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Blending type Approximations by Kantorovich variant of α-Baskakov operators

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Abstract In this manuscript, we present a new sequence of operators, $i.e.$, Baskakov-Kantorovich operators depending on two parameters $\alpha \in [0, 1]$ and $\rho > 0$ to approximate a class of Lebesgue measurable functions on $[0.\infty)$. Next, we give basic results and discuss the rapidity of convergence and order of approximation. Further, Graphical and numerical analysis are presented. Moreover, local and global approximation properties are discussed in terms of first and second order modulus of smoothness, Peetre's K-functional and weight functions for these sequences in different spaces of functions. Lastly, $A-$Statistical approximation results are obtained.

1 Introduction

Bernstein (1912) [1] proposed the Bernstein polynomials as follows:

$$B_n(f;x) = \sum_{\nu=0}^{n} p_{n,\nu}(x) f\left(\frac{\nu}{n}\right),\ n \in \mathbb{N}, \tag{1.1}$$

where $p_{n,\nu}(x) = \binom{n}{k} x^{\nu}(1-x)^{\nu-n}$. For the operators given by (1.1), he showed that $B_n(f;x)$ converges to f uniformly where $f \in C[0, 1]$. Several researchers, $e.g.$, Mursaleen et al. ([2], [3],), Nasiruzamman et al. [6], Acar et al. ([7], [8]), Mohiuddine et al. [9, 10, 11] Ana et al. [12], İçöz et al. ([13]), [14]), Kajla et al. ([15], [16] [17] [18]) constructed new sequences of linear positive operators to investigate the rapidity of convergence and order of approximation in different functional spaces in terms of several generating functions. In the recent past, for $g \in [0, 1], m \in \mathbb{N}$ and $\alpha \in [-1, 1]$, Chen et al. [19] constructed a sequence of new linear positive operators as:

$$T_{m,\alpha}(g;y) = \sum_{i=0}^{m} g\left(\frac{i}{m}\right) p^{\alpha}_{m,i}(y) \qquad (y \in [0,1]),, \tag{1.2}$$

where $p^{(\alpha)}_{1,0} = 1 - y, \quad p^{(\alpha)}_{1,1} = y$ and

$$\begin{aligned} p^{\alpha}_{m,i}(y) \quad = \quad & \left[(1-\alpha)y\binom{m-2}{i} + (1-\alpha)(1-y)\binom{m-2}{i-2} + \alpha y(1-y)\binom{m}{i}\right] \\ & y^{i-1}(1-y)^{m-i-1} \quad (m \geq 2). \end{aligned} \tag{1.3}$$

The operators defined in (1.2) are named as $\alpha-$Berstein operator of order m.

Remark 1.1. One can not that for $\alpha = 1$, the relation (1.2) reduces to classical Bernstein operators [1].

The bivariate version of $\alpha-$ Bernstein-Durrmeyer operators were developed and investigated by Micláu s and Kajla [15] where they studied GBS operator of $\alpha-$Bernstein-Durrmeyer operators. Further, Kajla and Acar [16] proposed the classical case of these linear positive operators. While the Kantorovich variant of $\alpha-$Bernstein operators developed by Mohiuddine et al. [9].

Later, Aral and Erbay [21] introduced a parametric extension of Baskakov operators as: for every $f \in C_B[0,\infty)$ where $C_B[0,\infty)$ stands for the continuous and bounded function, we have

$$L_{n,\alpha}(f;u) = \sum_{k=0}^{\infty} \mathcal{Q}_{n,k}^{(\alpha)}(u) f\left(\frac{k}{n}\right), \tag{1.4}$$

where $n \geq 1, u \in [0,\infty)$ and

$$\begin{aligned} \mathcal{Q}_{n,k}^{(\alpha)}(u) &= \frac{u^{k-1}}{(1+u)^{n+k-1}} \left\{ \frac{\alpha u}{1+u} \binom{n+k-1}{k} - (1-\alpha)(1+u)\binom{n+k-3}{k-2} \right. \\ &+ \left. (1-\alpha)u\binom{n+k-1}{k} \right\}, \end{aligned} \tag{1.5}$$

with $\binom{n-3}{-2} = \binom{n-2}{-1} = 0$. The operators defined in (1.4) are restricted for the space of continuous functions only. To approximate in Lebesgue measurable functional spaces, Ilarslan et al. [24] gave Kanotorovich Baskakov operators based on shape parameter α as:

$$K_{n,\alpha}(f;u) = (n+1)\sum_{k=0}^{\infty} \mathcal{Q}_{n,k}^{(\alpha)}(u) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} g(s)ds, \tag{1.6}$$

where $\mathcal{Q}_{n,k}^{(\alpha)}(u)$ is given by (1.5). Motivating by the above development, we construct a new sequence of positive linear operators as follows:

$$K_{n,\alpha}^{\rho}(f;u) = \sum_{k=0}^{\infty} \mathcal{Q}_{n,k}^{(\alpha)}(u) \int_0^1 g\left(\frac{k+t^{\rho}}{n+1}\right) dt, \tag{1.7}$$

where $\rho > o$ and $\mathcal{Q}_{n,k}^{(\alpha)}(u)$ is given by (1.5).
In the subsequent sections, we investigate basic Lemmas, rate of convergence, order of approximation, locally and globally approximation results in terms of modulus of continuity, Peetre‘s K-functional, second order modulus of smoothness, Lipschitz class and Lipschitz maximul function, weighted modulus of continuity. In the last section, $A-$statistical approximation properties are expressed.

