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Existence results on infinite systems of nonlinear Caputo Fractional integrodifferential inclusions for convex-compact multivalued maps

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Abstract The aim of the present paper is to discuss the existence of solutions of infinite systems for a boundary value problem of nonlinear fractional integrodifferential inclusions with integral boundary conditions involving Caputo fractional derivative for multivalued cases. Our results are based on nonlinear alternative of Leray-Schauder type. Finally, we give an illustrative example to verify the effectiveness and applicability of the main results.

1 Introduction

In this paper, we study the existence of solutions of the following infinite system of nonlinear fractional integrodifferential inclusions with integral boundary conditions (1.1) of order α $(1 < \alpha \leq 2)$ together with Caputo fractional derivative involving convex-compact multivalued mappings

$$\begin{cases} {}^cD^\alpha x_i(t) \in F_i(t, \sum_{j=i}^{i+k} x_j(t), (\vartheta \sum_{j=i}^{i+k} x_j)(t)),\ t \in [0,1], \\ ax_i(0) + bx_i'(0) = \int_0^1 q_1(x_i(s))ds,\ ax_i(1) + bx_i'(1) = \int_0^1 q_2(x_i(s))ds, \end{cases} \quad . \tag{1.1}$$

where ${}^cD^\alpha$ denotes the Caputo fractional derivative of order α, $F_i : [0,1] \times \mathbb{R} \times \mathbb{R} \to P(\mathbb{R})$ is a convex-compact L^1-Carath$\acute{e}$odory multivalued map, $i = 1, 2, \ldots$, $q_1, q_2 : \mathbb{R} \to \mathbb{R}$ are two mappings, $a > 0$, $b \geq 0$, $k \geq 1$ is a fixed natural number and for $\gamma : [0,1] \times [0,1] \to [0,\infty)$, ϑ is defined by

$$(\vartheta \sum_{j=i}^{i+k} x_j)(t) = \int_0^1 \gamma(t,s) \sum_{j=i}^{i+k} x_j(s)ds.$$

The fractional calculus, an active branch of mathematics analysis, is as old as the classical calculus which we know today. The original ideas of fractional calculus can be traced back to the end of the seventeenth century when the classical differential and integral calculus theories were created and developed by Newton and Leibniz [14]. The motivation for studying fractional differential equations comes from the fact that the theory of fractional differential equations has essentially been attracted by the enormous numbers of interesting and novel applications arising in physics, chemistry, biology, engineering, finance and other areas which have been developed in the last few decades. To focus on some applications, we refer the reader to the more recent results, e.g., works of Kilbas et al. [19] and Caponetto et al. [6] (control theory), Metzler et al. [22] (relaxation in filled polymer networks), Podlubny et al. [27] (heat propagation), Shaw et al. [29] (modeling of viscoelastic materials). They are often applied as approximation to partial differential equations, which represent much of the continuum phenomena. Many forms of these equations are possible. Some of the applications are unsteady aerodynamics and aero elastic phenomena, viscoelasticity, visco elastic panel in super sonic gas flow, fluid dynamics, electrodynamics of complex medium, many models of population growth, polymer rheology, neural network modeling, sandwich system identification, materials with fading memory, mathematical modeling of the diffusion of discrete particles in a turbulent fluid, heat conduction in materials with memory, theory of lossless transmission lines, theory of population dynamics, compartmental systems, nuclear reactors, and mathematical modeling of a hereditary phenomena. For details, see [3, 23, 26, 28] and the references therein.

Boundary value problems of fractional-order differential equations and inclusions supplemented with several kinds of conditions such as classical, nonlocal, multipoint, periodic/antiperiodic, fractional-order, and integral boundary conditions have extensively been investigated by many researchers. In particular, the study of nonlocal boundary value problems finds interesting applications in physical and chemical processes, where the classical initial/boundary conditions fail to describe some peculiar phenomena occurring inside the domain. Integral boundary conditions have various applications in applied fields such as blood flow problems, chemical engineering, thermoelasticity, underground water flow, and so forth.

Fixed point theory for multivalued mappings is an important topic of multivalued analysis and finds several applications to differential and integral inclusions, control theory and optimization. The multivalued analogue of the Schauder fixed point theorem due to Himmelberg [18] is useful for proving the existence theorems for such problems in the multivalued analysis. Several well-known fixed point theorems for single-valued mappings such as those of Banach and Schauder have been extended to multivalued mappings in Banach spaces; see, for example, the monographs of Gorniewicz et al. [4, 15]. Over the last decades, a lot of researches has been devoted to the study of the existence of common fixed points for pairs of single-valued and multivalued mappings in ordered Banach spaces [11, 12, 13, 20].

Infinite systems of differential equations find numerous applications in describing of several real world problems, e.g. neural nets, branching processes, dissociation of polymers and a lot of others. Recently several existence results have been studied with the help of fixed point theory, see [7, 8, 9, 17, 24, 25, 30] etc.

The paper is organized as follows. Section 2 contains some related preliminary materials from fractional calculus and multivalued analysis. In Section 3, we prove the existence result for convex-compact multivalued maps involved in the problem (1.1) by applying nonlinear alternative of Lerray-Schauder type. Finally, we give an example illustrating the obtained result.

