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Asymptotic behavior of semi-canonical third-order nonlinear functional differential equations

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Abstract Asymptotic properties of solutions of the third-order functional differential equation

$$\mathcal{L}y(t) + f(t)y^{\beta}(\varphi(t)) = 0,$$

where $\mathcal{L}y(t) = (p(t)(q(t)y'(t))')'$ is a semi-canonical differential operator, are studied. The main goal is to transform the semi-canonical operator into canonical form; which simplifies the investigation of the oscillatory properties of solutions. Examples are established to illustrate the significance of the results obtained.

1 Introduction

The present paper investigates the asymptotic behavior of solutions of the semi-canonical third-order functional differential equation

$$\mathcal{L}y(t) + f(t)y^{\beta}(\varphi(t)) = 0, \quad t \geq t_0 > 0, \tag{1.1}$$

where $\mathcal{L}$ is the differential operator as below

$$\mathcal{L}y(t) = (p(t)(q(t)y'(t))')', \tag{1.2}$$

and β is the ratio of positive odd integers.

Throughout the paper, and without further mention, the following terms will be accepted:

(i) $q \in C^{(2)}([t_0,\infty),(0,\infty))$, $p \in C^1([t_0,\infty),(0,\infty))$, and $f \in C([t_0,\infty),(0,\infty))$;

(ii) $\varphi \in C^1([t_0,\infty),\mathbb{R})$, $\varphi'(t) \geq 0$, and $\lim_{t\to\infty} \varphi(t) = \infty$;

(iii) The operator $\mathcal{L}$ is in semi-canonical form, namely,

$$\int_{t_0}^{\infty} \frac{1}{p(t)} dt = \infty \ \text{ and } \ \int_{t_0}^{\infty} \frac{1}{q(t)} dt < \infty. \tag{1.3}$$

Recall that a solution of (1.1) is a nontrivial real-valued function y satisfying (1.1) for $t \geq t_y$ for some $t_y \geq t_0$ such that $y \in C^1([t_y,\infty),\mathbb{R})$, $qy' \in C^1([t_y,\infty),\mathbb{R})$, and $p(qy')' \in C^1([t_y,\infty),\mathbb{R})$. Solutions vanishing identically in some neighborhood of infinity will be excluded from our consideration and we suppose that (1.1) possesses such solutions. Such a solution $y(t)$ of (1.1) is said to be *oscillatory* if it has arbitrarily large zeros on $[t_y,\infty)$; otherwise it is called *non-oscillatory*.

In the past several years many papers have appeared in the literature dealing with the oscillatory and asymptotic behavior of solutions of (1.1), see, for example ([1]–[31]) and the references cited therein. However, most of the papers are dedicated to canonical type equations, that is,

$$\int_{t_0}^{\infty} \frac{1}{p(t)} dt = \int_{t_0}^{\infty} \frac{1}{q(t)} dt = \infty.$$

This is due to the fact that the investigation of oscillatory properties of canonical type equations is much simpler than non-canonical ones. In this paper, we introduce a technique to convert the semi-canonical equation (1.1) to a canonical equation, and then we obtain some novel criteria for the oscillatory and asymptotic behavior of solutions of (1.1).

When considering nonoscillatory solution of (1.1), we may restrict our attention only to positive ones; since if $y(t)$ is a solution of (1.1), then $-y(t)$ is also a solution. It follows from a well-known result in [21, 24] that the positive solutions of semi-canonical equation (1.1) belong to the following class:

Lemma 1.1. *Assume that y is an eventually positive solution of (1.1) satisfying (1.3). Then there exists $t_1 \in [t_0, \infty)$ such that y satisfies one of the following three cases:*

(I) $y'(t) > 0, \quad (q(t)y'(t))' > 0, \quad (p(t)(q(t)y'(t))')' < 0,$

(II) $y'(t) < 0, \quad (q(t)y'(t))' > 0, \quad (p(t)(q(t)y'(t))')' < 0,$

(III) $y'(t) < 0, \quad (q(t)y'(t))' < 0, \quad (p(t)(q(t)y'(t))')' < 0,$

for $t \geq t_1$.

Therefore, from the above lemma, it is clear that if we wish to obtain oscillation criteria for a semi-canonical equation (1.1), we have to eliminate three above mentioned cases. To overcome this, we present a simple condition that converts equation (1.1) to a canonical form that will simplify the analysis of its solutions. In the process, we consider delay, advanced and ordinary differential equations.