2 Basic Estimates

Lemma 2.1. *[21] Let $e_i(t) = t^i$, $i = 0,1,2$ be the test functions. Then, for the operators $L_{n,\alpha}(,;,)$ given in (1.4) , we have*

$$\begin{aligned} L_{n,\alpha}(e_0;u) &= 1, \\ L_{n,\alpha}(e_1;u) &= \left(1+\frac{2}{n}(\alpha-1)\right)u + \frac{\lambda+1}{n}, \\ L_{n,\alpha}(e_2;u) &= u^2\left(1+\frac{4\alpha-3}{n}\right) + u\left(\frac{2\lambda+3}{n} + \frac{4\alpha-4+(2\lambda+3)(\alpha-1)}{n^2}\right) \\ &+ \frac{\lambda^2+3\lambda+2}{n^2}. \end{aligned}$$

Lemma 2.2. *For the operator defined in (1.7), we have*

$$K_{n,\alpha}^{\rho}(e_0;u) = 1,$$

$$K_{n,\alpha}^{\rho}(e_1;u) = \frac{n+2(\alpha-1)}{n+1}u + \frac{(\lambda+1)(\rho+1)+1}{2(\rho+1)(n+1)},$$

$$\begin{aligned} K_{n,\alpha}^{\rho}(e_2;u) &= \left(1+\frac{4\alpha-3}{n}\right)\frac{n^2u^2}{(n+1)^2} + \frac{[(\rho+1)(n(2\lambda+3)+(\alpha-1)(2\lambda+7))]+4(\alpha-1)}{(\rho+1)(n+1)^2}u \\ &+ \frac{2n(2\rho+1)+(\lambda+1)(2\rho+1)((\lambda+2)(\rho+1)+2)+\rho+1}{(2\rho+1)(\rho+1)(n+1)^2}. \end{aligned}$$

Proof. Using Lemma 2.1, one can easily prove Lemma 2.2 □

Lemma 2.3. *Let* $e_k(s) = (e_1(s) - u)^k = \psi_u^k(s)$, $k \in \mathbb{N}$ *be the central moments of* $K_{n,\alpha}^{\rho}(.;.)$ *constructed in* (1.7). *Then*

$$K_{n,\alpha}^{\rho}((e_1(s)-u);u) = \frac{2\alpha-3}{n+1}u + \frac{(\lambda+1)(\rho+1)+1}{(\rho+1)(n+1)},$$

$$\begin{aligned} K_{n,\alpha}^{\rho}((e_1(s)-u)^2;u) &= \left[\left(1+\frac{4\alpha-3}{n}\right)\frac{n^2}{(n+1)^2} - \frac{2n+4\alpha-1}{n+1}+1\right]u^2 \\ &+ \frac{[(\rho+1)(n(2\lambda+3)+(\alpha-1)(2\lambda+7)-2(\lambda+1))]+\alpha-6}{(\rho+1)(n+1)^2}u \\ &+ \frac{2n(2\rho+1)+(\lambda+1)(2\rho+1)((\lambda+2)(\rho+1)+2)+\rho+1}{(2\rho+1)(\rho+1)(n+1)^2}. \end{aligned}$$

Proof. In the light of Lemma 2.2, we easily prove Lemma 2.3. □

3 Rate of convergence of $K_{n,\alpha}^{\rho}(.;.)$

Definition 3.1. [9]Let $f \in C[0,\infty)$. Then, modulus of continuity for a uniformly continuous function f is defined as

$$\omega(f;\delta) = \sup_{|t_1-t_2|\le\delta} |f(t_1)-f(t_2)|, \qquad t_1,t_2 \in [0,\infty).$$

For a uniformly continuous function f in $C[0,\infty)$ and $\delta > 0$, we get

$$|f(t_1)-f(t_2)| \le \left(1+\frac{(t_1-t_2)^2}{\delta^2}\right)\omega(f;\delta). \tag{3.1}$$

Theorem 3.2. *Let* $K_{n,\alpha}^{\rho}(.;.)$ *be sequence of operators proposed by* (1.7). *Then,* $K_{n,\alpha}^{\rho}$ *converges to* f *uniformly on each bounded subset of* $[0,\infty)$ *where* $f \in C[0,\infty)\bigcap\left\{f : u \ge 0, \frac{f(u)}{1+u^2}\right.$ *converges as* $\left. u \to \infty\right\}$.

Proof. To prove this result, it is adequate to prove that

$$K_{n,\alpha}^{\rho}(e_i;u) \to e_i(u), \text{ for } i \in \{0,1,2\}.$$

Using Lemma 2.2, it is clear that $K_{n,\alpha}^{\rho}(e_i;u) \to e_i(u)$ for $i = 0,1,2$ as $n \to \infty$. Hence, Theorem 3.2 is proved. □

Theorem 3.3. *(See [22]) Let* $L : C([a,b]) \to B([a,b])$ *be a linear and positive operator and let* φ_x *be the function defined by*

$$\varphi_x(t) = |t-x|, \ (x,t) \in [a,b]\times[a,b].$$

If $f \in C_B([a,b])$ *for any* $x \in [a,b]$ *and any* $\delta > 0$, *the operator* L *verifies:*

$$\begin{aligned} |(Lf)(x)-f(x)| &\le |f(x)||(Le_0)(x)-1| \\ &+ \{(Le_0)(x)+\delta^{-1}\sqrt{(Le_0)(x)(L\varphi_x^2)(x)}\}\omega_f(\delta). \end{aligned}$$

Theorem 3.4. *Let the operators* $K_{n,\alpha}^{\rho}(.;.)$ *be introduced by* (1.7) *and* $f \in C_B[0,\infty)$, *we have*

$$|K_{n,\alpha}^{\rho}(f;u)-f(u)| \le 2\omega(f;\delta),$$

where $\delta = \sqrt{K_{n,\alpha}^{\rho}(\psi_u^2;u)}$.

