delta$.

First, in order to prove our results, we recall the following two lemmas.

Lemma 2.1. *[11] Let $w(z) = w_1 z + w_2 z^2 + \cdots$ be in the class $\mathbb{B}$. Then, for any complex number s*

$$|w_2 - s w_1^2| \leq \max\{1; \ |s|\}. \tag{2.1}$$

The result is sharp for the function $w(z) = z^2$ or $w(z) = z$.

Lemma 2.2. *[2] Let $w(z) = w_1 z + w_2 z^2 + \cdots$ be in the class $\mathbb{B}$. Then,*

$$|w_2 - s w_1^2| \leq \begin{cases} -s & : \ s \leq -1 \\ 1 & : \ -1 \leq s \leq 1 \\ s & : s \geq 1. \end{cases} \tag{2.2}$$

For $s < -1$ or $s > 1$, equality holds if and only if $w(z) = z$ or one of its rotations. For $-1 < s < 1$, equality holds if and only if $w(z) = z^2$ or one of its rotations. Equality holds for $s = -1$ if and only if

$$w(z) = z \frac{\xi + z}{1 + \xi z}, \qquad 0 \leq \xi \leq 1$$

or one of its rotations, while for $s = 1$, equality holds if and only if

$$w(z) = -z \frac{\xi + z}{1 + \xi z}, \qquad 0 \leq \xi \leq 1$$

or one of its rotations.

In our first theorem, we find the bound for the coefficient functional $\Phi_\eta(f) = |a_3 - \eta a_2^2|$, with complex η for the function class $\mathscr{S}_{n,\,q}^{\nu,\,\lambda}(\varsigma,\ \chi)$.

Theorem 2.3. *Let $0 \leq \lambda < 1$, $\varsigma \in \mathbb{C}\backslash\{0\}$, $\nu > 0$ and $n \geq 0$. Also, let $\chi(z) = 1 + \chi_1 z + \chi_2 z^2 + \cdots$, $\chi_1 > 0$. If f of the form (1.1) is in the class $\mathscr{S}_{n,\,q}^{\nu,\,\lambda}(\varsigma,\ \chi)$, then*

$$|a_2| \leq \frac{|\varsigma|\chi_1}{2^n(1-\lambda)q\nu} \tag{2.3}$$

$$|a_3| \leq \frac{|\varsigma|\chi_1}{3^{n-1}(1-\lambda)q(1+q)(2\nu^2+1)} \max\left\{1, \ \left|\frac{1+q\lambda}{(1-\lambda)q}\varsigma\chi_1 + \frac{\chi_2}{\chi_1}\right|\right\} \tag{2.4}$$

and for $\eta \in \mathbb{C}$

$$|a_3 - \eta a_2^2| \leq \frac{|\varsigma|\chi_1 \max\left\{1, \ \left|\frac{\varsigma\chi_1}{(1-\lambda)q}\left(1 + q\lambda - \frac{3^{n-1}(2\nu^2+1)(1+q)\eta}{2^{2n}\nu^2}\right) + \frac{\chi_2}{\chi_1}\right|\right\}}{3^{n-1}(1-\lambda)q(1+q)(2\nu^2+1)}. \tag{2.5}$$

Inequalities hold if

$$\frac{z\mathscr{D}_q\mathcal{F}(z)}{\lambda z\mathscr{D}_q\mathcal{F}(z) + (1-\lambda)\mathcal{F}(z)} = 1 + \varsigma[\chi(z) - 1]$$

or

$$\frac{z\mathscr{D}_q\mathcal{F}(z)}{\lambda z\mathscr{D}_q\mathcal{F}(z) + (1-\lambda)\mathcal{F}(z)} = 1 + \varsigma[\chi(z^2) - 1],$$

where $\mathcal{F}(z)$ is given by (1.3).

Proof. If $f \in \mathscr{S}_{n,\,q}^{\nu,\,\lambda}(\varsigma,\ \chi)$ then, there exists an analytic function $w(z) = w_1 z + w_2 z^2 + \cdots$ in $\mathbb{B}$ such that

$$1 + \frac{1}{\varsigma}\left(\frac{z\mathscr{D}_q\mathcal{F}(z)}{\lambda z\mathscr{D}_q\mathcal{F}(z) + (1-\lambda)\mathcal{F}(z)} - 1\right) = \chi(w(z)), \qquad z \in \Delta. \tag{2.6}$$