2 Main Results

The following symbols will be used to facilitate readability:

$$Q(t) := \int_t^\infty \frac{1}{q(s)} ds, \quad b(t) := q(t)Q^2(t), \quad a(t) := \frac{p(t)}{Q(t)},$$

$$F(t) := Q^\beta(\varphi(t))f(t), \quad \mu(t) := \int_{t_1}^t \frac{1}{a(s)} ds, \text{ and } \eta(t) := \int_{t_1}^t \frac{\mu(s)}{b(s)} ds$$

for $t \geq t_1$ for some $t_1 \geq t_0$.

Theorem 2.1. *Assume that*

$$\int_{t_0}^\infty \frac{1}{a(t)} dt = \infty. \tag{2.1}$$

Then the semi-canonical operator $\mathcal{L}$ has the unique canonical representation as below

$$\mathcal{L}y(t) = \left(\frac{p(t)}{Q(t)} \left(q(t)Q^2(t) \left(\frac{y(t)}{Q(t)}\right)'\right)'\right)'. \tag{2.2}$$

Proof. Direct calculation shows that

$$\begin{aligned} \frac{p(t)}{Q(t)} \left(q(t)Q^2(t) \left(\frac{y(t)}{Q(t)}\right)'\right)' &= \frac{p(t)}{Q(t)} \left[Q(t)q(t)y'(t) + y(t)\right]' \\ &= \frac{p(t)}{Q(t)} \left[Q(t)(q(t)y'(t))'\right] \\ &= p(t)(q(t)y'(t))'. \end{aligned}$$

Therefore

$$\left(\frac{p(t)}{Q(t)} \left(q(t)Q^2(t) \left(\frac{y(t)}{Q(t)}\right)'\right)'\right)' = (p(t)(q(t)y'(t))')'.$$

From (2.1) we observe that

$$\int_{t_0}^\infty \frac{Q(t)}{p(t)} dt = \infty,$$

and since

$$\int_{t_0}^{\infty} \frac{1}{q(t)Q^2(t)}dt = \lim_{t\to\infty}\left(\frac{1}{Q(t)} - \frac{1}{Q(t_0)}\right) = \infty,$$

we deduce that (2.2) is in canonical form and from [27] this canonical form is unique. □

From Theorem 2.1, it follows that under condition (2.1), equation (1.1) takes the equivalent form

$$\left(a(t)\left(b(t)\left(\frac{y(t)}{Q(t)}\right)'\right)'\right)' + f(t)y^{\beta}(\varphi(t)) = 0.$$

Letting $z(t) = \frac{y(t)}{Q(t)}$ and using the notation $F(t)$, we immediately arrive at the following conclusion.

Theorem 2.2. *Let (2.1) be satisfied. Then semi-canonical differential equation (1.1) possesses a solution $y(t)$ if and only if the canonical equation*

$$(a(t)(b(t)z'(t))')' + F(t)z^{\beta}(\varphi(t)) = 0 \tag{2.3}$$

has the solution $z(t) = \frac{y(t)}{Q(t)}$.

Corollary 2.3. *Let (2.1) holds. Then semi-canonical differential equation (1.1) has an eventually positive solution if and only if canonical equation (2.3) has an eventually positive solution.*

Corollary 2.3 clearly simplifies examination of (1.1) since for (2.3), and so we are concerned with only two classes of eventually positive solutions, i.e, either

$$z(t) > 0,\ \ b(t)z'(t) < 0,\ \ a(t)(b(t)z'(t))' > 0,\ \ (a(t)(b(t)z'(t))')' < 0$$

and in this case we say $z \in \mathcal{N}_0$, or

$$z(t) > 0,\ \ b(t)z'(t) > 0,\ \ a(t)(b(t)z'(t))' > 0,\ \ (a(t)(b(t)z'(t))')' < 0$$

then we say $z \in \mathcal{N}_2$.