2 Preliminaries

Let us recall some essential definitions and auxiliary facts in fractional calculus and multivalued maps which play a key role in the forthcoming analysis. For more details, we refer to [10, 15, 19, 26].

Definition 2.1. ([19, 26]). The fractional derivative of order α for a continuously differentiable function $f : [0, \infty) \to \mathbb{R}$ in the sense of Caputo is defined as

$$^{c}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds, \quad \alpha > 0,$$

where Γ is the gamma function defined by $\Gamma(r) = \int_0^\infty t^{r-1} e^{-t} dt$ and $n = [\alpha] + 1$ ($[\alpha]$ denotes the integer part of the real number α).

Lemma 2.2. *([1]). For any $f \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, q_1, $q_2 \in C([0,1], \mathbb{R})$, the unique solution of the boundary value problem*

$$\begin{cases} {}^{c}D^{\alpha} x(t) = f(t, x(t), (\vartheta x)(t)),\ t \in [0,1],\ (1 < \alpha \leq 2), \\ ax(0) + bx'(0) = \int_0^1 q_1(x(s)) ds,\ ax(1) + bx'(1) = \int_0^1 q_2(x(s)) ds, \end{cases}$$

is given by

$$\begin{aligned} x(t) &= \int_0^t \left(\frac{a(t-s)^{\alpha-1} + (b-at)(1-s)^{\alpha-1}}{a\Gamma(\alpha)} + \frac{b(b-at)(1-s)^{\alpha-2}}{a^2\Gamma(\alpha-1)} \right) f(s, x(s), (\vartheta x)(s)) ds \\ &\quad + \int_t^1 \left(\frac{(b-at)(1-s)^{\alpha-1}}{a\Gamma(\alpha)} + \frac{b(b-at)(1-s)^{\alpha-2}}{a^2\Gamma(\alpha-1)} \right) f(s, x(s), (\vartheta x)(s)) ds \\ &\quad + \frac{1}{a^2} \left((a(1-t)+b) \int_0^1 q_1(x(s)) ds + (b+at) \int_0^1 q_2(x(s)) ds \right), \end{aligned}$$

where $a > 0$, $b \geq 0$, $\gamma : [0,1] \times [0,1] \to [0,\infty)$ *and* ϑ *is defined by*

$$(\vartheta x)(t) = \int_0^t \gamma(t,s)x(s)ds.$$

Throughout this paper, unless otherwise mentioned, let E be a Banach space and let $P(E)$ or 2^E denote the class of all subsets of E. We denote the set

$$P_k(E) = \{A \subset E, A \text{ is nonempty and has a property } k\}.$$

Here, k may be the property k =closed (in short cl), or k =compact (in short cp), or k =convex (in short cv), or k =bounded (in short bd) etc. Also, $P_{cp,cv}(E)$ denotes the class of all compact and convex subsets of E.
A correspondence $F : E \to P_k(E)$ is called a multivalued operator or multivalued mapping on E into E. The multivalued map F is convex (closed) valued if $F(x)$ is convex (closed) for all $x \in E$. A point $x \in E$ is called a fixed point of F if $x \in Fx$. F is bounded on bounded sets if $F(B) = \bigcup_{x\in B} F(x)$ is bounded in E for any bounded set B of E (*i.e.*, $\sup_{x\in B} \big\{ \sup\{\|y\| : y \in F(x)\}\big\} < \infty$).
A function $H_d : P_k(E) \times P_k(E) \to \mathbb{R}_+$ defined by

$$H_d(A,B) = \max\big\{ \sup_{a\in A} D(a,B), \sup_{b\in B} D(b,A)\big\}$$

satisfies all the conditions of a metric on $P_k(E)$ and is called a Hausdorff-Pompeiu metric on E, where $D(a,B) = \inf\{\|a-b\| : b \in B\}$. It is known that the hyperspace $(P_{cl}(E), H_d)$ is a generalized metric space (see [5]).

Definition 2.3. ([2]). Let E be a separable Banach space, X be a nonempty subset of E, and $F : X \to P_k(E)$ a multivalued map. We say that:
(1) F is upper semi-continuous (u.s.c) on X if for each $x \in X$ the set Fx is a nonempty, closed subset of X and if, for each open subset U of X containing $F(x)$, there exists an open neighborhood V of x such that $F(V) \subset U$.
(2) F is said to be completely continuous if $F(B)$ is relatively compact for every bounded set B of X.
(3) If F is upper semi-continuous, it is said to be condensing map if, for any subset $B \subset X$ with $\chi(B) \neq 0$, we have

$$\chi(F(B)) < \chi(B),$$

where χ denotes the Kuratowski measure of noncompactness.

