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Generalization of prime ideals in $M\Gamma$-groups

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Abstract In this paper, we consider a module over a Γ-nearring (also known as, $M\Gamma$-group) G. We introduce the notions $(i, 2)$-absorbing, $i \in \{c, 3\}$ ideals of G, as a generalization of i-prime ideal of G. We prove several properties and exhibit examples that indicate these classes are different from the existing classes of prime ideals. Further, we prove the properties such as homomorphic images, inverse images of $(c, 2)$-absorbing ideals of G, and the properties involving Notherian quotients of $(i, 2)$-absorbing ideals of G, $i \in \{c, 3\}$.

1 Introduction

A module over a nearring (or N-group) is a generalization of a module over an arbitrary ring. Precisely, it is an action of a group (not necessarily abelian) over a nearring. Juglal et.al. [19] generalized prime ideal in modules over rings to modules over nearrings in several ways. They obtained some characterizations of prime nearring modules and exhibited interesting properties. More importantly, the connection between a prime ideal of module over a nearring and the corresponding annihilators in a nearring were investigated. Badawi [1] introduced the idea of 2-absorbing ideal of a commutative ring with identity, as a generalization of a prime ideal in a commutative ring, and explored several properties. Darani and Soheilnia [14] introduced a weakly 2-absorbing submodule of a module over a commutative ring with identity. 2-absorbing modules over non-commutative rings were studied by Groenewald and Nguyen [15]. Indeed, there are several means to generalize these to prime ideals of nearrings. Prime ideals in nearrings have been introduced in Holcombe [18], and later studied by [19, 4, 5]. The concept of Γ-ring was introduced by Nobusawa [25], and a generalization of this concept namely, a Γ-nearring was introduced by Bhavanari [2]. Further, the module structure over a Γ-nearring was explored in Bhavanari [2, 8]. The comprehensive study of Γ-nearrings is due to [6, 12, 13, 10, 11, 21]. Recently, Hamsa et.al. [17] defined quasi associativity in $\Theta\Gamma$-N-groups (as generalized $M\Gamma$-groups), and proved the fundamental isomorphism theorems. In this paper, we introduce $(i, 2)$-absorbing ideals ($i \in \{c, 3\}$) of a module over gamma nearrings and illustrate that these classes are different from the classes of prime ideals introduced in [27]. In section 2, we prove properties such as homomorphic images, inverse images of $(c, 2)$-absorbing ideals of G, and the properties involving Notherian quotients of $(c, 2)$-absorbing ideals of G. Section 3 deals with few results involving the properties of $(3, 2)$-absorbing ideals of G. Throughout, M denotes a right gamma nearring, and G stands for an $M\Gamma$-group. Further, all undefined notations and conventions will be used as in [27, 5]. We assume M to be zero-symmetric whenever necessary.

2 Preliminaries

The notion of Γ-nearring was defined by Bhavanari [2, 7] as a generalization of a nearring (Pilz[27]) and a Γ-ring (Nobusawa [25]).

Definition 2.1. Let $(M, +)$ be a group (not necessarily abelian) and Γ a non-empty set. Then

M is said to be a Γ-nearring if there exists a mapping $M \times \Gamma \times M \to M$ (denote the image of (m_1, γ, m_2) by $m_1\gamma_1 m_2$ for $m_1, m_2 \in M$ and $\gamma_1 \in \Gamma$) satisfying the following conditions:

(i) $(m_1 + m_2)\gamma_1 m_3 = m_1\gamma_1 m_3 + m_2\gamma_2 m_3$ and

(ii) $(m_1\gamma_1 m_2)\gamma_2 m_3 = m_1\gamma_1(m_2\gamma_2 m_3)$,

for all $m_1, m_2, m_3 \in M$ and for all $\gamma_1, \gamma_2 \in \Gamma$.

If $\Gamma = \{\cdot\}$, then M becomes a nearring.
The following definition is a generalization of an N-group defined in Pilz [27].

Definition 2.2. Let M be a Γ-nearring. An additive group G is said to be an $M\Gamma$-group (or $M\Gamma$-module or Γ-nearring module over M) if there exists a mapping $M \times \Gamma \times G \to G$ (denote the image of (m, γ, g) by $m\gamma g$ for $m \in M$ and $\gamma \in \Gamma, g \in G$) satisfying the following conditions:

(i) $(m_1 + m_2)\gamma_1 g = m_1\gamma_1 g + m_2\gamma_1 g$ and

(ii) $(m_1\gamma_1 m_2)\gamma_2 g = m_1\gamma_1(m_2\gamma_2 g)$,

for all $m_1, m_2 \in M$ and for all $\gamma_1, \gamma_2 \in \Gamma$ and $g \in G$.

Example 2.3. Let $G = \mathbb{Z}_4 = \{0, 1, 2, 3\}$, the ring of integers modulo 4 and $X = \{a, b\}$. Write $M = \{g | g : X \to G, g(a) = 0\} = \{g_0, g_1, g_2, g_3\}$, where $g_i(a) = 0$, $g_i(b) = i$, for $0 \le i \le 3$. Let $\Gamma = \{f_1, f_2, f_3, f_4\}$ where each $f_i : G \to X$ defined by $f_1(i) = a$ $(0 \le i \le 3)$, $f_2(i) = a$ $(0 \le i \le 2)$, $f_3(i) = a$, for $i \in \{0, 2, 3\}$, $f_3(1) = b$, $f_4(i) = a$ if $i \in \{0, 2\}$ and $f_4(i) = b$ if $i \notin \{0, 2\}$. For $g \in M$, $f \in \Gamma, x \in G$ write $gfx = g(f(x))$. Now G becomes an $M\Gamma$-group.

Definition 2.4. [8] A normal subgroup H of G is said to be an ideal of G if $m\gamma(g+h)-m\gamma g \in H$ for $m \in M, \gamma \in \Gamma, g \in G$ and $h \in H$.