Theorem 2.4. *Let $\varphi(t) \le t$, $\beta > 1$, and (2.1) hold. Assume that*

$$\int_{t_0}^{\infty} \frac{1}{b(s)}\int_{s}^{\infty} \frac{1}{a(v)}\int_{v}^{\infty} F(u)du dv ds = \infty \tag{2.4}$$

and

$$\limsup_{t\to\infty}\left(\frac{1}{\mu^{\beta}(\varphi(t))}\int_{t_0}^{\varphi(t)} F(s)\eta^{\beta}(\varphi(s))\mu(s)ds + \frac{1}{\mu^{\beta-1}(\varphi(t))}\int_{\varphi(t)}^{t} F(s)\eta^{\beta}(\varphi(s))ds + \mu(\varphi(t))\int_{t}^{\infty}\frac{F(s)\eta^{\beta}(\varphi(s))}{\mu^{\beta}(\varphi(s))}ds\right) = \infty. \tag{2.5}$$

Then every non-oscillatory solution $y(t)$ of (1.1) satisfies $\lim_{t\to\infty}\frac{y(t)}{Q(t)} = 0$.

Proof. Let $y(t)$ be a non-oscillatory solution of equation (1.1), say $y(t) > 0$, and $y(\varphi(t)) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$. Then, from Corollary 2.3, the corresponding function $z(t) = \frac{y(t)}{Q(t)}$ is a positive solution of (2.3) and so either $z \in \mathcal{N}_0$ or $z \in \mathcal{N}_2$ for $t \ge t_1$.

Let's consider the case when $z \in \mathcal{N}_2$. Then, we observe that

$$b(t)z'(t) \ge \int_{t_1}^{t} a^{-1}(s)a(s)(b(s)z'(s))'ds \ge a(t)(b(t)z'(t))'\mu(t),$$

which implies that

$$\left(\frac{b(t)z'(t)}{\mu(t)}\right)' \le 0. \tag{2.6}$$

It follows from (2.6) that

$$z(t) \geq \int_{t_1}^{t} \frac{b(s)z'(s)}{\mu(s)} \frac{\mu(s)}{b(s)} ds \geq \frac{b(t)z'(t)}{\mu(t)} \eta(t).$$

From this and (2.3), we observe that $x(t) = b(t)z'(t)$ is a positive increasing solution of the retarded differential inequality

$$(a(t)x'(t))' + \frac{F(t)\eta^{\beta}(\varphi(t))}{\mu^{\beta}(\varphi(t))} x^{\beta}(\varphi(t)) \leq 0, \tag{2.7}$$

and also from (2.6) we deduce that $x(t)/\mu(t)$ is nonincreasing. An integration of (2.7) from t to ∞ gives

$$x'(t) \geq \frac{1}{a(t)} \int_{t}^{\infty} \frac{F(s)\eta^{\beta}(\varphi(s))}{\mu^{\beta}(\varphi(s))} x^{\beta}(\varphi(s)) ds,$$

from which, we see that

$$\begin{aligned} x(t) \geq \int_{t_1}^{t} \frac{1}{a(s)} \int_{s}^{\infty} \frac{F(u)\eta^{\beta}(\varphi(u))}{\mu^{\beta}(\varphi(u))} x^{\beta}(\varphi(u)) du ds \\ = \int_{t_1}^{t} \frac{1}{a(s)} \int_{s}^{t} \frac{F(u)\eta^{\beta}(\varphi(u))}{\mu^{\beta}(\varphi(u))} x^{\beta}(\varphi(u)) du ds \\ + \int_{t_1}^{t} \frac{1}{a(s)} \int_{t}^{\infty} \frac{F(u)\eta^{\beta}(\varphi(u))}{\mu^{\beta}(\varphi(u))} x^{\beta}(\varphi(u)) du ds \\ = \int_{t_1}^{t} \frac{F(s)\eta^{\beta}(\varphi(s))\mu(s)}{\mu^{\beta}(\varphi(s))} x^{\beta}(\varphi(s)) ds \\ + \mu(t) \int_{t}^{\infty} \frac{F(s)\eta^{\beta}(\varphi(s))}{\mu^{\beta}(\varphi(s))} x^{\beta}(\varphi(s)) ds. \end{aligned} \tag{2.8}$$

Hence,

$$\begin{aligned} x(\varphi(t)) \geq \int_{t_1}^{\varphi(t)} \frac{F(s)\eta^{\beta}(\varphi(s))\mu(s)}{\mu^{\beta}(\varphi(s))} x^{\beta}(\varphi(s)) ds + \mu(\varphi(t)) \int_{\varphi(t)}^{t} \frac{F(s)\eta^{\beta}(\varphi(s))}{\mu^{\beta}(\varphi(s))} x^{\beta}(\varphi(s)) ds \\ + \mu(\varphi(t)) \int_{t}^{\infty} \frac{F(s)\eta^{\beta}(\varphi(s))}{\mu^{\beta}(\varphi(s))} x^{\beta}(\varphi(s)) ds. \end{aligned}$$