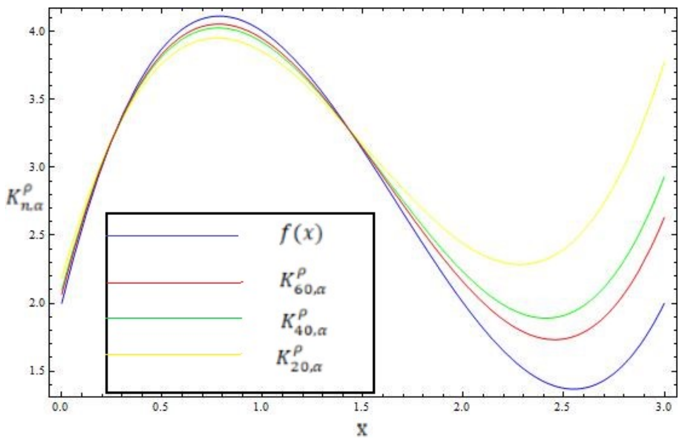

Figure 1. $K^{\rho}_{n,\alpha}(,;,)$ converges to $f(x)$ for $n = 20, 40, 60$

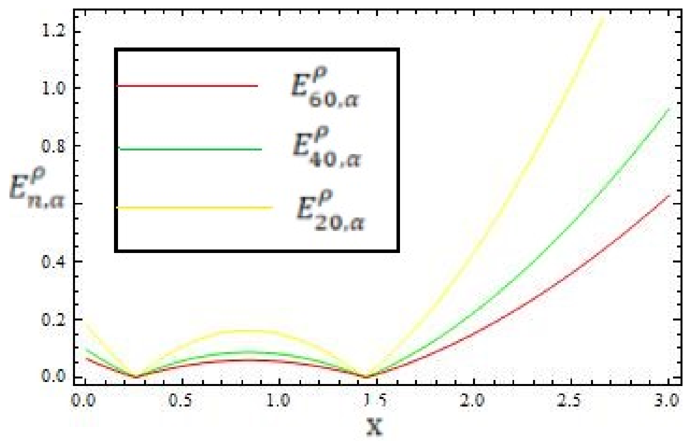

Figure 2.

Proof. In view of Theorem 3.3, Lemma 2.2 and Lemma 2.3, one has

$$|K^{\rho}_{n,\alpha}(f;u) - f(u)| \leq \{1 + \delta^{-1}\sqrt{K^{\rho}_{n,\alpha}(f;u)(\psi_u^2;u)}\}\omega(f;\delta).$$

On choosing $\delta = \sqrt{K^{\rho}_{n,\alpha}(\psi_u^2;u)}$, we completes the proof of this result. □

Example 3.5. For the values of $\rho = 0.5, \alpha = 0.9$ and $f(x) = x^3 - 5x^2 + 6x + 2$, the sequence of operators $K^{\rho}_{n,\alpha}(.;.)$ given by (1.7) converges to $f(x)$ for differen values of n as:

Example 3.6. In the graph, we investigate error $E^{\rho}_{n,\alpha}$ behaviour for the operators introduced by (1.7), $\rho = 0.5, \alpha = 0.9$ as:

Example 3.7. Here, we discuss numerical behaviour for different values of x as:

4 Pointwise Approximation results

Here, we recall some notions from [23]as: Let $C_B[0,\infty)$ be the space of real valued continuous and bounded functions equipped with the norm $\|f\| = \sup_{0\leq u<\infty} |f(u)|$. For any $f \in C_B[0,\infty)$ and $\delta > 0$, Peetre's K-functional is defined as

$$K_2(g,\delta) = inf\{\|f - h\| + \delta\|h''\| : h \in C_B^2[0,\infty)\} \tag{4.1}$$

x	$E^{\rho}_{20,\alpha}(f;x)$	$E^{\rho}_{40,\alpha}(f;x)$	$E^{\rho}_{60,\alpha}(f;x)$
0.3	0.0229166396	0.0130876655	0.0091142386
0.6	0.1364699277	0.0730024230	0.0497954983
0.9	0.1605849260	0.0860739832	0.0587509659
1.2	0.1000371450	0.0547116263	0.0375914988
1.5	0.0403979051	0.0186753674	0.0120720456
1.8	0.2559447144	0.1316777179	0.0886288103
2.1	0.5418277724	0.2818861450	0.1904679378
2.4	0.8932715689	0.4668913684	0.3159785708
2.7	1.305500594	0.6842841079	0.4635498522
3	1.773739337	0.9316550833	0.6315709244

where $C_B^2[0,\infty) = \{h \in C_B[0,\infty) : h', h'' \in C_B[0,\infty)\}$. From DeVore and Lorentz [[23], p.177, Theorem 2.4], there exists a absolute constant $C > 0$ in such a way

$$K_2(f;\delta) \leq C\omega_2(f;\sqrt{\delta}).$$

Lemma 4.1. *Consider the auxiliary operators as:*

$$\widehat{\mathcal{K}}_{n,\alpha}(f;u) = K^{\rho}_{n,\alpha}(f;u) + f(u) - f\left(\frac{n+2(\alpha-1)}{n+1}u + \frac{1}{2(n+1)}\right).$$