Definition 2.5. [27] A proper ideal P of M is called prime if for any two ideals S and T of M with $ST \subseteq P$ implies that $S \subseteq P$ or $T \subseteq P$; and P is completely prime (denoted as, c-prime) if $st \in P$ implies $s \in P$ or $t \in P$. In case of commutative rings, the notions prime and c-prime will coincide.

Definition 2.6. [3] An ideal I of G is said to have insertion of factors property (denoted as, IFP) if $x \in M, g \in G$ with $xg \in I$ then $xng \in I$, for all $n \in M$. If (0_G) is an IFP ideal, then we call G an IFP $M\Gamma$-group.

3 Completely 2-absorbing ideals

The following definition generalizes the completely prime (abbr. c-prime) ideal of an Γ-nearring.

Definition 3.1. A proper ideal I of M is called c-prime if whenever $m,\ m_1 \in M$ and $\gamma \in \Gamma$ with $m\gamma m_1 \in I$, then $m \in I$ or $m_1 \in I$.

Definition 3.2. A proper ideal I of G with $M\Gamma G \nsubseteq I$ is called c-prime if whenever $m \in M$, $g \in G$ and $\gamma \in \Gamma$ with $m\gamma g \in I$, then $m\Gamma G \subseteq I$ or $g \in I$.

The following definition is analogous to the Notation 0.1 given in [9].

Notation 3.1. [9] For any non-empty subset A of G we write

$A^0 = \{x - y | x, y \in A\}$;
$A^* = \{g + x - g | x \in M,\ g \in G\}$;
$A^+ = \{m\gamma(g + x) - m\gamma g | m \in M, g \in G, \gamma \in \Gamma, x \in A\}$.
Let X be a non-empty subset of G and write $X_0 = X$, and $X_{i+1} = X_i^0 \cup X^* \cup X_i^+$ for all integers $i \ge 0$. Then $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$ and clearly $\cup_{i=0}^{\infty} X_i$ is the ideal generated by X, denoted by $\langle X \rangle$. If $X = \{g\}$, then we denote it as $\langle g \rangle$.

Definition 3.3. A proper ideal I of G with $M\Gamma G \nsubseteq I$ is called strictly c-prime if $x\gamma g \in I$ implies $x\Gamma\langle g \rangle \subseteq I$ or $g \in I$.

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

·	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	b	b	b	b
c	c	c	c	c

Table 1.

Remark 3.4. Every c-prime ideal is strictly c-prime.

Proof. Suppose I is c-prime. Let $x \in M$, $\gamma \in \Gamma$ and $g \in G$ such that $x\gamma g \in I$. Since I is c-prime, $x\Gamma G \subseteq I$ or $g \in I$. Now $x\Gamma\langle g\rangle \subseteq x\Gamma G \subseteq I$ or $g \in I$. Therefore, I is strictly c-prime. The converse of Remark 3.4 need not be true. We show this in Example 3.8. □

Now we introduce the notion of $(c, 2)$-absorbing ideal as a generalization of c-prime ideal.

Definition 3.5. An ideal I of M is called completely 2-absorbing (abbr. $(c, 2)$-absorbing) if whenever $x,\ y,\ z \in M$, $\gamma \in \Gamma$ with $x\gamma y\gamma z \in I$ then $x\gamma y \in I$ or $y\gamma z \in I$ or $x\gamma z \in I$. M is called $(c, 2)$-absorbing if the ideal (0) is $(c, 2)$-absorbing.

Notation 3.2. Let I be an ideal of G. We denote the Noetherian quotient as $(I : \Gamma G) = \{x \in M : x\Gamma G \subseteq I\}$.

We generalize the definition 3.5 as follows, which is a key notion in this paper.

Definition 3.6. A proper ideal I of G is said to be a completely 2-absorbing (abbr. $(c, 2)$-absorbing) ideal if whenever $x, y \in M$, $\gamma \in \Gamma$ and $g \in G$ with $x\gamma y\gamma g \in I$, then $x\gamma y \in (I : \Gamma G)$ or $x\gamma g \in I$ or $y\gamma g \in I$.

Example 3.7. Refer to the nearring M given in E-23, page 408 of Pilz [27], where $M = \{0, a, b, c\}$. Consider M itself is an $M\Gamma$-group with $\Gamma = \{\cdot\}$ as defined in Table 1. Then clearly, M is not zero-symmetric. It can be easily seen that, $\{0\}$ is c-prime and $(c, 2)$-absorbing.

Example 3.8. Take $G = \mathbb{Z}_8$, $\Gamma = \{\cdot_8\}$ and $M = \mathbb{Z}$, nearring of integers. Then G is an $M\Gamma$-group. Clearly $I_1 = \{0, 4\}$ and $I_2 = \{0, 2, 4, 6\}$ are ideals of G. It can be verified that I_1 is a $(c, 2)$-absorbing ideal of G but not c-prime, since $2 \cdot_8 6 \in I_1$ and neither $2\mathbb{Z}_8 \subseteq I_1$ nor $6 \in I_1$. The ideal I_2 is c-prime as well as $(c, 2)$-absorbing. However, I_1 is strictly c-prime since $2 \cdot_8 \langle 6\rangle = \{0, 4\} \subseteq I_1$.

Example 3.9. Take $G = \mathbb{Z}_6$ and $M = \mathbb{Z}$, nearring of integers. Then G is an $M\Gamma$-group. Clearly $I_1 = \{0\}$ and $I_2 = \{0, 2, 4\}$ are ideals of G. It can be verified that I_1 is $(c, 2)$-absorbing, but not c-prime since $2 \cdot 3 \in I_1$ and neither $2\mathbb{Z}_6 \subseteq I_1$ nor $3 \in I_1$. The ideal I_2 is c-prime as well as $(c, 2)$-absorbing.

Example 3.10. Let $M = (S_3, +, \cdot)$ be a nearring (given in H-11, page 410 of Pilz ([27])), which is not zero-symmetric and non-commutative. Consider M as an $M\Gamma$-group over itself, where we define $\Gamma = \{\cdot\}$ as follows in Table 2: Clearly $I = \{0\}$ is $(c, 2)$-absorbing but not c-prime, since $x \cdot a \in I$, and $a \notin I$, $x \cdot G \nsubseteq I$.