Using the monotonicity properties of $x(t)$ and $x(t)/\mu(t)$, we obtain

$$\begin{aligned} x(\varphi(t)) \geq \frac{x^{\beta}(\varphi(t))}{\mu^{\beta}(\varphi(t))} \int_{t_1}^{\varphi(t)} F(s)\eta^{\beta}(\varphi(s))\mu(s) ds + \frac{x^{\beta}(\varphi(t))}{\mu^{\beta-1}(\varphi(t))} \int_{\varphi(t)}^{t} F(s)\eta^{\beta}(\varphi(s)) ds \\ + \mu(\varphi(t)) x^{\beta}(\varphi(t)) \int_{t}^{\infty} \frac{F(s)\eta^{\beta}(\varphi(s))}{\mu^{\beta}(\varphi(s))} ds, \end{aligned}$$

or

$$\begin{aligned} x^{1-\beta}(\varphi(t)) \geq \frac{1}{\mu^{\beta}(\varphi(t))} \int_{t_1}^{\varphi(t)} F(s)\eta^{\beta}(\varphi(s))\mu(s) ds + \frac{1}{\mu^{\beta-1}(\varphi(t))} \int_{\varphi(t)}^{t} F(s)\eta^{\beta}(\varphi(s)) ds \\ + \mu(\varphi(t)) \int_{t}^{\infty} \frac{F(s)\eta^{\beta}(\varphi(s))}{\mu^{\beta}(\varphi(s))} ds. \end{aligned} \tag{2.9}$$

Since $x(t)$ is positive and increasing, there exists a constant $B > 0$ such that $x(t) \geq B$, and so we have $x^{1-\beta}(\varphi(t)) \leq B^{1-\beta}$. Using this in (2.9) yields

$$\begin{aligned} B^{1-\beta} \geq \frac{1}{\mu^{\beta}(\varphi(t))} \int_{t_1}^{\varphi(t)} F(s)\eta^{\beta}(\varphi(s))\mu(s) ds + \frac{1}{\mu^{\beta-1}(\varphi(t))} \int_{\varphi(t)}^{t} F(s)\eta^{\beta}(\varphi(s)) ds \\ + \mu(\varphi(t)) \int_{t}^{\infty} \frac{F(s)\eta^{\beta}(\varphi(s))}{\mu^{\beta}(\varphi(s))} ds. \end{aligned}$$

Now take the lim sup as $t \to \infty$ of the resulting inequality, we obtain a contradiction to (2.5).

Next, we assume that $z \in \mathcal{N}_0$. Then $\lim_{t\to\infty} z(t) = m \geq 0$. We assert that $m = 0$. Otherwise, we have $z(t) \geq m > 0$. Integrating (2.3) from t to ∞ gives

$$a(t)(b(t)z'(t))' \geq \int_t^\infty F(s)z^\beta(\varphi(s))ds \geq m^\beta \int_t^\infty F(s)ds.$$

Hence

$$-b(t)z'(t) \geq m^\beta \int_t^\infty \frac{1}{a(u)} \int_u^\infty F(s)dsdu,$$

from which

$$z(t_1) \geq m^\beta \int_{t_1}^\infty \frac{1}{b(u)} \int_u^\infty \frac{1}{a(v)} \int_v^\infty F(s)dsdvdu,$$

which yields a contradiction to (2.4) and so the following is followed

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} \frac{y(t)}{Q(t)} = 0.$$

This completes the proof of the theorem. □

Theorem 2.5. *Let $\varphi(t) \leq t$, $\beta = 1$, and (2.1) hold. If (2.4) and*

$$\limsup_{t\to\infty} \left(\frac{1}{\mu(\varphi(t))} \int_{t_0}^{\varphi(t)} F(s)\eta(\varphi(s))\mu(s)ds + \int_{\varphi(t)}^t F(s)\eta(\varphi(s))ds + \mu(\varphi(t)) \int_t^\infty \frac{F(s)\eta(\varphi(s))}{\mu(\varphi(s))} ds \right) > 1 \quad (2.10)$$

are fulfilled, then the conclusion of Theorem 2.4 remains intact.