Note that a completely continuous multivalued map is a condensing map.
If $F : E \to P_k(E)$ is a multivalued operator, then graph of the operator F is defined by

$$Gr(F) = \{(x,y) \in E \times E \mid y \in Fx\}.$$

It is known that if the multivalued map F is completely continuous with nonempty compact values, then F is upper semi-continuous if and only if F has a closed graph [2] (*i.e.*, $x_n \to x$ and $y_n \to y$ s.t $y_n \in F(x_n) \Rightarrow y \in F(x)$).
Let J be a compact real interval, and $F : J \times \mathbb{R} \times \mathbb{R} \to P(\mathbb{R})$ be a multivalued map with nonempty and compact values. Assign to F the multivalued operator $S_{F,\gamma} : C(J,\mathbb{R}) \to P(L^1(J,\mathbb{R}))$ by letting

$$S_{F,\gamma}(y) = \big\{ f \in L^1(J,\mathbb{R}) : f(t) \in F(t, y(t), (\vartheta y)(t)) \text{ a.e. for each } t \in J \big\},$$

where $\gamma : [0,1] \times [0,1] \to [0,\infty)$ is a mapping and ϑ is defined by $(\vartheta y)(t) = \int_0^1 \gamma(t,s)y(s)ds$. The operator $S_{F,\gamma}$ is called the Niemytzki operator associated with F.

Definition 2.4. ([10]). A multivalued map $F : [0,T] \times \mathbb{R} \times \mathbb{R} \to P(\mathbb{R})$ is said to be Carathéodory if

(i) $t \to F(t,x,y)$ is measurable for each $x, y \in \mathbb{R}$;
(ii) $(x,y) \to F(t,x,y)$ is upper semi-continuous for almost all $t \in [0,T]$.
Further a Carathéodory function F is called L^1-Carathéodory if
(iii) For each $k > 0$, there exists $h_k \in L^1([0,T], \mathbb{R}_+)$ such that

$$\|F(t,x,y)\| = \sup\{|v|, v \in F(t,x,y)\} \leq h_k(t)$$

for all $|x|, |y| \leq k$ and for *a.e.* $t \in [0,T]$.

Definition 2.5. If $x_i \in C([0,1], \mathbb{R})$ $(i = 1, 2, \ldots)$ with its Caputo derivative of order α $(1 < \alpha \leq 2)$, existing on $[0,1]$ then $x = (x_i)$ is a solution of the problem (1.1) if for each $i \in \mathbb{N}$ there exists a function $f_i \in L^1([0,1], \mathbb{R})$ such that $f_i(t) \in F_i(t, \sum_{j=i}^{i+k} x_j(t), (\vartheta \sum_{j=i}^{i+k} x_j)(t))$ *a.e.* on $[0,1]$ and

$$\begin{aligned} x_i(t) \quad = \quad & \int_0^t \left(\frac{a(t-s)^{\alpha-1} + (b-at)(1-s)^{\alpha-1}}{a\Gamma(\alpha)} + \frac{b(b-at)(1-s)^{\alpha-2}}{a^2\Gamma(\alpha-1)} \right) f_i(s)ds \\ & + \int_t^1 \left(\frac{(b-at)(1-s)^{\alpha-1}}{a\Gamma(\alpha)} + \frac{b(b-at)(1-s)^{\alpha-2}}{a^2\Gamma(\alpha-1)} \right) f_i(s)ds \\ & + \frac{1}{a^2} \left((a(1-t)+b) \int_0^1 q_1(x_i(s))ds + (b+at) \int_0^1 q_2(x_i(s))ds \right). \end{aligned}$$

In the sequel, we use the following important lemmata. They play a crucial role in the proof of the main results.

Lemma 2.6. *([21]). Let I be a compact real interval. If $dim(E) < \infty$ and $F : I \times E \times E \to P(E)$ is compact and convex, then $S_{F,\gamma}(y) \neq \emptyset$ for all $y \in E$.*

Lemma 2.7. *([21]). Let I be a compact real interval. Let $F : I \times E \times E \to P(E)$ be an L^1-Carathéodory multivalued mapping with nonempty, compact and convex values, and $S_{F,\gamma} \neq \emptyset$, then for any linear continuous mapping $\Lambda : L^1(I,E) \to C(I,E)$, the convex compact multifunction $\Lambda o S_{F,\gamma} : C(I,E) \to P(C(I,E))$ has a closed graph.*

Lemma 2.8. *(Nonlinear alternative for Kakutani maps [16]). Let E be a Banach space, C be a closed convex subset of E, U be an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to P_{cp,cv}(C)$ is an upper semi-continuous compact map. Then either*
(i) F has a fixed point in $\overline{U}$, or
(ii) there exist $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.

3 Main results

In this section, we investigate the solvability of infinite system of nonlinear Caputo fractional integrodifferential inclusions (1.1). Also, we give an illustrative example to verify the effectiveness and applicability of our results.
We assume that the following conditions are satisfied:
(A_1) $F_i : [0,1] \times \mathbb{R} \times \mathbb{R} \to P_{cp,cv}(\mathbb{R})$ is an L^1-Carathéodory multivalued map with nonempty values.
(A_2) For each $r > 0$ there exist functions $m_{i,r} \in L^1([0,1], \mathbb{R}_+)$, $i = 1, 2, \ldots$ and there exist continuous increasing functions $\Psi_{i,j} : [0, \infty) \to (0, \infty)$, $i = 1, 2, \ldots$ $j = 1, 2$ such that for each $(t,x,y) \in [0,1] \times \mathbb{R} \times \mathbb{R}$