Example 3.11. Consider the nearring M, given in E-3, page 408 of Pilz [27], where $M = \{0, a, b, c\}$ with operations $+$ as given in Table 1 and $\cdot$ defined as in Table 3.
Observe that M is zero-symmetric. Let $I = \{0\}$. Then $ba = 0 \in I$ but $b \notin I$ and $a \notin I$. Therefore, $I = \{0\}$ is not c-prime. However, $I = \{0\}$ is $(c, 2)$-absorbing. Thus, M is $(c, 2)$-prime.

Example 3.12. Let $G = D_8 = \langle\{\sigma, s \mid \sigma^4 = s^2 = e,\ \sigma s = s\sigma^{-1}\}\rangle$ $= \{e, \sigma, \sigma^2, \sigma^3, s, s\sigma, s\sigma^2, s\sigma^3\}$, where σ is the rotation in an anti-clockwise direction about the origin through $\frac{\pi}{2}$ radians and s is the reflection about the line of symmetry. Take $G = M$ (listed as no. K(139) on p. 418 of Pilz [27]), $\Gamma = \{*\}$, and has $+$ and $*$ given in Table 4 (also, refer to Hamsa et.al. [17]):

+	0	a	b	c	x	y
0	0	a	b	c	x	y
a	a	0	y	x	c	b
b	b	x	0	y	a	c
c	c	x	y	0	b	a
x	x	b	c	a	y	0
y	y	c	a	b	0	x

·	0	a	b	c	x	y
0	0	0	0	0	0	0
a	a	a	a	a	a	a
b	a	a	c	b	b	c
c	a	a	b	c	c	b
x	0	0	y	x	x	y
y	0	0	x	y	y	x

Table 2.

·	0	a	b	c
0	0	0	0	0
a	0	0	a	a
b	0	0	b	b
c	0	0	c	c

Table 3.

$+$	e	σ	σ^2	σ^3	s	$s\sigma$	$s\sigma^2$	$s\sigma^3$
e	e	σ	σ^2	σ^3	s	$s\sigma$	$s\sigma^2$	$s\sigma^3$
σ	σ	σ^2	σ^3	e	$s\sigma^3$	s	$s\sigma$	$s\sigma^2$
σ^2	σ^2	σ^3	e	σ	$s\sigma^2$	$s\sigma^3$	s	$s\sigma$
σ^3	σ^3	e	σ	σ^2	$s\sigma$	$s\sigma^2$	$s\sigma^3$	s
s	s	$s\sigma$	$s\sigma^2$	$s\sigma^3$	e	σ	σ^2	σ^3
$s\sigma$	$s\sigma$	$s\sigma^2$	$s\sigma^3$	s	σ^3	e	σ	σ^2
$s\sigma^2$	$s\sigma^2$	$s\sigma^3$	s	$s\sigma$	σ^2	σ^3	e	σ
$s\sigma^3$	$s\sigma^3$	s	$s\sigma$	$s\sigma^2$	σ	σ^2	σ^3	e

$*$	e	σ	σ^2	σ^3	s	$s\sigma$	$s\sigma^2$	$s\sigma^3$
e	e	e	e	e	e	e	e	e
σ	e	σ	σ^2	σ^3	s	$s\sigma$	$s\sigma^2$	$s\sigma^3$
σ^2	e	σ^2	e	σ^2	e	e	e	e
σ^3	e	σ^3	σ^2	e	s	$s\sigma$	$s\sigma^2$	$s\sigma^3$
s	e	s	e	s	$s\sigma^2$	e	$s\sigma^2$	e
$s\sigma$	e	$s\sigma$	σ^2	$s\sigma^3$	σ^2	$s\sigma$	e	$s\sigma^3$
$s\sigma^2$	e	$s\sigma^2$	e	$s\sigma^2$	$s\sigma^2$	e	$s\sigma^2$	e
$s\sigma^3$	e	$s\sigma^3$	σ^2	$s\sigma$	σ^2	$s\sigma$	e	$s\sigma^3$

Table 4.

Then G is an $M\Gamma$-group, where M is non-abelian. $I = \{e, \sigma^2\}$ is $(c, 2)$-absorbing but not c-prime, since $s\sigma * s = \sigma^2 \in I$, but $s \notin I$ and $s\sigma * G \nsubseteq I$.

Definition 3.13. [8] For two $M\Gamma$-groups G and G', a group homomorphism $\phi : G \to G'$ is said to be an $M\Gamma$-homomorphism if $\phi(m\gamma g) = m\gamma\phi(g)$, for all $m \in M$, $\gamma \in \Gamma$ and $g \in G$.

Theorem 3.14. *Let ϕ be an $M\Gamma$-homomorphism of G onto G'. If I is a $(c, 2)$-absorbing ideal of G containing ker ϕ, then $\phi(I)$ is a $(c, 2)$-absorbing ideal of G'.*

Proof. Let $m_1, m_2 \in M$, $g' \in G'$ with $m_1\gamma m_2\gamma g' \in \phi(I)$. Then $m_1\gamma m_2\gamma g' = \phi(x)$ for some $x \in I$. Since ϕ is an $M\Gamma$-epimorphism and $g' \in G'$, it follows that $\phi(g) = g'$ for some $g \in G$.