Proof. The proof is followed by putting $\beta = 1$ in Theorem 2.4, so the details are omitted. □

Theorem 2.6. *Let $\varphi(t) \leq t$, $0 < \beta < 1$, and (2.1) hold. If (2.4) and*

$$\limsup_{t\to\infty} \left(\frac{1}{\mu(\varphi(t))} \int_{t_0}^{\varphi(t)} F(s)\eta^\beta(\varphi(s))\mu(s)ds + \int_{\varphi(t)}^t F(s)\eta^\beta(\varphi(s))ds + \mu^\beta(\varphi(t)) \int_t^\infty \frac{F(s)\eta^\beta(\varphi(s))}{\mu^\beta(\varphi(s))} ds \right) = \infty \quad (2.11)$$

hold, then every non-oscillatory solution $y(t)$ of (1.1) satisfies $\lim_{t\to\infty} \frac{y(t)}{Q(t)} = 0$.

Proof. Let $y(t)$ be a non-oscillatory solution of equation (1.1), say $y(t) > 0$, and $y(\varphi(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Then, from Corollary 2.3, the corresponding function $z(t) = \frac{y(t)}{Q(t)}$ is a positive solution of (2.3) and so either $z \in \mathcal{N}_0$ or $z \in \mathcal{N}_2$ for $t \geq t_1$.

First, we assume that $z \in \mathcal{N}_2$. Proceeding as in the proof of Theorem 2.4, we are lead to (2.9). Now dividing (2.9) by $\mu^{1-\beta}(\varphi(t))$, we observe that

$$\left(\frac{x(\varphi(t))}{\mu(\varphi(t))}\right)^{1-\beta} \geq \frac{1}{\mu(\varphi(t))} \int_{t_1}^{\varphi(t)} F(s)\eta^\beta(\varphi(s))\mu(s)ds + \int_{\varphi(t)}^t F(s)\eta^\beta(\varphi(s))ds + \mu^\beta(\varphi(t)) \int_t^\infty \frac{F(s)\eta^\beta(\varphi(s))}{\mu^\beta(\varphi(s))} ds. \quad (2.12)$$

Since $x(\varphi(t))/\mu(\varphi(t))$ is decreasing and $0 < \beta < 1$, there exists a constant $M_1 > 0$ such that

$$\left(\frac{x(\varphi(t))}{\mu(\varphi(t))}\right)^{1-\beta} \leq M_1^{1-\beta}.$$

Using this in (2.12) and taking the lim sup as $t \to \infty$, we establish a contradiction with (2.11), and therefore $z \notin \mathcal{N}_2$.

Next, we assume that $z \in \mathcal{N}_0$. Proceeding similarly to the proof of Theorem 2.4, we again see that condition (2.4) implies $\lim_{t\to\infty} \frac{y(t)}{Q(t)} = 0$. This completes the proof. □

Remark 2.7. The assertions of Theorems 2.4–2.6 can be restated as follows:

If $y(t)$ is a non-oscillatory solution of (1.1), then for any $B > 0$ is a constant, we have $|y(t)| \leq BQ(t)$.

The method used for obtaining asymptotic criteria for retarded functional differential equations (RFDE) can be used also for advanced type functional differential equations (AFDE) as well.

Theorem 2.8. *Let $\varphi(t) \geq t$, $\beta > 1$, and (2.1) hold. If (2.4) and*

$$\limsup_{t\to\infty} \left(\frac{1}{\mu^\beta(\varphi(t))} \int_{t_0}^{t} F(s)\eta^\beta(\varphi(s))\mu(s)ds + \int_t^{\varphi(t)} \frac{F(s)\eta^\beta(\varphi(s))\mu(s)}{\mu^\beta(\varphi(s))} ds + \mu(\varphi(t)) \int_{\varphi(t)}^{\infty} \frac{F(s)\eta^\beta(\varphi(s))}{\mu^\beta(\varphi(s))} ds \right) = \infty, \tag{2.13}$$

then the coclusion of Theorem 2.4 holds.

Proof. Let $y(t)$ be a non-oscillatory solution of equation (1.1), say $y(t) > 0$, and $y(\varphi(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Then, from Corollary 2.3, the corresponding function $z(t) = \frac{y(t)}{Q(t)}$ is a positive solution of (2.3) and so either $z \in \mathcal{N}_0$ or $z \in \mathcal{N}_2$ for $t \geq t_1$.