Then, for $f \in C_B^2[0,\infty)$ one has

$$|\widehat{\mathcal{K}}^*_{n,\alpha}(f;u) - f(u)| \leq \xi^u_n \|h''\|,$$

where

$$\xi^u_n = K^{\rho}_{n,\alpha}(\psi^2_u;u) + \left(K^{\rho}_{n,\alpha}(\psi^1_u;u)\right)^2.$$

Proof. From (4.2), we get

$$\widehat{\mathcal{K}}^*_{n,\alpha}(1;u) = 1,\ \widehat{\mathcal{K}}^*_{n,\alpha}(\psi_u;u) = 0 \text{ and } |\widehat{\mathcal{K}}^*_{n,\alpha}(f;u)| \leq 3\|f\|. \tag{4.2}$$

In the direction of Taylor's series, for $g \in C_B^2[0,\infty)$, we have

$$g(t) = g(u) + (t-u)g'(u) + \int\limits_u^t (u-v)g''(v)dv. \tag{4.3}$$

Applying operators (1.7) in (4.3) both the sides, one get

$$\widehat{\mathcal{K}}^*_{n,\alpha}(h;u) - h(u) = h'(u)\widehat{\mathcal{K}}^*_{n,\alpha}(t-u;u) + \widehat{\mathcal{K}}^*_{n,\alpha}\left(\int_u^t (t-v)h''(v)dv;u\right),$$

with the help of (4.2) and (4.3)), we get

$$\begin{aligned}
\widehat{\mathcal{K}}^*_{n,\alpha}(f;u) - h(u) &= \widehat{\mathcal{K}}^*_{n,\alpha}\left(\int_u^t (t-v)h''(v)dv;u\right)\\
&= K^{\rho}_{n,\alpha}\left(\int_u^t (t-v)h''(v)dv;u\right)\\
&- \int_u^{\frac{n+2(\alpha-1)}{n+1}u + \frac{(\lambda+1)(\rho+1)+1}{2(\rho+1)(n+1)}} \left(\frac{n+2(\alpha-1)}{n+1}u + \frac{(\lambda+1)(\rho+1)+1}{2(\rho+1)(n+1)} - v\right)\\
&g''(v)dv.
\end{aligned}$$

$$|\widehat{\mathcal{K}}^{*}_{n,\alpha}(f;u)-f(u)|\leq\left|\left(\int_u^t(t-v)h''(v)dv;u\right)\right|$$
$$+\left|\int_u^{\frac{n+2(\alpha-1)}{n+1}u+\frac{(\lambda+1)(\rho+1)+1}{2(\rho+1)(n+1)}}\left(\frac{n+2(\alpha-1)}{n+1}u+\frac{(\lambda+1)(\rho+1)+1}{2(\rho+1)(n+1)}-v\right)h''(v)dv\right|. \tag{4.4}$$

Since

$$\left|\int_u^t(t-v)h''(v)dv\right|\leq(t-v)^2\parallel h''\parallel. \tag{4.5}$$

Then

$$\left|\int_u^{K^{\rho}_{n,\alpha}(e_1;u)}\left(K^{\rho}_{n,\alpha}(e_1;u)-v\right)h''(v)dv\right|\leq\left(K^{\rho}_{n,\alpha}(t-v;u)\right)^2\parallel h''\parallel. \tag{4.6}$$

In the light of (4.4), (4.5) and (4.6), we get

$$\widehat{\mathcal{K}}^{*}_{n,\alpha}(h;u)-h(u)|\leq\xi^u_n\|h''\|,$$

□

which completes the proof of this Lemma.

Theorem 4.2. *Let $f\in C_B^2[0,\infty)$ and operators $K^{\rho}_{n,\alpha}(.;.)$ be constructed in (1.7). Then, there exists a constant $C>0$ such that*

$$|\,K^{\rho}_{n,\alpha}(f;u)-f(u)\,|\leq C\omega_2\left(f;\frac{1}{2}\sqrt{\xi^u_n}\right)+\omega(f;K^{\rho}_{n,\alpha}(\psi_u;u),$$

where ξ^u_n is defined in Lemma 4.1.