Now, $\phi(m_1\gamma m_2\gamma g) = m_1\gamma m_2\gamma\phi(g) = m_1\gamma m_2\gamma g' = \phi(x)$, and hence, $\phi(m_1\gamma m_2\gamma g - x) = 0$ in G'. Since $x \in I$ and $m_1\gamma m_2\gamma g - x \in ker\ \phi \subseteq I$, we have, $m_1\gamma m_2\gamma g \in I$. Since I is $(c,2)$-absorbing, we have, $m_1\gamma m_2\gamma \in (I : \Gamma G)$ or $m_1\gamma g \in I$ or $m_2\gamma g \in I$, which implies $m_1\gamma m_2\gamma G \subseteq I$ or $m_1\gamma g \in I$ or $m_2\gamma g \in I$. Therefore, $\phi(m_1\gamma m_2\gamma G) \subseteq \phi(I)$ or $\phi(m_1\gamma g) \in \phi(I)$ or $\phi(m_2\gamma g) \in \phi(I)$. Now since ϕ is an $M\Gamma$-homomorphism, we have $m_1\gamma m_2\gamma\phi(G) \subseteq \phi(I)$ or $m_1\gamma\phi(g) \in \phi(I)$ or $m_2\gamma\phi(g) \in \phi(I)$. Therefore, $m_1\gamma m_2\gamma G' \in \phi(I)$ or $m_1\gamma g' \in \phi(I)$ or $m_2\gamma g' \in \phi(I)$. That is, $m_1\gamma m_2\gamma \in (\phi(I) : \Gamma G')$ or $m_1\gamma g' \in \phi(I)$ or $m_2\gamma g' \in \phi(I)$, proves $\phi(I)$ is a $(c,2)$-absorbing ideal of G' .

□

Theorem 3.15. *Let $h : G \to G'$ be an $M\Gamma$-epimorphism. If I' is a $(c,2)$-absorbing ideal of G', then $h^{-1}(I')$ is a $(c,2)$-absorbing ideal of G.*

Proof. Suppose I' is $(c,2)$-absorbing ideal of G'. Let $a, b \in M, g \in G$ and $\gamma \in \Gamma$ be such that $a\gamma b\gamma g \in h^{-1}(I')$. Since h is a $M\Gamma$-homomorphism, it follows that $a\gamma b\gamma h(g) = h(a\gamma b\gamma g) \in I'$. Again, since I' is $(c,2)$-absorbing, it follows that, $a\gamma g' \in I'$ or $b\gamma g' \in I'$ or $a\gamma b \in (I' : \Gamma G')$, where $g' = h(g)$.
Case (i): If $a\gamma b \in (I' : \Gamma G')$, then $a\gamma b\Gamma G' \subseteq I'$ and $h^{-1}(a\gamma b\Gamma G') \subseteq h^{-1}(I')$. Now, $a\gamma b\Gamma h^{-1}(G') = a\gamma b\Gamma G \subseteq h^{-1}(I')$, shows that $a\gamma b \in (h^{-1}(I') : \Gamma G)$.
Case (ii): If $a\gamma g' \in I'$, then $a\gamma h^{-1}(g') \in h^{-1}(I')$, shows that $a\gamma g \in h^{-1}(I')$. Similarly, if $b\gamma g' \in I'$, then $b\gamma g \in h^{-1}(I')$. Thus, $h^{-1}(I')$ is $(c,2)$-absorbing.

□

Lemma 3.16. *[19] For any ideal I of G, $(I : \Gamma G)$ is an ideal of M.*

For completeness, we provide the proof of the following lemma.

Lemma 3.17. *If I is a c-prime ideal of G, then $(I : \Gamma G)$ is a c-prime ideal in M.*

Proof. Since $I \neq G$, $1 \notin (I : \Gamma G)$, and so $(I : \Gamma G)$ is proper. Suppose that $a\gamma b \in (I : \Gamma G)$ and $b \notin (I : \Gamma G)$. Now since, $b\Gamma G \nsubseteq I$, there exists $g \in G$ and $\gamma \in \Gamma$ with $b\gamma g \notin I$, and so $a\gamma(b\gamma g) = (a\gamma b)\gamma g \in I$. Since I is c-prime in G, it follows that $a\Gamma G \subseteq I$, and hence $a \in (I : \Gamma G)$. □

Theorem 3.18. *Let M be zero-symmetric. If I is a c-prime ideal of G, then I is $(c,2)$-absorbing in G.*

Proof. Let I be a c-prime ideal of G, and let $a, b \in M, g \in G$ and $\gamma \in \Gamma$ with $a\gamma b\gamma g \in I$. Since I is c-prime, it follows that $a\gamma b\Gamma G \subseteq I$ or $g \in I$. This implies $a\gamma b \in (I : \Gamma G)$ or $g \in I$. If $a\gamma b \in (I : \Gamma G)$, then it is clear. If $g \in I$, we have $a\gamma g = a\gamma(0 + g) - a\gamma 0 \in I$ or $b\gamma g = b\gamma(0 + g) - b\gamma 0 \in I$. Therefore, I is a $(c,2)$-absorbing ideal in G. □

We define monogenic and locally monogenic $M\Gamma$-groups similar to those of given for an N-group in Pilz [27], and Ke and Meyer [20] respectively.

Definition 3.19. G is monogenic if there exists $g \in G$ and $\gamma \in \Gamma$ such that $M\gamma g = G$.

Definition 3.20. G is weakly monogenic if there exists $g \in G$ such that $M\Gamma g = G$. In this case we write $G = \langle g \rangle$.

Remark 3.21. Every monogenic $M\Gamma$-group is weakly monogenic.

Proof. Suppose G is monogenic by $g \in G$ and $\gamma \in \Gamma$. To show G is weakly monogenic. Clearly $M\Gamma g \subseteq G$. Let $g_1 \in G$. Then there exists $m \in M$ such that $m\gamma g = g_1$. Now, $g_1 = m\gamma g \in M\Gamma g$. Therefore $G \subseteq M\Gamma g$. Hence $M\Gamma g = G$. □

Definition 3.22. G is called locally monogenic if for every $S \subseteq G$, where S is finite, there exists $a \in G$ and $\gamma \in \Gamma$ such that $S \subseteq M\gamma a$.

It is obvious that every locally monogenic $M\Gamma$-group is monogenic.

Note 3.3. Let I be an ideal of G. Then $(I : \Gamma G) \subseteq (I : \gamma g)$ for all $g \in G \backslash I$ and $\gamma \in \Gamma$. Further, equality holds if M is zero-symmetric and G is weakly monogenic by g.