From the definition of $\mathcal{F}(z)$ given in (1.3), we have

$$\mathcal{F}(z) = z + A_2 z^2 + A_3 z^3 + \cdots,$$

where

$$A_2 = 2^n \nu a_2, \qquad A_3 = 3^{n-1}(2\nu^2+1)a_3. \tag{2.7}$$

Since

$$\begin{aligned}&\frac{z\mathscr{D}_q\mathcal{F}(z)}{\lambda z\mathscr{D}_q\mathcal{F}(z)+(1-\lambda)\mathcal{F}(z)}\\ = \ & 1+(1-\lambda)\left\{[2]_q-1\right\}A_2z\\ &+\left((1-\lambda)\left\{[3]_q-1\right\}A_3-\left\{\lambda[2]_q+(1-\lambda)\right\}(1-\lambda)\left\{[2]_q-1\right\}A_2^2\right)z^2\\ &+\cdots.\end{aligned}$$

But, $[2]_q=1+q$, $[3]_q=1+q+q^2$. Then we have

$$\frac{z\mathscr{D}_q\mathcal{F}(z)}{\lambda z\mathscr{D}_q\mathcal{F}(z)+(1-\lambda)\mathcal{F}(z)}=1+(1-\lambda)qA_2z+(1-\lambda)q\left((1+q)A_3-(1+q\lambda)A_2^2\right)z^2+\cdots.$$

and

$$\chi(w(z))=1+\chi_1w_1z+(\chi_1w_2+\chi_2w_1^2)z^2+\cdots.$$

From (2.6) we have

$$\begin{aligned}A_2&=\frac{\varsigma\chi_1w_1}{(1-\lambda)q}\\ A_3&=\frac{\varsigma\chi_1}{(1-\lambda)q(1+q)}\left[w_2+\left(\frac{(1+q\lambda)}{(1-\lambda)q}\varsigma\chi_1+\frac{\chi_2}{\chi_1}\right)w_1^2\right].\end{aligned}\tag{2.8}$$

Using (2.7), we have

$$|a_2|=\frac{|\varsigma|\chi_1|w_1|}{(1-\lambda)q2^n\nu}\leq\frac{|\varsigma|\chi_1}{(1-\lambda)q2^n\nu}$$

and

$$|a_3|=\frac{|\varsigma|\chi_1}{(1-\lambda)q(q+1)3^{n-1}(2\nu^2+1)}\left|w_2-\left(-\frac{1+q\lambda}{(1-\lambda)q}\varsigma\chi_1-\frac{\chi_2}{\chi_1}\right)w_1^2\right|.\tag{2.9}$$

The inequality (2.4) follows by an application of Lemma 2.1 with

$$s=-\frac{1+q\lambda}{(1-\lambda)q}\varsigma\chi_1-\frac{\chi_2}{\chi_1}.$$

Now,

$$a_3-\eta a_2^2=\frac{\varsigma\chi_1\left\{w_2-\left[\frac{-\varsigma\chi_1}{(1-\lambda)q}\left(1+q\lambda-\frac{3^{n-1}(2\nu^2+1)(1+q)}{2^{2n}\nu^2}\eta\right)-\frac{\chi_2}{\chi_1}\right]w_1^2\right\}}{3^{n-1}(1-\lambda)q(1+q)(2\nu^2+1)}.\tag{2.10}$$

Applying Lemma 2.1 with

$$s=-\frac{\varsigma\chi_1}{(1-\lambda)q}(1+q\lambda-\frac{3^{n-1}(2\nu^2+1)(1+q)}{2^{2n}\nu^2}\eta)-\frac{\chi_2}{\chi_1},$$

we obtain inequality (2.8). Thus, the proof is completed. □

In the next theorem, we consider Fekete-Szegö problem for the class $\mathscr{S}_{n,\,q}^{\nu;\,\lambda}(\varsigma,\,\chi)$ for $\eta\in\mathbb{R}$.