$$\|F_i(t,x,y)\| = \sup\{|v| : v \in F_i(t,x,y)\} \leq m_{i,r}(t)[\Psi_{i,1}(|x|) + \Psi_{i,2}(|y|)], \ \sup_{i \in \mathbb{N}} \|m_{i,r}\|_{L^1} < \infty,$$

where $\sup_{i \in \mathbb{N}} \Psi_{i,1}(r) < \infty$ and $\sup_{i \in \mathbb{N}} \Psi_{i,2}(\vartheta(r)) < \infty$.
(A_3) There exist continuous increasing functions $\phi_{i,j} : [0, \infty) \to (0, \infty)$ $(i = 1, 2, \ldots, j = 1, 2)$ and $p_j \in L^1([0,1], \mathbb{R}_+)$, $j = 1, 2$ such that $(\phi_{i,j})_{i=1}^{\infty}$ are point-wise bounded and for each $(t, (x_i)) \in [0,1] \times (C([0,1], \mathbb{R}))^{\infty}$,

$$|q_j(x_i(t))| \leq p_j(t) \phi_{i,j}(\sup_{i \in \mathbb{N}} \sup_{t \in [0,1]} \sum_{j=i}^{i+k} |x_j(t)|).$$

(A_4) There exists a positive real number M, such that

$$\frac{M}{\Psi_{i,1}(\sup\limits_{i\in\mathbb{N}}\sup\limits_{t\in[0,1]}\sum\limits_{j=i}^{i+k}|x_j(t)|)+\Psi_{i,2}\big(\zeta((x_i))\big)+\frac{a+b}{a^2}\big(\eta((x_i))\big)}>1,$$

where $\zeta((x_i)) := |\vartheta\sup\limits_{i\in\mathbb{N}}\sup\limits_{t\in[0,1]}\sum\limits_{j=i}^{i+k} x_j(t)|\big(\frac{a+2b}{a\Gamma(\alpha)}+\frac{2b^2}{a^2\Gamma(\alpha-1)}\|m_{i,r}\|_{L^1}\big)$ and

$$\eta((x_i)) := \phi_{i,1}(\sup_{i\in\mathbb{N}}\sup_{t\in[0,1]}\sum_{j=i}^{i+k}|x_j(t)|)\|p_1\|_{L^1}+\phi_{i,2}(\sup_{i\in\mathbb{N}}\sup_{t\in[0,1]}\sum_{j=i}^{i+k}|x_j(t)|)\|p_2\|_{L^1}.$$

Theorem 3.1. *Suppose that the assumptions (A_1)-(A_4) are satisfied. Then the infinite system of Caputo fractional integrodifferential inclusions with integral boundary conditions (1.1) has at least one solution on $[0,1]$.*

Proof. By Lemma 2.6, for $y=(y_i)\in(C([0,1],\mathbb{R}))^\infty$ $S_{F_i,\gamma}(\sum\limits_{j=i}^{i+k} y_j)$ is nonempty. Let us transform the problem into a fixed point problem. Consider the multivalued map $\Theta : (C([0,1],\mathbb{R}))^\infty \to P((C([0,1],\mathbb{R}))^\infty)$ defined as follows. For $y=(y_i)\in(C([0,1],\mathbb{R}))^\infty$, $\Theta(y)$ is the set of all $(x_i)\in(C([0,1],\mathbb{R}))^\infty$ such that

$$\begin{aligned} x_i(t) &= \int_0^t\Big(\frac{a(t-s)^{\alpha-1}+(b-at)(1-s)^{\alpha-1}}{a\Gamma(\alpha)}+\frac{b(b-at)(1-s)^{\alpha-2}}{a^2\Gamma(\alpha-1)}\Big)g_i(s)ds\\ &+\int_t^1\Big(\frac{(b-at)(1-s)^{\alpha-1}}{a\Gamma(\alpha)}+\frac{b(b-at)(1-s)^{\alpha-2}}{a^2\Gamma(\alpha-1)}\Big)g_i(s)ds\\ &+\frac{1}{a^2}\Big((a(1-t)+b)\int_0^1 q_1(y_i(s))ds+(b+at)\int_0^1 q_2(y_i(s))ds\Big), \end{aligned}$$

where $g_i\in S_{F_i,\gamma}(\sum\limits_{j=i}^{i+k} y_j)$. We show that Θ satisfies the assumptions of Lemma 2.8. The proof will be given in several steps:

Step 1: For each $y=(y_i)\in(C([0,1],\mathbb{R}))^\infty$, $\Theta(y)$ is convex. Let $(h_\ell^i)\in\Theta(y)$ for $\ell=1,2$, then

$$\begin{aligned} h_\ell^i(t) &= \int_0^t\Big(\frac{a(t-s)^{\alpha-1}+(b-at)(1-s)^{\alpha-1}}{a\Gamma(\alpha)}+\frac{b(b-at)(1-s)^{\alpha-2}}{a^2\Gamma(\alpha-1)}\Big)g_\ell^i(s)ds\\ &+\int_t^1\Big(\frac{(b-at)(1-s)^{\alpha-1}}{a\Gamma(\alpha)}+\frac{b(b-at)(1-s)^{\alpha-2}}{a^2\Gamma(\alpha-1)}\Big)g_\ell^i(s)ds\\ &+\frac{1}{a^2}\Big((a(1-t)+b)\int_0^1 q_1(y_i(s))ds+(b+at)\int_0^1 q_2(y_i(s))ds\Big), \end{aligned}$$