Proof. Let $x \in (I : \Gamma G)$. Then $x\Gamma G \subseteq I$, implies $x\gamma g \in I$, for all $g \in G$ and $\gamma \in \Gamma$. That is, $x \in (I : \gamma g)$. Therefore, $(I : \Gamma G) \subseteq (I : \gamma g)$. To show the equality, let M be zero-symmetric and G be weakly monogenic by g. That is, $M\Gamma g = G$. Let $x \in (I : \gamma g)$. Then $x\gamma g \in I$ implies that $x\gamma\langle g\rangle \subseteq I$. This shows that $x\gamma G \subseteq I$. Since γ is arbitrary, we get $x\Gamma G \subseteq I$. Therefore $x \in (I : \Gamma G)$. □

Theorem 3.23. *Let G be monogenic over $M = M_0$, and I an ideal of G. If $(I : \Gamma G)$ is a $(c, 2)$-absorbing ideal in M, then I is a $(c, 2)$-absorbing ideal of G.*

Proof. Let $(I : \Gamma G)$ be a $(c, 2)$-absorbing ideal of M. To prove I is a $(c, 2)$-absorbing ideal of G, take $x, y \in M$, $\gamma \in \Gamma$ and $g \in G$ with $x\gamma y\gamma g \in I$. Since G is monogenic, we have $G = M\gamma_1 g_1$ for some $\gamma_1 \in \Gamma$ and $g_1 \in G$. Now, $g = a\gamma_1 g_1 \in I$, for some $a \in M$. Then, $x\gamma y\gamma a\gamma_1 g_1 = x\gamma y\gamma g \in I$ implies $x\gamma y\gamma a \in (I : \gamma g_1)$. Since $M = M_0$, by Note 3.3, we have $x\gamma y\gamma a \in (I : \Gamma G)$. Since $(I : \Gamma G)$ is $(c, 2)$-absorbing, we have $x\gamma y \in (I : \Gamma G)$ or $x\gamma a \in (I : \Gamma G)$ or $y\gamma a \in (I : \Gamma G)$. That is, $x\gamma y \in (I : \Gamma G)$ or $x\gamma a\gamma_1 g_1 \in I$ or $y\gamma a\gamma_1 g_1 \in I$, implies $x\gamma y \in (I : \Gamma G)$ or $x\gamma g \in I$ or $y\gamma g \in I$. Therefore, I is $(c, 2)$-absorbing. □

The following definition is similar to that of defined for N-groups by Meldrum [24].

Definition 3.24. M distributes over G, if $m\gamma(g_1 + g_2) = m\gamma g_1 + m\gamma g_2$ for all $m \in M$, $\gamma \in \Gamma$, $g_1, g_2 \in G$.

Lemma 3.25. *If G be locally monogenic. Then M is distributive implies M distributes over G.*

Proof. Suppose G is locally monogenic and M is distributive. Take $m \in M$ and $g_1, g_2 \in G$. Since G is locally monogenic and $\{g_1, g_2\} \subseteq G$, there exist $\gamma \in \Gamma$ and $g \in G$ such that $\{g_1, g_2\} \subseteq M\gamma g$. Therefore, $g_1 = m_1\gamma g$ and $g_2 = m_2\gamma g$, for some $m_1, m_2 \in M$. Now, $m\gamma(g_1 + g_2) = m\gamma(m_1\gamma g + m_2\gamma g) = m\gamma((m_1 + m_2)\gamma g) = (m\gamma(m_1 + m_2))\gamma g = (m\gamma m_1 + m\gamma m_2)\gamma g = m\gamma m_1\gamma g + m\gamma m_2\gamma g = m\gamma g_1 + m\gamma g_2$, shows that M distributes over G. □

Theorem 3.26. *Let G be locally monogenic over a distributive Γ-nearring M. If I is a $(c, 2)$-absorbing ideal of G, then $(I : \Gamma G)$ is a $(c, 2)$-absorbing ideal of M.*

Proof. Suppose that I is a $(c, 2)$-absorbing ideal of G. To show $(I : \Gamma G)$ is $(c, 2)$-absorbing in M, let $a, b, c \in M$ and $\gamma \in \Gamma$ such that $a\gamma b\gamma c \in (I : \Gamma G)$. Assume that $a\gamma c \notin (I : \Gamma G)$, $b\gamma c \notin (I : \Gamma G)$. In this case, we show that $a\gamma b \in (I : \Gamma G)$. Since $a\gamma c\Gamma G \nsubseteq I$ and $b\gamma c\Gamma G \nsubseteq I$, we have $a\gamma c\gamma_1 g_1 \notin I$ and $b\gamma c\gamma_2 g_2 \notin I$ for some $g_1, g_2 \in G$ and $\gamma_1, \gamma_2 \in \Gamma$. Further, since $a\gamma b\gamma c \in (I : \Gamma G)$, we have $a\gamma b\gamma c\alpha g \in I$ for all $g \in G$ and $\alpha \in \Gamma$. In particular, $a\gamma b\gamma c\gamma_1 g_1$, $a\gamma b\gamma c\gamma_2 g_2 \in I$, and since I is an additive subgroup of G, it follows that $a\gamma b\gamma c\gamma_1 g_1 + a\gamma b\gamma c\gamma_2 g_2 \in I$. Now, by Lemma 3.25, $a\gamma b\gamma(c\gamma_1 g_1 + c\gamma_2 g_2) \in I$. Since I is $(c, 2)$-absorbing, it follows that $a\gamma(c\gamma_1 g_1 + c\gamma_2 g_2) \in I$ or $b\gamma(c\gamma_1 g_1 + c\gamma_2 g_2) \in I$ or $a\gamma b \in (I : \Gamma G)$. If $a\gamma(c\gamma_1 g_1 + c\gamma_2 g_2) \in I$, then $a\gamma c\gamma_1 g_1 + a\gamma c\gamma_2 g_2 \in I$. Now if $a\gamma c\gamma_2 g_2 \in I$, then $a\gamma c\gamma_1 g_1 = a\gamma c\gamma_1 g_1 + a\gamma c\gamma_2 g_2 - a\gamma c\gamma_2 g_2 \in I$, a contradiction. Therefore, $a\gamma(c\gamma_2 g_2) \notin I$. Also $b\gamma(c\gamma_2 g_2) \notin I$ but $a\gamma b\gamma(c\gamma_2 g_2) \in I$ (here, $c\gamma_2 g_2 \in G$). Since I is $(c, 2)$-absorbing, we have $a\gamma b \in (I : \Gamma G)$. In a similar way, if $b\gamma(c\gamma_1 g_1 + c\gamma g_2) \in I$, then $b\gamma c\gamma_1 g_1 + b\gamma c\gamma_2 g_2 \in I$. If $b\gamma c\gamma_1 g_1 \in I$, then since $b\gamma c\gamma_1 g_1 + b\gamma c\gamma_2 g_2 = i_1$, for some $i \in I$, we get $b\gamma c\gamma_2 g_2 = i - b\gamma c\gamma_1 g_1 \in I$, a contradiction. Therefore, $b\gamma c\gamma_1 g_1 \notin I$, also $a\gamma c\gamma_1 g_1 \notin I$, but $a\gamma b\gamma(c\gamma_1 g_1) \in I$. Since I is $(c, 2)$-absorbing, it follows that $a\gamma b \in (I : \Gamma G)$. □