Theorem 2.4. *Let* $0\leq\lambda<1$, $\varsigma>0$, $\nu>0$ *and* $n\geq0$. *Let* $\chi(z)=1+\chi_1z+\chi_2z^2+\cdots$, $\chi_1>0,\chi_2\in\mathbb{R}$. *If a function* f, *given by* (1.1) *belongs to the class* $\mathscr{S}_{n,\,q}^{\nu;\,\lambda}(\varsigma,\,\chi)$, *then for* $\eta\in\mathbb{R}$

$$|a_3-\eta a_2^2|\leq\begin{cases}\dfrac{\varsigma\chi_1\left[\frac{\varsigma\chi_1}{(1-\lambda)q}\left(1+q\lambda-\eta\frac{3^{n-1}(2\nu^2+1)(q+1)}{2^{2n}\nu^2}\right)+\frac{\chi_2}{\chi_1}\right]}{3^{n-1}(1-\lambda)q(1+q)(2\nu^2+1)} & if\ \eta\leq\rho_1\\[2ex] \dfrac{\varsigma\chi_1}{3^{n-1}(1-\lambda)q(1+q)(2\nu^2+1)} & if\ \rho_1\leq\eta\leq\rho_2\\[2ex] \dfrac{-\varsigma\chi_1\left[\frac{\varsigma\chi_1}{(1-\lambda)q}\left(1+q\lambda-\eta\frac{3^{n-1}(2\nu^2+1)(1+q)}{2^{2n}\nu^2}\right)+\frac{\chi_2}{\chi_1}\right]}{3^{n-1}(1-\lambda)q(1+q)(2\nu^2+1)} & if\ \eta\geq\rho_2,\end{cases}\tag{2.11}$$

where

$$\rho_1 = \frac{2^{2n}\nu^2}{3^{n-1}(2\nu^2+1)(1+q)}\left[1+q\lambda+\frac{(1-\lambda)q}{\varsigma\chi_1}\left(\frac{\chi_2}{\chi_1}-1\right)\right]$$

and

$$\rho_2 = \frac{2^{2n}\nu^2}{3^{n-1}(2\nu^2+1)(1+q)}\left[1+q\lambda+\frac{(1-\lambda)q}{\varsigma\chi_1}\left(\frac{\chi_2}{\chi_1}+1\right)\right].$$

For each η there exists a function in $\mathscr{S}_{n,\,q}^{\nu,\,\lambda}(\varsigma,\ \chi)$ such that equality holds.

Proof. If

$$\frac{-\varsigma\chi_1}{(1-\lambda)q}\left(1+q\lambda-\eta\frac{3^{n-1}(2\nu^2+1)(1+q)}{2^{2n}\nu^2}\right)-\frac{\chi_2}{\chi_1}\leq -1,$$

then

$$\eta \leq \frac{2^{2n}\nu^2\left[1+q\lambda+\frac{(1-\lambda)q}{\varsigma\chi_1}\left(\frac{\chi_2}{\chi_1}-1\right)\right]}{3^{n-1}(2\nu^2+1)(1+q)} \qquad (\eta\leq\rho_1).$$

Making use of (2.10) and Lemma 2.2, we have

$$|a_3-\eta a_2^2| \leq \frac{\varsigma\chi_1}{3^{n-1}(4\nu^2+2)}[\varsigma\chi_1(1-\eta\frac{3^{n-1}(2\nu^2+1)}{2^{2n-1}\nu^2})+\frac{\chi_2}{\chi_1}].$$

$$\left|a_3-\eta a_2^2\right| \leq \frac{\varsigma\chi_1\left[\frac{\varsigma\chi_1}{(1-\lambda)q}\left(1+q\lambda-\frac{3^{n-1}(2\nu^2+1)(1+q)}{2^{2n}\nu^2}\eta\right)+\frac{\chi_2}{\chi_1}\right]}{(1-\lambda)q(1+q)3^{n-1}(2\nu^2+1)}$$

for

$$-1 \leq \frac{-\varsigma\chi_1}{(1-\lambda)q}\left(1+q\lambda-\eta\frac{3^{n-1}(2\nu^2+1)}{2^{2n}\nu^2}(1+q)\right)-\frac{\chi_2}{\chi_1}\leq 1$$

we have

$$\frac{2^{2n}\nu^2}{3^{n-1}(2\nu^2+1)(1+q)}\left[1+q\lambda+\frac{(1-\lambda)q}{\varsigma\chi_1}\left(\frac{\chi_2}{\chi_1}-1\right)\right] \leq \eta \leq \frac{2^{2n}\nu^2}{3^{n-1}(2\nu^2+1)(1+q)}\left[1+q\lambda+\frac{(1-\lambda)q}{\varsigma\chi_1}\left(\frac{\chi_2}{\chi_1}+1\right)\right] \qquad (\rho_1\leq\eta\leq\rho_2)$$

and (2.10) together with Lemma 2.2 yield

$$|a_3-\eta a_2^2| \leq \frac{\varsigma\chi_1}{3^{n-1}(1-\lambda)q(1+q)(2\nu^2+1)}.$$