Proof. For $h\in C_B^2[0,\infty)$ and $f\in C_B[0,\infty)$ and by the definition of $\widehat{\mathcal{K}}^{*}_{n,\alpha}(.;.)$, we have

$$|K^{\rho}_{n,\alpha}(f;u)-f(u)|\leq|\widehat{\mathcal{K}}^{*}_{n,\alpha}(f-h;u)|+|(f-h)(u)|+|\widehat{\mathcal{K}}^{*}_{n,\alpha}(h;u)-h(u))|$$
$$+\left|f(K^{\rho}_{n,\alpha}(e_1;u))-g(u)\right|.$$

Lemma 4.1 and relations in (4.2), we yield

$$\begin{aligned}|K^{\rho}_{n,\alpha}(f;u)-f(u)|&\leq4\|f-h\|+|\widehat{\mathcal{K}}^{*}_{n,\alpha}(h;u)-h(u)|\\&+\left|f(K^{\rho}_{n,\alpha}(e_1;u))-g(u)\right|\\&\leq4\|f-h\|+\xi^u_n\|h''\|+\omega\Big(f;K^{\rho}_{n,\alpha}(\psi_x;u)\Big).\end{aligned}$$

With the aid of Peetre's K-functional, we have

$$|K^{\rho}_{n,\alpha}(f;u)-f(u)|\leq C\omega_2\left(f;\frac{1}{2}\sqrt{\xi^u_n}\right)+\omega(f;K^{\rho}_{n,\alpha}(\psi_x;u).$$

We arrive at the desired result. □

Here, we consider the Lipschitz type space [28] as

$$Lip_M^{k_1,k_2}(\gamma):=\left\{f\in C_B[0,\infty):|f(t)-f(u)|\leq M\frac{|t-u|^\gamma}{(t+k_1u+k_2u^2)^{\frac{\gamma}{2}}}:u,t\in(0,\infty)\right\},$$

where $M\geq 0$ is a real valued constant number, $k_1,k_2>0$, $\rho>0$ and $\gamma\in(0,1]$.

Theorem 4.3. *For $f\in Lip_M^{k_1,k_2}(\gamma)$, one yield*

$$|K_{n,\alpha}^\rho(f;u)-f(u)|\leq M\left(\frac{\eta_n^*(u)}{k_1u+k_2u^2}\right)^{\frac{\gamma}{2}}, \tag{4.7}$$

where $u>0$ and $\eta_n^*(u)=K_{n,\alpha}^\rho(\psi_u^2;u)$.

Proof. For $\gamma=1$, we have

$$\begin{aligned}|K_{n,\alpha}^\rho(f;u)-f(u)|&\leq K_{n,\alpha}^\gamma(|f(t)-f(u)|)(u)\\&\leq MK_{n,\alpha}^\rho\left(\frac{|t-u|}{(t+k_1u+k_2u^2)^{\frac{1}{2}}};u\right).\end{aligned}$$

Since $\frac{1}{t+k_1u+k_2u^2}<\frac{1}{k_1u+k_2u^2}$ for all $t,u\in(0,\infty)$, we get

$$\begin{aligned}|K_{n,\alpha}^\rho(f;u)-f(u)|&\leq\frac{M}{(k_1u+k_2u^2)^{\frac{1}{2}}}(K_{n,\alpha}^\rho((t-u)^2;u))^{\frac{1}{2}}\\&\leq M\left(\frac{\eta_n^*(u)}{k_1u+k_2u^2}\right)^{\frac{1}{2}}.\end{aligned}$$

This implies that for $\gamma=1$, this result stand good. Now, for $\gamma\in(0,1)$ and using Hölder's inequality with $p=\frac{2}{\gamma}$ and $q=\frac{2}{2-\gamma}$, one obtain

$$\begin{aligned}|K_{n,\alpha}^\rho(f;u)-f(u)|&\leq\left(K_{n,\alpha}^\rho((|f(t)-f(u)|)^{\frac{2}{\gamma}};u)\right)^{\frac{\gamma}{2}}\\&\leq M\left(K_{n,\alpha}^\rho\left(\frac{|t-u|^2}{(t+k_1u+k_2u^2)};u\right)\right)^{\frac{\gamma}{2}}.\end{aligned}$$

Since $\frac{1}{t+k_1u+k_2u^2}<\frac{1}{k_1u+k_2u^2}$ for all $t,u\in(0,\infty)$, we obtain

$$\begin{aligned}|K_{n,\alpha}^\rho(f;u)-f(u)|&\leq M\left(\frac{\mathcal{P}_n^{\mu,\nu}\left(|t-u|^2;u\right)}{k_1u+k_2u^2}\right)^{\frac{\gamma}{2}}\\&\leq M\left(\frac{\eta_n^*(u)}{k_1u+k_2u^2}\right)^{\frac{\gamma}{2}}.\end{aligned}$$

Hence, we arrive at the desired result. □

5 Global Approximations

From [26], we recall some notation to prove the global approximation results.
For the weight function $1+u^2$ and $0\leq u<\infty$, we have
$B_{1+u^2}[0,\infty)=\{f(u):|f(u)|\leq M_f(1+u^2),\ M_f$ is constant depending on $f\}$.
$C_{1+u^2}[0,\infty)\subset B_{1+u^2}[0,\infty)$ space of continuous functions endowed with the norm $\|f\|_{1+u^2}=\sup\limits_{u\in[0,\infty)}\frac{|f|}{1+u^2}$.
and

$$C_{1+u^2}^k[0,\infty)=\{f\in C_{1+u^2}:\lim_{u\to\infty}\frac{f(u)}{1+u^2}=k,\ \text{where } k \text{ is a constant}\}.$$

Theorem 5.1. *Let the $K^{\rho}_{n,\alpha}(.;.)$ be the operators given by (1.7) and $K^{\rho}_{n,\alpha}(.;.) : C^{k}_{1+u^2}[0,\infty) \to B_{1+u^2}[0,\infty)$. Then, we have*

$$\lim_{n\to\infty} \|K^{\rho}_{n,\alpha}(f;u) - f\|_{1+u^2} = 0,$$

where $f \in C^{k}_{1+u^2}[0,\infty)$.