Definition 3.27. We say that G is called connected if for any g_1, $g_2 \in G$, there exist $g \in G$, and m_1, $m_2 \in M$ and $\gamma \in \Gamma$ such that $g_1 = m_1\gamma g$ and $g_2 = m_2\gamma g$.

Note 3.4. G is locally monogenic $\implies$ G is connected $\implies$ G is monogenic.

Proof. Suppose G is locally monogenic. Let $g_1, g_2 \in G$. Now, $S = \{g_1, g_2\} \subseteq G$ implies there exist $g \in G$ and $\gamma \in \Gamma$ such that $\{g_1, g_2\} \subseteq M\gamma g$. Therefore, $g_1 = m_1\gamma g$ and $g_2 = m_2\gamma g$, for some $m_1, m_2 \in M$, thus G is connected.
Suppose G is connected. Let $g_1 \in G$, and since $0 \in G$, there exist $g \in G$, $m_1, m_2 \in M$ and $\gamma \in \Gamma$ such that $g_1 = m_1\gamma g$ and $0 = m_2\gamma g$. Therefore, G is monogenic. □

Corollary 3.28. *Let G be connected over a distributive Γ-nearring M. If I is a $(c, 2)$-absorbing ideal of G, then $(I : \Gamma G)$ is a $(c, 2)$-absorbing ideal of M.*

Proof. Since G is connected, we have G is monogenic, and hence the proof follows from Theorem 3.26. □

Definition 3.29. An ideal I of G is said to have IFP, if for any $m \in M$, $\gamma \in \Gamma$, and $g \in G$, $m\gamma g \in I$ implies $m\gamma m'\gamma g \in I$, for all $m' \in M$, and G is said to have IFP, if $\langle 0 \rangle$ is an IFP ideal.

Proposition 3.30. *Let G be locally monogenic over a distributive Γ-nearring M. If an ideal I of G is $(c,2)$-absorbing with IFP, then $(I : \gamma g)$ is $(c,2)$-absorbing in M for all $g \in G \backslash I$ and $\gamma \in \Gamma$.*

Proof. Let $g \in G\backslash I$ and $\gamma \in \Gamma$. If $1 \in (I : \gamma g)$, then $1\gamma g = g \in I$, a contradiction. Therefore, $(I : \gamma g) \neq M$ and hence $(I : \gamma g)$ is proper. Since $0_M\gamma g = 0_G \in I$, and so $0 \in (I : \gamma g)$, hence $(I : \gamma g) \neq \phi$. Clearly, $(I : \gamma g)$ is an ideal of M. To show that $(I : \gamma g)$ is $(c,2)$-absorbing, let $a, b, c \in M$ and $\gamma_1 \in \Gamma$ with $a\gamma_1 b\gamma_1 c \in (I : \gamma g)$. We need to show $a\gamma_1 b \in (I : \gamma g)$ or $b\gamma_1 c \in (I : \gamma g)$ or $a\gamma_1 c \in (I : \gamma g)$. Since $a\gamma_1 b\gamma_1 c \in (I : \gamma g)$, we have $a\gamma_1(b\gamma_1 c)\gamma g = a\gamma_1 b\gamma_1 c\gamma_1 g \in I$. Since I is a $(c,2)$-absorbing ideal of G, it follows that $a\gamma g \in I$ or $b\gamma_1 c\gamma g \in I$ or $a\gamma_1 b\gamma_1 c \in (I : \Gamma G)$.
Case (i): If $a\gamma g \in I$, then since I has IFP, we get $a\gamma m\gamma g \in I$ for all $m \in M$. In particular, $a\gamma b\gamma g \in I$ and so $a\gamma b \in (I : \gamma g)$.
Case (ii): If $b\gamma_1 c\gamma g \in I$, then clearly, $b\gamma_1 c \in (I : \Gamma G)$.
Case (iii): If $a\gamma_1 b\gamma_1 c \in (I : \Gamma G)$, then the proof follows from Theorem 3.26. □

Theorem 3.31. *Let I, J be ideals of G with $J \subseteq I$. Then I is a $(c,2)$-absorbing ideal of G if and only if $\frac{I}{J}$ is a $(c,2)$-absorbing ideal of $\frac{G}{J}$.*