Finally, if

$$\frac{-\varsigma\chi_1}{(1-\lambda)q}\left(1+q\lambda-\eta\frac{3^{n-1}(2\nu^2+1)}{2^{2n}\nu^2}(1+q)\right)-\frac{\chi_2}{\chi_1}\geq 1$$

then

$$\eta \geq \frac{2^{2n}\nu^2}{3^{n-1}(2\nu^2+1)(1+q)}\left[1+q\lambda+\frac{(q-\lambda)q}{\varsigma\chi_1}\left(\frac{\chi_2}{\chi_1}+1\right)\right] \qquad (\eta\geq\rho_2).$$

It follows from (2.10) and Lemma 2.2 that

$$|a_3-\eta a_2^2| \leq \frac{-\varsigma\chi_1}{3^{n-1}(1-\lambda)q(1+q)(2\nu^2+1)}\left[\frac{\varsigma\chi_1}{(1-\lambda)q}\left(1+q\lambda-\eta\frac{3^{n-1}(2\nu^2+1)(1+q)}{2^{2n}\nu^2}\right)+\frac{\chi_2}{\chi_1}\right].$$

The sharpness of the result follows from the sharpness of inequalities in Lemma 2.2. The proof of our theorem is completed. □

If we take $\chi(z)=\frac{1+z}{1-z}$ in Theorem 2.3 and Theorem 2.4, we obtain the following results.

Corollary 2.5. *Let* $0 \le \lambda < 1, \varsigma \in \mathbb{C} \setminus \{0\}, \nu > 0$ *and* $n \ge 0$. *If* f *of the form* (1.1) *is in the class* $\mathscr{S}_{n,\,q}^{\nu,\,\lambda}(\varsigma)$, *then*

$$|a_2| \le \frac{|\varsigma|}{2^{n-1}(1-\lambda)q\nu}$$

$$|a_3| \le \frac{2|\varsigma|}{3^{n-1}(1-\lambda)q(1+q)(2\nu^2+1)} \max\left\{1, \left|\frac{2\varsigma(1+q\lambda)}{(1-\lambda)q} + 1\right|\right\}$$

and for $\eta \in \mathbb{C}$

$$|a_3 - \eta a_2^2| \le \frac{2|\varsigma| \max\left\{1, \left|\frac{2\varsigma}{(1-\lambda)q}\left(1+q\lambda - \frac{3^{n-1}(2\nu^2+1)(1+q)\eta}{2^{2n}\nu^2}\right)+1\right|\right\}}{3^{n-1}(1-\lambda)q(1+q)(2\nu^2+1)}.$$

The inequalities are sharp.

Corollary 2.6. *Let* $0 \le \lambda < 1, \varsigma > 0, \nu > 0$ *and* $n \ge 0$. *If a function* f, *given by* (1.1) *belongs to the class* $\mathscr{S}_{n,\,q}^{\nu,\,\lambda}(\varsigma)$, *then for* $\eta \in \mathbb{R}$

$$|a_3 - \eta a_2^2| \le \begin{cases} \dfrac{2\varsigma\left[\dfrac{2\varsigma}{(1-\lambda)q}\left(1+q\lambda - \eta\dfrac{3^{n-1}(2\nu^2+1)(q+1)}{2^{2n}\nu^2}\right)+1\right]}{3^{n-1}(1-\lambda)q(1+q)(2\nu^2+1)} & if\ \eta \le \rho_3 \\[2ex] \dfrac{2\varsigma}{3^{n-1}(1-\lambda)q(1+q)(2\nu^2+1)} & if\ \rho_3 \le \eta \le \rho_4 \\[2ex] \dfrac{-2\varsigma\left[\dfrac{2\varsigma}{(1-\lambda)q}\left(1+q\lambda - \eta\dfrac{3^{n-1}(2\nu^2+1)(1+q)}{2^{2n}\nu^2}\right)+1\right]}{3^{n-1}(1-\lambda)q(1+q)(2\nu^2+1)} & if\ \eta \ge \rho_4, \end{cases}$$

where

$$\rho_3 = \frac{2^{2n}\nu^2(1+q\lambda)}{3^{n-1}(2\nu^2+1)(1+q)}$$

and

$$\rho_4 = \frac{2^{2n}\nu^2}{3^{n-1}(2\nu^2+1)(1+q)}\left[1+q\lambda+\frac{(1-\lambda)q}{\varsigma}\right].$$

The result is sharp.