where $g_\ell^i\in S_{F_i,\gamma}(\sum\limits_{j=i}^{i+k} y_j)$, for $\ell=1,2$ and $i=1,2,\dots$. Let $0\le\beta\le1$. For all $t\in[0,1]$ we have

$$\begin{aligned} &\beta h_1^i+(1-\beta)h_2^i(t)\\ &= \int_0^t\Big(\frac{a(t-s)^{\alpha-1}+(b-at)(1-s)^{\alpha-1}}{a\Gamma(\alpha)}+\frac{b(b-at)(1-s)^{\alpha-2}}{a^2\Gamma(\alpha-1)}\Big)(\beta g_1^i(s)+(1-\beta)g_2^i(s))ds\\ &+\int_t^1\Big(\frac{(b-at)(1-s)^{\alpha-1}}{a\Gamma(\alpha)}+\frac{b(b-at)(1-s)^{\alpha-2}}{a^2\Gamma(\alpha-1)}\Big)(\beta g_1^i(s)+(1-\beta)g_2^i(s))ds\\ &+\frac{1}{a^2}\Big((a(1-t)+b)\int_0^1 q_1(y_i(s))ds+(b+at)\int_0^1 q_2(y_i(s))ds\Big). \end{aligned}$$

The set $S_{F_i,\gamma}(\sum\limits_{j=i}^{i+k} y_j)$ is convex because F_i is convex. Hence $(\beta h_1^i+(1-\beta)h_2^i)\in\Theta(y)$.

Step 2: We show that Θ maps bounded sets (balls) into bounded sets in $(C([0,1],\mathbb{R}))^\infty$. For a positive number r, let $B_r = \{y = (y_i) \in (C([0,1],\mathbb{R}))^\infty : \sup_{i\in\mathbb{N}} \sup_{t\in[0,1]} \sum_{j=i}^{i+k} |y_j(t)| \le r\}$. Then, for each $h_r = (h_{i,r}) \in \Theta(y)$, $y = (y_i) \in B_r$, there exists a function $g_i \in S_{F_i,\gamma}(\sum_{j=i}^{i+k} y_j)$ such that

$$\begin{aligned} h_{i,r}(t) &= \int_0^t \Big(\frac{a(t-s)^{\alpha-1} + (b-at)(1-s)^{\alpha-1}}{a\Gamma(\alpha)} + \frac{b(b-at)(1-s)^{\alpha-2}}{a^2\Gamma(\alpha-1)}\Big) g_i(s)ds \\ &\quad + \int_t^1 \Big(\frac{(b-at)(1-s)^{\alpha-1}}{a\Gamma(\alpha)} + \frac{b(b-at)(1-s)^{\alpha-2}}{a^2\Gamma(\alpha-1)}\Big) g_i(s)ds \\ &\quad + \frac{1}{a^2}\Big((a(1-t)+b)\int_0^1 q_1(y(s))ds + (b+at)\int_0^1 q_2(y(s))ds\Big), \end{aligned}$$

where $g_i \in S_{F_i,\gamma}(\sum_{j=i}^{i+k} y_j)$. In view of (A_2) and (A_3) we obtain

$$\begin{aligned} |h_{i,r}(t)| &\le \Psi_{i,1}(\sup_{i\in\mathbb{N}}\sup_{t\in[0,1]} \sum_{j=i}^{i+k} |y_j(t)|) + \Psi_{i,2}(|\vartheta(\sup_{i\in\mathbb{N}}\sup_{t\in[0,1]} \sum_{j=i}^{i+k} |y_j(t)|)|)\big(\frac{a+2b}{a\Gamma(\alpha)} + \frac{2b^2}{a^2\Gamma(\alpha-1)}\|m_{i,r}\|_{L^1}\big) \\ &\quad + \phi_{i,1}(\sup_{i\in\mathbb{N}}\sup_{t\in[0,1]} \sum_{j=i}^{i+k} |y_j(t)|)\frac{a+b}{a^2}\|p_1\|_{L^1} + \phi_{i,2}(\sup_{i\in\mathbb{N}}\sup_{t\in[0,1]} \sum_{j=i}^{i+k} |y_j(t)|)\frac{a+b}{a^2}\|p_2\|_{L^1}. \end{aligned}$$

Thus,

$$\begin{aligned} \sup_{i\in\mathbb{N}}\sup_{t\in[0,1]} \sum_{j=i}^{i+k} |h_{j,r}(t)| &\le (\Psi_{i,1}(r) + \Psi_{i,2}(\vartheta(r)))\big(\frac{a+2b}{a\Gamma(\alpha)} + \frac{2b^2}{a^2\Gamma(\alpha-1)}\|m_{i,r}\|_{L^1}\big) \\ &\quad + \phi_{i,1}(r)\frac{a+b}{a^2}\|p_1\|_{L^1} + \phi_{i,2}(r)\frac{a+b}{a^2}\|p_2\|_{L^1}. \end{aligned}$$