Proof. To prove this result, it is sufficient to show that

$$\lim_{n\to\infty} \|K^{\rho}_{n,\alpha}(e_i;u) - u^i\|_{1+u^2} = 0, \; i = 0,1,2.$$

From Lemma 2.2, we get

$$\|K^{\rho}_{n,\alpha}(e_0;u) - u^0\|_{1+u^2} = \sup_{u\in[0,\infty)} \frac{|K^{\rho}_{n,\alpha}(1;u) - 1|}{1+u^2} = 0 \text{ for } i = 0.$$

For $i = 1$

$$\begin{aligned}
\|K^{\rho}_{n,\alpha}(e_1;u) - u^1\|_{1+u^2} &= \sup_{u\in[0,\infty)} \frac{\frac{n+2(\alpha-1)}{n+1}u + \frac{(\lambda+1)(\rho+1)+1}{2(\rho+1)(n+1)}}{1+u^2} \\
&= \left(\frac{n+2(\alpha-1)}{n+1} - 1\right) \sup_{u\in[0,\infty)} \frac{u}{1+u^2} \\
&+ \frac{(\lambda+1)(\rho+1)+1}{2(\rho+1)(n+1)} \sup_{u\in[0,\infty)} \frac{1}{1+u^2}.
\end{aligned}$$

Which implies that $\|K^{\rho}_{n,\alpha}(e_1;u) - u^1\|_{1+u^2} \to 0$ an $n \to \infty$.
Similarly, we see that $\|K^{\rho}_{n,\alpha}(e_2;u) - u^2\|_{1+u^2} \to 0$ as $n \to \infty$. □

Theorem 5.2. *For $f \in C^{k}_{\gamma}[0,\infty)$ and γ is positive real number. Then*

$$\lim_{n\to\infty} \sup_{u\in[0,\infty)} \frac{|K^{\rho}_{n,\alpha}(f;u) - f(u)|}{(1+u^2)^{1+\gamma}} = 0.$$

Proof. Since $|f(u)| \leq \|f\|_{\rho}(1+u^2)$, for any fixed real number $u_0 > 0$, we have

$$\begin{aligned}
\sup_{u\in[0,\infty)} \frac{|K^{\rho}_{n,\alpha}(f;u) - f(u)|}{(1+u^2)^{1+\gamma}} &\leq \sup_{u\leq u_0} \frac{|K^{\rho}_{n,\alpha}(f;u) - f(u)|}{(1+u^2)^{1+\gamma}} + \sup_{u\geq u_0} \frac{|K^{\rho}_{n,\alpha}(f;u) - f(u)|}{(1+u^2)^{1+\gamma}} \\
&\leq \|K^{\rho}_{n,\alpha}(f;u) - f(u)|\|_{C[0,u_0]} \\
&+ \|f\|_{\rho} \sup_{u\geq u_0} \frac{|K^{\rho}_{n,\alpha}(1+t^2;u)|}{(1+u^2)^{1+\gamma}} + \sup_{u\geq u_0} \frac{|f(u)|}{(1+u^2)^{1+\gamma}} \\
&= P_1 + P_2 + P_3, say. \qquad (5.1)
\end{aligned}$$

We have

$$P_3 = \sup_{u\geq u_0} \frac{|f(u)|}{(1+u^2)^{1+\gamma}} \leq \sup_{u\geq u_0} \frac{\|f\|_{\rho}(1+u^2)}{(1+u^2)^{1+\gamma}} \leq \frac{\|f\|_{\rho}}{(1+u_0^2)^{\gamma}}.$$

In view of Lemma 2.2, we have

$$\lim_{n\to\infty} \sup_{u\in[u_0,\infty)} \frac{K^{\rho}_{n,\alpha}(1+t^2;u)}{1+u^2} = 1.$$

Therefore, for arbitrary $\epsilon > 0$, there corresponds $n_1 \in \mathbb{N}$ in such a way

$$\sup_{u\in[u_0,\infty)} \frac{K^{\rho}_{n,\alpha}(1+t^2;u)}{1+u^2} \leq \frac{(1+u_0^2)^{\gamma}}{\|f\|_{\rho}} \frac{\epsilon}{3} + 1, \text{ for all } n \geq n_1.$$

Hence

$$P_2 = ||f||_{1+u^2} \sup_{u\in[u_0,\infty)} \frac{K^{\rho}_{n,\alpha}(1+t^2;u)}{(1+u^2)^{1+\gamma}} \leq \frac{||f||_{1+u^2}}{(1+u_0^2)^{\gamma}} + \frac{\epsilon}{3}, \text{ for all } n \geq n_1. \tag{5.2}$$

Therefore,

$$P_2 + P_3 < 2\frac{||f||_{1+u^2}}{(1+u^2)^{\gamma}} + \frac{\epsilon}{3}.$$

Now, choosing u_0 to be so large that $\frac{||f||_{1+u^2}}{(1+u^2)^{\gamma}} < \frac{\epsilon}{6}$, we get

$$P_2 + P_3 < \frac{2\epsilon}{3} \text{ for all } n \geq n_1. \tag{5.3}$$

From Theorem 6.1, there corresponds $n_2 > n$ such that

$$P_1 = \|K^{\rho}_{n,\alpha}(f;\cdot) - f\|_{C[0,u_0]} < \frac{\epsilon}{3} \text{ for all } n_2 \geq n. \tag{5.4}$$