First, we assume that $z \in \mathcal{N}_2$. Proceeding as in the proof of Theorem 2.4, we again lead to (2.8) for $t \geq t_1$. From (2.8),

$$x(\varphi(t)) \geq \int_{t_1}^{t} \frac{F(s)\eta^\beta(\varphi(s))\mu(s)}{\mu^\beta(\varphi(s))} x^\beta(\varphi(s))ds + \int_t^{\varphi(t)} \frac{F(s)\eta^\beta(\varphi(s))\mu(s)}{\mu^\beta(\varphi(s))} x^\beta(\varphi(s))ds + \mu(\varphi(t)) \int_{\varphi(t)}^{\infty} \frac{F(s)\eta^\beta(\varphi(s))}{\mu^\beta(\varphi(s))} x^\beta(\varphi(s))ds. \tag{2.14}$$

Using the monotonicity properties of $x(t)$ and $x(t)/\mu(t)$, we observe from (2.14) that

$$x(\varphi(t)) \geq \frac{x^\beta(\varphi(t))}{\mu^\beta(\varphi(t))} \int_{t_1}^{t} F(s)\eta^\beta(\varphi(s))\mu(s)ds + x^\beta(\varphi(t)) \int_t^{\varphi(t)} \frac{F(s)\eta^\beta(\varphi(s))\mu(s)}{\mu^\beta(\varphi(s))} ds + \mu(\varphi(t))x^\beta(\varphi(t)) \int_{\varphi(t)}^{\infty} \frac{F(s)\eta^\beta(\varphi(s))}{\mu^\beta(\varphi(s))} ds,$$

or

$$x^{1-\beta}(\varphi(t)) \geq \frac{1}{\mu^\beta(\varphi(t))} \int_{t_1}^{t} F(s)\eta^\beta(\varphi(s))\mu(s)ds + \int_t^{\varphi(t)} \frac{F(s)\eta^\beta(\varphi(s))\mu(s)}{\mu^\beta(\varphi(s))} ds + \mu(\varphi(t)) \int_{\varphi(t)}^{\infty} \frac{F(s)\eta^\beta(\varphi(s))}{\mu^\beta(\varphi(s))} ds. \tag{2.15}$$

Since $x(t)$ is increasing, there exists $M_2 > 0$ such that $x^{1-\beta}(\varphi(t)) \leq M_2^{1-\beta}$ for $t \geq t_1$. Using this in (2.15) and then letting lim sup as $t \to \infty$, we contradict (2.13).

In the case when $z \in \mathcal{N}_0$, as did in Theorem 2.4, we contradict (2.4). This completes the proof. □

Theorem 2.9. *Let $\varphi(t) \geq t$, $\beta = 1$, and (2.1) hold. If (2.4) and*

$$\limsup_{t\to\infty} \left(\frac{1}{\mu(\varphi(t))} \int_{t_0}^{t} F(s)\eta(\varphi(s))\mu(s)ds + \int_t^{\varphi(t)} \frac{F(s)\eta(\varphi(s))\mu(s)}{\mu(\varphi(s))} ds + \mu(\varphi(t)) \int_{\varphi(t)}^{\infty} \frac{F(s)\eta(\varphi(s))}{\mu(\varphi(s))} ds \right) > 1, \tag{2.16}$$

then every non-oscillatory solution $y(t)$ of (1.1) satisfies $\lim_{t\to\infty} \frac{y(t)}{Q(t)} = 0$.

Proof. The proof follows from Theorem 2.8 by setting $\beta = 1$ in (2.15). This completes the proof. □

Theorem 2.10. *Let* $\varphi(t) \geq t$, $0 < \beta < 1$, *and* (2.1) *hold. If* (2.4) *and*

$$\limsup_{t\to\infty} \left(\frac{1}{\mu(\varphi(t))} \int_{t_0}^{t} F(s)\eta^\beta(\varphi(s))\mu(s)ds + \frac{1}{\mu^{1-\beta}(\varphi(t))} \int_{t}^{\varphi(t)} \frac{F(s)\eta^\beta(\varphi(s))\mu(s)}{\mu^\beta(\varphi(s))}ds + \mu^\beta(\varphi(t)) \int_{\varphi(t)}^{\infty} \frac{F(s)\eta^\beta(\varphi(s))}{\mu^\beta(\varphi(s))}ds \right) = \infty, \quad (2.17)$$

then every non-oscillatory solution $y(t)$ *of* (1.1) *satisfies* $\lim_{t\to\infty} \frac{y(t)}{Q(t)} = 0$.