Step 3: Θ sends bounded sets of $(C([0,1],\mathbb{R}))^\infty$ into equicontinuous sets. For this purpose, let $t_1, t_2 \in [0,1]$ be such that $t_1 < t_2$ and $y = (y_i) \in B_r$, for each $h = (h_i) \in \Theta(y)$ we can write

$$\begin{aligned} &|\pi_i h(t_2) - \pi_i h(t_1)| \\ &\le \big(\Psi_{i,1}(\sup_{i\in\mathbb{N}}\sup_{t\in[0,1]} \sum_{j=i}^{i+k} |y_j(t)|) + \Psi_{i,2}(|\vartheta(\sup_{i\in\mathbb{N}}\sup_{t\in[0,1]} \sum_{j=i}^{i+k} y_j(t)|)\big) \times \\ &\quad \Big|\int_{t_1}^{t_2} \Big(\frac{a(t_2-s)^{\alpha-1} + (b-at_2)(1-s)^{\alpha-1}}{a\Gamma(\alpha)} + \frac{b(b-at_2)(1-s)^{\alpha-2}}{a^2\Gamma(\alpha-1)}\Big) m_{i,r}(s)ds\Big| \\ &\quad + \big(\Psi_{i,1}(\sup_{i\in\mathbb{N}}\sup_{t\in[0,1]} \sum_{j=i}^{i+k} |y_j(t)|) + \Psi_{i,2}(|\vartheta(\sup_{i\in\mathbb{N}}\sup_{t\in[0,1]} \sum_{j=i}^{i+k} |y_j(t)|)\big) \times \\ &\quad \Big|\int_0^{t_1} \Big(\frac{a((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}) + a(t_1-t_2)(1-s)^{\alpha-1}}{a\Gamma(\alpha)} + \frac{ba(t_1-t_2)(1-s)^{\alpha-2}}{a^2\Gamma(\alpha-1)}\Big) m_{i,r}(s)ds\Big| \\ &\quad + \big(\Psi_{i,1}(\sup_{i\in\mathbb{N}}\sup_{t\in[0,1]} \sum_{j=i}^{i+k} |y_j(t)|) + \Psi_{i,2}(|\vartheta(\sup_{i\in\mathbb{N}}\sup_{t\in[0,1]} \sum_{j=i}^{i+k} y_j(t))|\big) \times \\ &\quad \Big|\int_{t_2}^1 \Big(\frac{a(t_1-t_2)(1-s)^{\alpha-1}}{a\Gamma(\alpha)} + \frac{ba(t_1-t_2)(1-s)^{\alpha-2}}{a^2\Gamma(\alpha-1)}\Big) m_{i,r}(s)ds\Big| \\ &\quad + \big(\Psi_{i,1}(\sup_{i\in\mathbb{N}}\sup_{t\in[0,1]} \sum_{j=i}^{i+k} |y_j(t)|) + \Psi_{i,2}(|\vartheta(\sup_{i\in\mathbb{N}}\sup_{t\in[0,1]} \sum_{j=i}^{i+k} y_j(t))|)\big) \times \\ &\quad \Big|\int_{t_1}^{t_2} \Big(\frac{(b-at_1)(1-s)^{\alpha-1}}{a\Gamma(\alpha)} + \frac{b(b-at_1)(1-s)^{\alpha-2}}{a^2\Gamma(\alpha-1)}\Big) m_{i,r}(s)ds\Big| \\ &\quad + \frac{|t_1-t_2|}{a}\Big|\phi_{i,1}(\sup_{i\in\mathbb{N}}\sup_{t\in[0,1]} \sum_{j=i}^{i+k} |y_j(t)|)\int_0^1 p_1(s)ds + \phi_{i,2}(\sup_{i\in\mathbb{N}}\sup_{t\in[0,1]} \sum_{j=i}^{i+k} |y_j(t)|)\int_0^1 p_2(s)ds\Big|, \end{aligned}$$

where $g_i \in S_{F_i,\gamma}(\sum_{j=i}^{i+k} y_j)$, $y = (y_i) \in B_r$. The right-hand side of the above inequality converges to 0 as $t_1 \to t_2$. Now, by Steps (2), and (3) combined with the Arzelá-Ascoli Theorem, we conclude that Θ is completely continuous.

Step 4: We show that Θ has a closed graph. Let $(y_n) = (y_n^m)_{m=1}^\infty$ be a sequence in $(C([0,1],\mathbb{R}))^\infty$ converging to $y = (y_i)$ and consider a sequence $(h_n) = (h_n^m)_{m=1}^\infty$ such that $(h_n^m)_{m=1}^\infty \in \Theta((y_n^m)_{m=1}^\infty)$ converges to $h = (h^m)_{m=1}^\infty$. We prove that $h \in \Theta(y)$. We have

$$\begin{aligned} h_n(t) = \big(h_n^m(t)\big)_{m=1}^\infty &= \Big(\int_0^t \Big(\frac{a(t-s)^{\alpha-1} + (b-at)(1-s)^{\alpha-1}}{a\Gamma(\alpha)} + \frac{b(b-at)(1-s)^{\alpha-2}}{a^2\Gamma(\alpha-1)}\Big) g_n^m(s)ds \\ &+ \int_t^1 \Big(\frac{(b-at)(1-s)^{\alpha-1}}{a\Gamma(\alpha)} + \frac{b(b-at)(1-s)^{\alpha-2}}{a^2\Gamma(\alpha-1)}\Big) g_n^m(s)ds \\ &+ \frac{1}{a^2}\Big((a(1-t)+b)\int_0^1 q_1(y_n^m(s))ds + (b+at)\int_0^1 q_2(y_n^m(s))ds\Big)\Big)_{m=1}^\infty, \end{aligned}$$

where $g_n^m \in S_{F_m,\gamma}(\sum_{j=m}^{m+k} y_m^j)$.