From the Alexander transformation, we have that $\mathcal{F}(z)$ satisfies (1.8) if and only if $z\mathcal{F}'(z)$ satisfies (1.7), where $\mathcal{F}(z)$ is given by (1.3). Consequently, we can easily obtain coefficient bounds and a solution of the Fekete-Szegö problem for the class $\mathscr{C}_{n,\,q}^{\nu,\,\lambda}(\varsigma,\,\chi)$.

Theorem 2.7. *Let* $0 \le \lambda < 1, \varsigma \in \mathbb{C}\setminus\{0\}, \nu > 0$ *and* $n \ge 0$. *Also, let* $\chi(z) = 1+\chi_1 z+\chi_2 z^2+\cdots$, $\chi_1 > 0$. *If* f *is of the form* (1.1) *is in the class* $\mathscr{C}_{n,\,q}^{\nu,\,\lambda}(\varsigma,\,\chi)$, *then*

$$|a_2| \le \frac{|\varsigma|\chi_1}{2^{n+1}(1-\lambda)q\nu}$$

$$|a_3| \le \frac{|\varsigma|\chi_1}{3^{n}(1-\lambda)q(1+q)(2\nu^2+1)} \max\left\{1, \left|\frac{(1+q\lambda)}{(1-\lambda)q}\varsigma\chi_1 + \frac{\chi_2}{\chi_1}\right|\right\}$$

and for $\eta \in \mathbb{C}$

$$|a_3 - \eta a_2^2| \le \frac{|\varsigma|\chi_1 \max\left\{1, \left|\frac{\varsigma\chi_1}{(1-\lambda)q}\left(1+q\lambda - \frac{3^{n}(2\nu^2+1)(1+q)\eta}{2^{2n+2}\nu^2}\right)+\frac{\chi_2}{\chi_1}\right|\right\}}{3^{n}(1-\lambda)q(1+q)(2\nu^2+1)}.$$

The results are sharp.

Theorem 2.8. *Let* $0 \leq \lambda < 1, \varsigma > 0, \nu > 0$ *and* $n \geq 0$. *Let* $\chi(z) = 1+\chi_1 z+\chi_2 z^2+\cdots, \chi_1 > 0$. *If a function* f*, given by* (1.1) *belongs to the class* $\mathscr{C}_{n,\,q}^{\nu,\,\lambda}(\varsigma,\, \chi)$, *then for* $\eta \in \mathbb{R}$

$$|a_3 - \eta a_2^2| \leq \begin{cases} \dfrac{\varsigma\chi_1 \left[\dfrac{\varsigma\chi_1}{(1-\lambda)q}\left(1+q\lambda-\eta\dfrac{3^n(2\nu^2+1)(q+1)}{2^{2n+2}\nu^2}\right)+\dfrac{\chi_2}{\chi_1}\right]}{3^n(1-\lambda)q(1+q)(2\nu^2+1)} & if\ \eta \leq \rho_5 \\ \dfrac{\varsigma\chi_1}{3^n(1-\lambda)q(1+q)(2\nu^2+1)} & if\ \rho_5 \leq \eta \leq \rho_6 \\ \dfrac{-\varsigma\chi_1 \left[\dfrac{\varsigma\chi_1}{(1-\lambda)q}\left(1+q\lambda-\eta\dfrac{3^n(2\nu^2+1)(1+q)}{2^{2n+2}\nu^2}\right)+\dfrac{\chi_2}{\chi_1}\right]}{3^n(1-\lambda)q(1+q)(2\nu^2+1)} & if\ \eta \geq \rho_6, \end{cases}$$

where

$$\rho_5 = \frac{2^{2n+2}\nu^2}{3^n(2\nu^2+1)(1+q)}\left[1+q\lambda+\frac{(1-\lambda)q}{\varsigma\chi_1}\left(\frac{\chi_2}{\chi_1}-1\right)\right]$$

and

$$\rho_6 = \frac{2^{2n+2}\nu^2}{3^n(2\nu^2+1)(1+q)}\left[1+q\lambda+\frac{(1-\lambda)q}{\varsigma\chi_1}\left(\frac{\chi_2}{\chi_1}+1\right)\right].$$

The result is sharp.

The proofs of Theorem 2.7 and Theorem 2.8 are much akin to those of Theorems 2.3 and 2.4, respectively and therefore we omit the details involved.

Remark 2.9. If we put $\lambda = 0$, and $q \to 1$ in Theorems 2.3 to 2.8, then we obtain the corresponding results of first author with Raducanu [13].

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