Proof. Let J be an ideal of G. Then $\frac{G}{J}$ is an $M\Gamma$-group by natural way $m\gamma(g + J) = m\gamma g + J$ for all $m \in M, g \in G$ and $\gamma \in \Gamma$. Clearly, $\frac{I}{J}$ is an ideal of $\frac{G}{J}$. Let $a, b \in M$, $g \in G$ and $\gamma \in \Gamma$ such that $a\gamma b\gamma g + J \in \frac{I}{J}$. Then, $a\gamma b\gamma g + J = i + J$ for some $i \in I$. Then $a\gamma b\gamma g - i \in J \subseteq I$. Since I is an ideal of G, we have $a\gamma b\gamma g = a\gamma b\gamma g - i + i \in I$. Now, since I is $(c,2)$-absorbing, we have $a\gamma b \in (I : \Gamma G)$ or $a\gamma g \in I$ or $b\gamma g \in I$ shows that $a\gamma b\Gamma G \in I$ or $a\gamma g \in I$ or $b\gamma g \in I$. That is, $a\gamma b\Gamma\left(\frac{G}{J}\right) \subseteq \frac{I}{J}$ or $a\gamma(g + J) \in \frac{I}{J}$ or $b\gamma(g + J) \in \frac{I}{J}$. Hence, $\frac{I}{J}$ is a $(c,2)$-absorbing ideal of $\frac{G}{J}$.

Conversely, suppose that $\frac{I}{J}$ is a $(c,2)$-absorbing ideal of $\frac{G}{J}$. Since $\frac{I}{J} \neq \frac{G}{J}$, we have I is proper. Let $a,\ b \in M$ and $g \in G$ be such that $a\gamma b\gamma g \in I$. Then $a\gamma b\gamma(g + I) = a\gamma b\gamma g + J \in \frac{I}{J}$. Since $\frac{I}{J}$ is $(c,2)$-absorbing, we get $a\gamma(g + I) \in \frac{I}{J}$ or $b\gamma(g + I) \in \frac{I}{J}$ or $a\gamma b\left(\frac{G}{J}\right) \subseteq \frac{I}{J}$. This shows that, $a\gamma g \in I$ or $b\gamma g \in I$ or $a\gamma b \in (I : \Gamma G)$. Hence I is $(c,2)$-absorbing in G. □

We provide the notion of symmetric ideal which is analogous to the notion defined by [23].

Definition 3.32. An ideal I of G is said to be symmetric if for $a, b \in M, g \in G$, and $\gamma \in \Gamma$, $a\gamma b\gamma g \in I$ implies $b\gamma a\gamma g \in I$.

It is clear that if M is commutative, then every ideal of G is symmetric.

Theorem 3.33. *Let M be zero-symmetric. If $I = I_1 \cap I_2$ is symmetric where I_i, $(i = 1, 2)$ are c-prime ideals of G, then I is a $(c,2)$-absorbing ideal of G.*

Proof. Let I_1 and I_2 be c-prime ideals of G. To show $I = I_1 \cap I_2$ is a $(c,2)$-absorbing ideal of G, let $m_1, m_2 \in M$, $g \in G$ and $\gamma \in \Gamma$ with $m_1\gamma m_2\gamma g \in I$. Since $I = I_1 \cap I_2$ and $m_1\gamma m_2\gamma g \in I$, we have $m_1\gamma m_2\gamma g \in I_1$ and $m_1\gamma m_2\gamma g \in I_2$. Now, $(m_1\gamma m_2)\gamma g \in I_1$ and I_1 is c-prime, we have $m_1\gamma m_2\Gamma G \subseteq I_1$ or $g \in I_1$. This implies $m_1\gamma m_2 \in (I_1 : \Gamma G)$ or $g \in I_1$. Since I_1 is c-prime, by Lemma 3.17 we have $(I_1 : \Gamma G)$ is c-prime, and so $m_1 \in (I_1 : \Gamma G)$ or $m_2 \in (I_1 : \Gamma G)$ or $g \in I_1$.

In a similar argument, we get $m_1\gamma m_2\gamma g \in I_2$ implies $m_1 \in (I_2 : \Gamma G)$ or $m_2 \in (I_2 : \Gamma G)$ or $g \in I_2$.

Case (i): If $m_1 \in (I_1 : \Gamma G)$ and $m_1 \in (I_2 : \Gamma G)$, then $m_1 \in ((I_1 : \Gamma G) \cap (I_2 : \Gamma G)) = (I_1 \cap I_2 : \Gamma G)$ which implies $m_1\gamma g \in I_1 \cap I_2$, for all $g \in G$. Now, $m_2\gamma m_1\gamma g = m_2\gamma(0+m_1\gamma g) - m_2\gamma 0 \in I_1 \cap I_2$. Since I is symmetric, it follows that $m_1\gamma m_2\gamma g \in I_1 \cap I_2$ for all $g \in G$. That is, $m_1\gamma m_2 \in (I_1 \cap I_2 : \Gamma G)$. Therefore, in this case, we have obtained that $I_1 \cap I_2$ is $(c, 2)$-absorbing.

Case (ii): Let $m_1 \in (I_1 : \Gamma G)$ and $m_2 \in (I_2 : \Gamma G)$. Then by Lemma 3.16, we have $m_1\gamma m_2 \in (I_1 : \Gamma G) \cap (I_2 : \Gamma G) \subseteq (I_1 \cap I_2 : \Gamma G)$. Hence, $I_1 \cap I_2$ is a $(c, 2)$-absorbing ideal of G.

Case (iii): Let $m_1 \in (I_1 : \Gamma G)$ and $g \in I_2$. Then $m_1\gamma g \in I_1$ for all $g \in G$. Since $g \in I_2$ and I_2 is an ideal of G, it follows that $m_1\gamma g = m_1\gamma(0_G + g) - m_1\gamma 0_G \in I_2$. Therefore, $m_1\gamma g \in I_1 \cap I_2$. Hence, $I_1 \cap I_2$ is a $(c, 2)$-absorbing ideal of G.

□

From Theorem 3.18 and Theorem 3.33, we have the following Corollary.