Now, we define the continuous linear operator $\Lambda : L^1([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ by

$$\begin{aligned} g \to \Lambda(g)(t) &= \int_0^t \Big(\frac{a(t-s)^{\alpha-1} + (b-at)(1-s)^{\alpha-1}}{a\Gamma(\alpha)} + \frac{b(b-at)(1-s)^{\alpha-2}}{a^2\Gamma(\alpha-1)}\Big) g(s)ds \\ &+ \int_t^1 \Big(\frac{(b-at)(1-s)^{\alpha-1}}{a\Gamma(\alpha)} + \frac{b(b-at)(1-s)^{\alpha-2}}{a^2\Gamma(\alpha-1)}\Big) g(s)ds \\ &+ \frac{1}{a^2}\Big((a(1-t)+b)\int_0^1 q_1(y(s))ds + (b+at)\int_0^1 q_2(y(s))ds\Big). \end{aligned}$$

We have $h_n^m \in \Lambda o S_{F_m,\gamma}(\sum_{j=m}^{m+k} y_m^j)$. From Lemma 2.7, $\Lambda o S_{F_m,\gamma}$ has a closed graph, then $h_n^m \in \Lambda o S_{F,\gamma}(y)$. Thus, there exists $g^m \in S_{F_m,\gamma}(\sum_{j=m}^{m+k} y_m^j)$ such that $h^m(t) = \Lambda(g^m)(t)$ or equivalently $(h^m)_{m=1}^\infty = (\Lambda(g^m))_{m=1}^\infty \in N(y) = N((y_m)_{m=1}^\infty)$. It implies that $h = (h^m)_{m=1}^\infty \in \Theta(y)$. Consequently, Θ is upper semi-continuous.

If there exists $\lambda \in (0,1)$ such that $x = (x_i) \in \lambda\Theta(y)$, then there exists $g_i \in L^1([0,1],\mathbb{R})$ with $g_i \in S_{F_i,\gamma}(\sum_{j=i}^{i+k} y_j)$ such that for each $t \in [0,1]$ we have

$$\begin{aligned} x_i(t) &= \lambda^{-1}\Big[\int_0^t \Big(\frac{a(t-s)^{\alpha-1} + (b-at)(1-s)^{\alpha-1}}{a\Gamma(\alpha)} + \frac{b(b-at)(1-s)^{\alpha-2}}{a^2\Gamma(\alpha-1)}\Big) g_i(s)ds \\ &+ \int_t^1 \Big(\frac{(b-at)(1-s)^{\alpha-1}}{a\Gamma(\alpha)} + \frac{b(b-at)(1-s)^{\alpha-2}}{a^2\Gamma(\alpha-1)}\Big) g_i(s)ds \\ &+ \frac{1}{a^2}\Big((a(1-t)+b)\int_0^1 q_1(y_i(s))ds + (b+at)\int_0^1 q_2(y_i(s))ds\Big)\Big]. \end{aligned}$$

Taking into account condition (A_2), and using the computations in the second step for each $t \in [0,1]$, we obtain

$$\sup_{i\in\mathbb{N}} \sup_{t\in[0,1]} \sum_{j=i}^{i+k} |x_j(t)| \le \lambda^{-1}\Big[\Psi_{i,1}(\sup_{i\in\mathbb{N}} \sup_{t\in[0,1]} \sum_{j=i}^{i+k} |y_j(t)|) + \Psi_{i,2}(\zeta((y_i))) + \frac{a+b}{a^2}\eta((y_i))\Big].$$

So, we deduce that

$$\frac{\sup_{i\in\mathbb{N}} \sup_{t\in[0,1]} \sum_{j=i}^{i+k} |x_j(t)|}{\Psi_{i,1}(\sup_{i\in\mathbb{N}} \sup_{t\in[0,1]} \sum_{j=i}^{i+k} |y_j(t)|) + \Psi_{i,2}(\zeta((y_i))) + \frac{a+b}{a^2}\eta((y_i))} \le 1$$

In view of the condition (A_4), there exists M such that $\sup_{i\in\mathbb{N}} \sup_{t\in[0,1]} \sum_{j=i}^{i+k} |x_j(t)| \neq M$. Let us set

$$U = \{x = (x_i) \in (C([0,1],\mathbb{R}))^\infty : \sup_{i\in\mathbb{N}} \sup_{t\in[0,1]} \sum_{j=i}^{i+k} |x_j(t)| < M+1\}.$$

From the choice of U, there is no $x = (x_i) \in \partial U$ such that $x = (x_i) \in \lambda\Theta(y)$ for some $\lambda \in (0,1)$. Note that the operator $\Psi : \overline{U} \to P((C([0,1],\mathbb{R}))^\infty)$ is upper semi-continuous and completely continuous. Consequently, by the nonlinear alternative of Leray-Schauder type, we deduce that Θ has a fixed point $x = (x_i) \in \overline{U}$. That is a solution of the Caputo fractional differential inclusion (1.1). □

Further, we provide an example illustrating our considerations.