Let $n_3 = \max(n_1, n_2)$. Now, using (5.1), (5.3) and (5.4), we obtain

$$\sup_{u\in[0,\infty)} \frac{|K^{\rho}_{n,\alpha}(f;u) - f(u)|}{(1+u^2)^{1+\gamma}} < \epsilon.$$

Hence, the proof of Theorem 6.2 is done. □

6 A-Statistical approximation

Gadjiev et al [27] was the first who introduces Statistical approximation theorems in operators theory. Here, we recall some notation from [27], let $A = (a_{nk})$ be a non-negative infinite suitability matrix. For a given sequence $u := (u_k)$, the A-transform of u denoted by Au : $((Au)_n)$ is defined as

$$(Au)_n = \sum_{k=1}^{\infty} a_{nk}u_k,$$

provided the series converges for each n. A is said to be regular if $\lim(Au)_n = L$ whenever $\lim u = L$. Then $u = (u_n)$ is said to be a A-statistically convergent to L i.e. $st_A - \lim u = L$ if for every $\epsilon > 0$, $\lim_n \sum_{k:|u_k - L|\geq\epsilon} a_{nk} = 0$.

Theorem 6.1. *Let $A = (a_{nk})$ be a non-negative regular suitability matrix and $u \geq 0$. Then, we have*

$$st_A - \lim_n \|K^{\rho}_{n,\alpha}(f;u) - f\|_{1+u^2} = 0, \textit{ for all } f \in C^k_{1+u^2}[0,\infty).$$

Proof. From ([25], p. 191, Th. 3), it is sufficient to show that for $\lambda_1 = 0$

$$st_A - \lim_n \|K^{\rho}_{n,\alpha}(e_i;u) - e_i\|_{1+u^2} = 0, \text{ for } i \in \{0,1,2\}. \tag{6.1}$$

From Lemma 2.2, we have

$$\begin{aligned}
\|K^{\rho}_{n,\alpha}(e_1;u) - u\|_{1+u^2} &= \sup_{u\in[0,\infty)} \frac{1}{1+u^2}\left|\frac{n+2(\alpha-1)}{n+1}u + \frac{(\lambda+1)(\rho+1)+1}{2(\rho+1)(n+1)}\right| \\
&\leq \left|\frac{n+2(\alpha-1)}{n+1}\right| \sup_{u\in[0,\infty)} \frac{u}{1+u^2} \\
&+ \left|\frac{(\lambda+1)(\rho+1)+1}{2(\rho+1)(n+1)}\right| \sup_{u\in[0,\infty)} \frac{1}{1+u^2} \\
&\leq \left|\frac{n+2(\alpha-1)}{n+1}\right| + \left|\frac{(\lambda+1)(\rho+1)+1}{2(\rho+1)(n+1)}\right|.
\end{aligned}$$

We have

$$st_A - \lim_n \left| \frac{n+2(\alpha-1)}{n+1} \right| = st_A - \lim_n \left| \frac{(\lambda+1)(\rho+1)+1}{2(\rho+1)(n+1)} \right| = 0. \tag{6.2}$$

Now, for a given $\epsilon > 0$, we define the following sets

$$N_1 := \left\{ n : \|K^{\rho}_{n,\alpha}(e_1; u) - u\| \geq \epsilon \right\},$$

$$N_2 := \left\{ n : \left| \frac{n+2(\alpha-1)}{n+1} \right| \geq \frac{\epsilon}{2} \right\},$$

$$N_3 := \left\{ n : \left| \frac{(\lambda+1)(\rho+1)+1}{2(\rho+1)(n+1)} \right| \geq \frac{\epsilon}{2} \right\}.$$

This implies that $N_1 \subseteq N_2 \cup J_3$, which shows that $\sum_{k_1 \in N_1} a_{nk_1} \leq \sum_{k_1 \in N_2} a_{nk} + \sum_{k_1 \in N_3} a_{nk}$. Hence, from (6.2) we get

$$st_A - \lim_n \|K^{\rho}_{n,\alpha}(e_1; u) - u\|_{1+u^2} = 0. \tag{6.3}$$

Similarly, one can show that

$$st_A - \lim_n \|K^{\rho}_{n,\alpha}(e_2; u) - u^2\|_{1+u^2} = 0. \tag{6.4}$$

This completes the proof of Theorem 6.1. □

Next, we deal with the rate of A-Statistical convergence in terms of Peetre's K-functional for $K^{\rho}_{n,\alpha}$.