Proof. The proof is similar to that of Theorem 2.6 and hence the details are omitted. □

For the special case $\varphi(t) \equiv t$, we obtain the following results for the ordinary equation

$$(p(t)(q(t)y'(t))')' + f(t)y^\beta(t) = 0. \quad (2.18)$$

From the above theorems, we immediately get the following corollaries.

Corollary 2.11. *Assume that* $\beta > 1$, (2.1), *and* (2.4) *are satisfied. If*

$$\limsup_{t\to\infty} \left(\frac{1}{\mu^\beta(t)} \int_{t_0}^{t} F(s)\eta^\beta(s)\mu(s)ds + \mu(t) \int_{t}^{\infty} \frac{F(s)\eta^\beta(s)}{\mu^\beta(s)}ds \right) = \infty, \quad (2.19)$$

then every non-oscillatory solution $y(t)$ *of* (1.1) *satisfies* $\lim_{t\to\infty} \frac{y(t)}{Q(t)} = 0$.

Corollary 2.12. *Let* $\beta = 1$, (2.1), *and* (2.4) *are fullfiled. If*

$$\limsup_{t\to\infty} \left(\frac{1}{\mu(t)} \int_{t_0}^{t} F(s)\eta(s)\mu(s)ds + \mu(t) \int_{t}^{\infty} \frac{F(s)\eta(s)}{\mu(s)}ds \right) > 1, \quad (2.20)$$

then every non-oscillatory solution $y(t)$ *of* (1.1) *satisfies* $\lim_{t\to\infty} \frac{y(t)}{Q(t)} = 0$.

Corollary 2.13. *Let* $0 < \beta < 1$, (2.1), *and* (2.4) *are satisfied. If*

$$\limsup_{t\to\infty} \left(\frac{1}{\mu(t)} \int_{t_0}^{t} F(s)\eta^\beta(s)\mu(s)ds + \mu^\beta(t) \int_{t}^{\infty} \frac{F(s)\eta^\beta(s)}{\mu^\beta(s)}ds \right) = \infty, \quad (2.21)$$

then every non-oscillatory solution $y(t)$ *of* (1.1) *satisfies* $\lim_{t\to\infty} \frac{y(t)}{Q(t)} = 0$.

Based on each of the results obtained, it is possible to formulate an oscillation theorem. As an example, we do it for Corollary 2.12.

Theorem 2.14. *Let* $\beta = 1$. *If conditions* (2.1), (2.4) *and* (2.20) *hold, then any solution* $y(t)$ *of* (2.18) *is either oscillatory or satisfies* $\lim_{t\to\infty} \frac{y(t)}{Q(t)} = 0$.

3 Examples

Here we will demonstrate the applicability of the results obtained above through some examples.

Example 3.1. Consider semi-canonical differential equation of the third-order

$$\left(\frac{1}{t}\left(t^{3/2}y'(t)\right)'\right)' + \frac{1}{t^{13/6}}y^{5/3}\left(\frac{t}{2}\right) = 0, \quad t \geq 1. \quad (3.1)$$

Here $p(t) = 1/t$, $q(t) = t^{3/2}$, $f(t) = 1/t^{13/6}$, $\beta = 5/3$, and $\varphi(t) = t/2$. A simple computation shows that $Q(t) = 2/\sqrt{t}$, $a(t) = 1/2\sqrt{t}$, $b(t) = 4\sqrt{t}$, $F(t) = 4\sqrt{2}/t^3$, $\mu(t) \approx 4t^{3/2}/3$ and $\eta(t) \approx t^2/6$. The transformed equation

$$\left(\frac{1}{\sqrt{t}}(\sqrt{t}z'(t))'\right)' + \frac{2\sqrt{2}}{t^3}z^{5/3}\left(\frac{t}{2}\right) = 0$$

is canonical. It is easy to see that conditions (2.1) and (2.4) are satisfied. Further, condition (2.5) becomes

$$\limsup_{t\to\infty}\left[\frac{4^{1/3}}{t^{5/2}}\int_1^{t/2} s^{11/6}ds+\frac{2^{5/6}}{t}\int_{t/2}^{t} s^{1/3}ds+\frac{t^{3/2}}{2^{5/6}}\int_t^{\infty}\frac{1}{s^{13/6}}ds\right]=\infty,$$

that is, (2.5) holds. Now, all terms of Theorem 2.4 are provided. Hence, every non-oscillatory solution $y(t)$ of (3.1) satisfies

$$\lim_{t\to\infty}\sqrt{t}y(t)=0.$$

Example 3.2. Consider semi-canonical differential equation of the third-order

$$\left(\frac{1}{\sqrt{t}}\left(t^2y'(t)\right)'\right)'+\frac{4^{1/7}}{t^{9/7}}y^{5/7}\left(\frac{t}{2}\right)=0,\quad t\geq 1. \tag{3.2}$$