Example 3.2. Consider the Caputo fractional differential inclusion

$$ {}^cD^{\frac{3}{2}} x_i(t) \in \big[1, \frac{e^{-2t}}{\sqrt{7+t}}(|\sin(\sum_{j=i}^{i+k} x_j(t))| + |\cos((\vartheta \sum_{j=i}^{i+k} x_j)(t))|) + \frac{2}{\sqrt{7}}\big],\ t \in [0,1] \tag{3.1} $$

with the integral boundary value conditions

$$ \frac{1}{2}x_i(0) + \frac{1}{4}x_i'(0) = \int_0^1 \frac{ds}{2(1+|x_i(s)|)^2}, \quad \frac{1}{2}x_i(1) + \frac{1}{4}x_i'(1) = \int_0^1 \frac{ds}{3(1+e^{2|x_i(s)|})}, $$

where $x = (x_i) \in (C([0,1],\mathbb{R}))^\infty$, k is a natural number and ϑ is defined by $(\vartheta x)(t) = \int_0^1 \frac{e^t+s}{3} x_i(s)ds$.
Put $\alpha = \frac{3}{2}$, $a = \frac{1}{2}$, $b = \frac{1}{4}$, $\gamma(t,s) = \frac{e^t+s}{3}$ and $F_i : [0,1] \times \mathbb{R} \times \mathbb{R} \to P(\mathbb{R})$ defined by

$$ F_i(t, \sum_{j=i}^{i+k} x_j(t), (\vartheta \sum_{j=i}^{i+k} x_j)(t)) = \big[1, \frac{e^{-2t}}{\sqrt{7+t}}(|\sin(\sum_{j=i}^{i+k} x_j(t))| + |\cos((\vartheta \sum_{j=i}^{i+k} x_j)(t))|) + \frac{2}{\sqrt{7}}\big] $$

for each $(x_i), (y_i) \in (C([0,1],\mathbb{R}))^\infty$ and $t \in [0,1]$. Note that,

$$ \|F_i(t, \sum_{j=i}^{i+k} x_j(t), (\vartheta \sum_{j=i}^{i+k} x_j)(t))\| = \sup\big\{|v|, v \in F_i(t, \sum_{j=i}^{i+k} x_j(t), (\vartheta \sum_{j=i}^{i+k} x_j)(t))\big\} \leq \frac{4}{\sqrt{7}}, $$

with $m_{i,r}(t) = \frac{1}{\sqrt{7}}$ $(r > 0)$, and $\psi_{i,1}(s) = \psi_{i,2}(s) = 2$. Here

$$ |q_1(x_i(t))| = |\frac{1}{2(1+|x_i(t)|)^2}| \leq \frac{1}{2},\ p_1(t) = \frac{1}{2}, \quad \phi_{i,1}(s) = 1, $$

$$ |q_2(x_i(t)| = |\frac{1}{3(1+e^{2|x_i(t)|})}| \leq \frac{1}{6},\ p_2(t) = \frac{1}{2}, \quad \phi_{i,2}(s) = \frac{1}{3}. $$

Take $\zeta((x_i)) = |(\vartheta \sup_{i\in\mathbb{N}} \sup_{t\in[0,1]} \sum_{j=i}^{i+k} x_j(t))|(\frac{a+2b}{a\Gamma(\alpha)} + \frac{2b^2}{a^2\Gamma(\alpha-1)}\|m_{i,r}\|_{L^1})$ and

$\eta((x_i)) = \phi_{i,1}(\sup_{i\in\mathbb{N}} \sup_{t\in[0,1]} \sum_{j=i}^{i+k} |x_j(t)|)\|p_1\|_{L^1} + \phi_{i,2}(\sup_{i\in\mathbb{N}} \sup_{t\in[0,1]} \sum_{j=i}^{i+k} |x_j(t)|)\|p_2\|_{L^1}$. In view of the condition

$$ \frac{M}{\Psi_{i,1}(\sup_{i\in\mathbb{N}} \sup_{t\in[0,1]} \sum_{j=i}^{i+k} |x_j(t)|) + \Psi_{i,2}(\zeta((x_i))) + \frac{a+b}{a^2}(\eta((x_i)))} > 1, $$

we find that $M \geq 12$. Thus, all the conditions of Theorem 3.1 are satisfied. So, the infinite system of nonlinear fractional integrodifferential inclusion (3.1) has at least one solution.

4 Conclusion

We believe, up to now, the authors have not investigated the solvability of fractional differential inclusions with integral boundary conditions for the case of the infinite systems in the multivalued case (inclusions). For further works, we suggest that the numerical methods can be applied for finding approximate solutions and error diagrams for infinite systems of fractional differential inclusions.

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