Theorem 6.2. *Let $f \in C^2_B[0, \infty)$. Then*

$$st_A - \lim_n \|K^{\rho}_{n,\alpha}(f; \cdot) - f\|_{C_B[0,\infty)} = 0.$$

Proof. Applying the Taylor's infinite series, we have

$$f(t) = f(u) + f'(u)(t-u) + \frac{1}{2} f''(\eta)(t-u)^2,$$

where $t \leq \eta \leq u$. Operating $K^{\rho}_{n,\alpha}$, we get

$$K^{\rho}_{n,\alpha}(f; u) - f(u) = f'(u) K^{\rho}_{n,\alpha}(t-u; u) + \frac{1}{2} f''(\eta) K^{\rho}_{n,\alpha}((t-u)^2; u).$$

This implies that

$$\begin{aligned} \|K^{\rho}_{n,\alpha}(f; \cdot) - f\|_{C_B[0,\infty)} &\leq \|f'\|_{C_B[0,\infty)} \|K^{\rho}_{n,\alpha}(e_1 -, .)\|_{C_B[0,\infty)} \\ &\quad + \|f''\|_{C_B[0,\infty)} \|K^{\rho}_{n,\alpha}(e_1 -, .)^2\|_{C_B[0,\infty)} \\ &= P_1 + P_2, say. \end{aligned} \tag{6.5}$$

From (6.3) and (6.4), one has

$$\begin{aligned} \lim_n \sum_{k \in \mathbb{N}: P_1 \geq \frac{\epsilon}{2}} a_{nk} &= 0, \\ \lim_n \sum_{k \in \mathbb{N}: P_2 \geq \frac{\epsilon}{2}} a_{nk} &= 0. \end{aligned}$$

From (6.5), we have

$$\lim_n \sum_{k \in \mathbb{N}: \|K^{\rho}_{n,\alpha}(f;\cdot) - f\|_{C_B[0,\infty)} \geq \epsilon} a_{nk} \leq \lim_n \sum_{k \in \mathbb{N}: P_1 \geq \frac{\epsilon}{2}} a_{nk} + \lim_n \sum_{k \in \mathbb{N}: P_2 \geq \frac{\epsilon}{2}} a_{nk}.$$

Thus $st_A - \lim_n \|K^{\rho}_{n,\alpha}(f; \cdot) - f\|_{C_B[0,\infty)} \to 0.$ as $n \to \infty$.

The proof is completed. □

Theorem 6.3. *Let $f \in C_B^2[0,\infty)$. Then*

$$\|K_{n,\alpha}^{\rho}(f;\cdot) - f\|_{C_B[0,\infty)} \leq M\omega_2(f;\sqrt{\delta}),$$

where $\delta = \|K_{n,\alpha}^{\rho}(e_1 - \cdot;\cdot)\|_{C_B[0,\infty)} + \|K_{n,\alpha}^{\rho}((e_1 - \cdot)^2;\cdot)\|_{C_B[0,\infty)}$, *and* $\|f\|_{C_B^2[0,\infty)} = \|f\|_{C_B[0,\infty)} + \|f'\|_{C_B[0,\infty)} + \|f''\|_{C_B[0,\infty)}$.

Proof. Let $g \in C_B^2[0,\infty)$. By (6.5), one yield

$$\begin{aligned}
\|K_{n,\alpha}^{\rho}(g) - g\|_{C_B[0,\infty)} &\leq \|g'\|_{C_B[0,\infty)}\|K_{n,\alpha}^{\rho}((e_1 - \cdot);\cdot)\|_{C_B[0,\infty)} \\
&\quad + \frac{1}{2}\|g''\|_{C_B[0,\infty)}K_{n,\alpha}^{\rho}((e_1 - \cdot)^2;\cdot)\|_{C_B[0,\infty)} \\
&\leq \delta\|g\|_{C_B^2[0,\infty)}.
\end{aligned} \tag{6.6}$$

For $f \in C_B[0,\infty)$ and $g \in C_B^2$ and in the light of equation (6.6), we get

$$\begin{aligned}
\|K_{n,\alpha}^{\rho}(f;\cdot) - f\|_{C_B[0,\infty)} &\leq \|K_{n,\alpha}^{\rho}(f;\cdot) - K_{n,\alpha}^{\rho}(g;\cdot)\|_{C_B[0,\infty)} \\
&\quad + \|K_{n,\alpha}^{\rho}(g;\cdot) - g\|_{C_B[0,\infty)} + \|g - f\|_{C_B[0,\infty)} \\
&\leq 2\|g - f\|_{C_B[0,\infty)} + \|K_{n,\alpha}^{\rho}(g;\cdot) - g\|_{C_B[0,\infty)} \\
&\leq 2\|g - f\|_{C_B[0,\infty)} + \delta\|g\|_{C_B^2}.
\end{aligned}$$

By the definition of Peetre's K-functional, we get

$$\|K_{n,\alpha}^{\rho}(f;\cdot) - f\|_{C_B[0,\infty)} \leq 2K_2(f;\delta)$$

and

$$\|K_{n,\alpha}^{\rho}(f;\cdot) - f\|_{C_B[0,\infty)} \leq M\big\{\omega_2(f;\sqrt{\delta}) + \min(1,\delta)\|f\|_{C_B[0,\infty)}\big\}.$$

From (6.3), we obtain that

$$st_A - \lim_n \delta = 0, \text{ thus } st_A - \lim_n \omega(f;\sqrt{\delta}) = 0.$$

this proves the desired result. □

7 Conclusion

The motive of the present paper is to give a better error estimation of convergence of Baskakov-Kantorovich operators using two two parameters $\alpha \in [0,1]$ and $\rho > 0$. This modification gives better error estimation for a class of function in comparison with $\alpha-$Baskakov-Kanotovich operators introduced by Ilarslan et al. [24]. Graphical and numerical comparision are investigated by these sequences. We study approximation properties using Peetre's K-functional and modulus of smoothness of second order. In the last section, A-statistical approximation properties are discussed.

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