Here $p(t)=1/\sqrt{t}$, $q(t)=t^2$, $f(t)=4^{1/7}/t^{9/7}$, $\varphi(t)=t/2$, and $\beta=5/7$. A simple calculation shows that $Q(t)=1/t$, $a(t)=\sqrt{t}$, $b(t)=1$, $F(t)=2/t^2$, $\mu(t)\approx 2\sqrt{t}$ and $\eta(t)=4t^{3/2}/3$. The transformed equation

$$\left(\sqrt{t}z''(t)\right)'+\frac{2}{t^2}z^{5/7}\left(\frac{t}{2}\right)=0$$

is canonical. It is easy to see that conditions (2.1) and (2.4) hold. Further, condition (2.11) becomes

$$\limsup_{t\to\infty}\left[\frac{8^{1/7}}{t^{1/2}}\int_1^{t/2}\frac{1}{s^{3/7}}ds+\frac{1}{2^{1/14}}\int_{t/2}^{t}\frac{1}{s^{13/14}}ds+\frac{t^{5/14}}{2^{-13/14}}\int_t^{\infty}\frac{1}{s^{9/7}}ds\right]=\infty,$$

that is, (2.11) is satisfied. Hence, by Theorem 2.6, every non-oscillatory solution $y(t)$ of (3.2) satisfies

$$\lim_{t\to\infty}ty(t)=0.$$

Example 3.3. Consider semi-canonical differential equation of the third-order

$$\left(\frac{1}{t}\left(t^2y'(t)\right)'\right)'+\frac{1}{t^2}y\left(\frac{t}{2}\right)=0,\quad t\geq 1. \tag{3.3}$$

Here $p(t)=1/t$, $q(t)=t^2$, $f(t)=1/t^2$, $\varphi(t)=t/2$, and $\beta=1$. A simple calculation shows that $Q(t)=1/t$, $a(t)=b(t)=1$, $F(t)=2/t^3$, $\mu(t)\approx t$ and $\eta(t)\approx t^2/2$. The transformed equation

$$z'''(t)+\frac{2}{t^3}z\left(\frac{t}{2}\right)=0$$

is canonical. Clearly conditions (2.1) and (2.4) are satisfied. Further, condition (2.10) becomes

$$\limsup_{t\to\infty}\left[\frac{2}{t}\int_1^{t/2}\frac{1}{2}ds+\int_{t/2}^{t}\frac{1}{2s}ds+\frac{t}{2}\int_t^{\infty}\frac{1}{s^2}ds\right]=1+\frac{1}{2}\log 2>1,$$

that is, (2.10) is satisfied. Hence, by Theorem 2.5, every non-oscillatory solution $y(t)$ of (3.3) satisfies

$$\lim_{t\to\infty}ty(t)=0.$$

Example 3.4. Consider the semi-canonical ordinary differential equation of third-order

$$\left(\frac{1}{t}\left(t^2y'(t)\right)'\right)'+\frac{6}{t^2}y(t)=0,\quad t\geq 1. \tag{3.4}$$

Here $p(t)=1/t$, $q(t)=t^2$, $f(t)=6/t^2$, $\varphi(t)=t$, and $\beta=1$. A simple calculation shows that $Q(t)=1/t$, $a(t)=b(t)=1$, $F(t)=6/t^3$, $\mu(t)\approx t$ and $\eta(t)\approx t^2/2$. Clearly all conditions of Corollary 2.12 hold, so any non-oscillatory solution y of (3.4) satisfies $\lim_{t\to\infty}ty(t)=0$. In fact, $y(t)=1/t^2$ is one such solution of (3.4).

4 Concluding Remarks

In the present paper, we introduced semi-canonical third-order differential operators and then we transform this type of operators to a canonical form. This approach was then used to study the asymptotic behavior of nonoscillatory solutions of (1.1). Results were obtained for the delay, advanced and ordinary cases of semi-canonical equations. Examples to illustrate the main results were presented.

Finally, we note that additional results on the behavior of solutions of (1.1) can be obtained from known results for canonical equations applied to (2.3) and (2.18).

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