Corollary 3.34. *Let M be zero-symmetric, and I_i, $(i = 1, 2)$ be $(c, 2)$-absorbing ideals of G, $I = I_1 \cap I_2$. If I is symmetric, then I is $(c, 2)$-absorbing.*

Theorem 3.35. *Let I be a $(c, 2)$-absorbing ideal of G. Then $(I : \Gamma G)$ is a c-prime ideal of M implies $(I : \gamma g)$ is a c-prime ideal of M for all $g \in G\backslash I$, $\gamma \in \Gamma$.*

Proof. Suppose $(I : \Gamma G)$ is a c-prime ideal of M. Let $m_1, m_2 \in M$, $g \in G\backslash I$ and $\gamma_1 \in \Gamma$ with $m_1\gamma m_2 \in (I : \gamma g)$. Then $m_1\gamma m_2\gamma g \in I$. Since I is $(c, 2)$-absorbing, we have $m_1\gamma g \in I$ or $m_2\gamma g \in I$ or $m_1\gamma m_2 \in (I : \Gamma G)$.
Case (i): If $m_1\gamma g \in I$, then $m_1 \in (I : \gamma g)$.
Case (ii): If $m_2\gamma g \in I$, then $m_2 \in (I : \gamma g)$.
Case (iii): Let $m_1\gamma m_2 \in (I : \Gamma G)$. Since $(I : \Gamma G)$ is c-prime, we have $m_1 \in (I : \Gamma G)$ or $m_2 \in (I : \Gamma G)$. Now, by Note 3.3, we get $m_1 \in (I : \gamma g)$ or $m_2 \in (I : \gamma g)$. Therefore, $(I : \gamma g)$ is a c-prime ideal of M. □

4 $(3, 2)$-absorbing ideal

In this section, we introduce a $(3, 2)$-absorbing ideal of G as a generalization of a 3-prime ideal of M as well as G.

Definition 4.1. An ideal I of G with $M\Gamma G \nsubseteq I$ is said to be 3-prime if whenever $m \in M$, $g \in G$ and $\gamma \in \Gamma$ with $m\gamma M\gamma g \subseteq I$, then $m\Gamma G \subseteq I$ or $g \in I$.

Definition 4.2. A proper ideal I of G is said to be a $(3, 2)$-absorbing ideal if whenever $m \in M$, $\gamma \in \Gamma$ and $g \in G$ with $m\gamma M\gamma g \subseteq I$, then $m\Gamma G \subseteq I$ or $g \in I$ or $M\gamma g \subseteq I$.

Lemma 4.3. *Every 3-prime ideal of G is $(3, 2)$-absorbing.*

Proof. Suppose I is 3-prime. Let $m \in M$, $\gamma \in \Gamma$ and $g \in G$ with $m\gamma M\gamma g \subseteq I$. Since I is 3-prime, we have $m\Gamma G \subseteq I$ or $g \in I$. □

Remark 4.4. The converse of Lemma 4.3 need not be true, in general. Consider the following example.

Example 4.5. (i) Take $M = \mathbb{Z}_8 = G$, nearring of integers, and $\Gamma = \{\gamma\}$ as given in Table 5. Then G is an $M\Gamma$-group. Here $I = \{0, 4\}$ is a $(3, 2)$-absorbing ideal of G but not 3-prime since $2\gamma M\gamma 3 = \{0\} \subseteq I$ but neither $2 \in I$ nor $3 \in I$.
(ii) Let $\Gamma = \{\gamma, \gamma_1\}$ in (i), given in Table 5 and 6. Then G is an $M\Gamma$-group. Here $I = \{0, 4\}$ is a $(i, 2)$-absorbing ideal of G but not i-prime, $i \in \{c, 3\}$, since $6\gamma 3 = \{0\} \subseteq I$, and $6\gamma_1 M\gamma_1 3 = \{0\} \subseteq I$ but neither $6 \in I$ nor $3 \in I$.

Lemma 4.6. *If M has unity, then every c-prime ideal of G is 3-prime.*

Proof. Suppose I is c-prime. To show I is 3-prime, let $m \in M$ and $g \in G$ such that $m\gamma M\gamma g \subseteq I$. If $g \in I$, then it is clear. Suppose $g \notin I$. Since $1 \in M$, we have $m\gamma g = m\gamma 1\gamma g \in m\gamma M\gamma g \subseteq I$, and I is c-prime implies $m\Gamma G \subseteq I$. Therefore I is 3-prime. □

γ	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	0	0	0	0	0	0	0
3	0	3	6	1	4	7	2	5
4	0	0	0	0	0	0	0	0
5	0	5	2	7	4	1	6	3
6	0	0	0	0	0	0	0	0
7	0	7	6	5	4	3	2	1

Table 5.

γ_1	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	0	0	0	0	0	0	0
3	0	3	6	1	4	7	2	5
4	0	0	0	0	0	0	0	0
5	0	1	2	3	4	5	6	7
6	0	0	0	0	0	0	0	0
7	0	3	6	1	4	7	2	5

Table 6.

Remark 4.7. The converse of Lemma 4.6 need not be true, in general (refer the Example 4.11).

Theorem 4.8. *Let M be a Γ-nearring with* 1. *If I is a $(c,2)$-absorbing ideal of G, then I is $(3,2)$-absorbing in G.*

Proof. Let I be a $(c,2)$-absorbing ideal of G, and let $m \in M, g \in G$ and $\gamma \in \Gamma$ with $m\gamma M\gamma g \subseteq I$. This implies $m\gamma m_1\gamma g \in I$ for all $m_1 \in M$. Since I is $(c,2)$-absorbing, it follows that $m\gamma m_1 \in (I : \Gamma G)$ or $m\gamma g \in I$ or $m_1\gamma g \in I$. That is, $m\gamma m_1\Gamma G \subseteq I$ or $m\gamma g \in I$ or $m_1\gamma g \in I$ for all $m_1 \in M$. Since $1 \in M$, we have $m\Gamma G \subseteq I$ or $m\gamma g \in I$ or $m_1\gamma g \in I$. Therefore, I is a $(3,2)$-absorbing ideal in G. □

Corollary 4.9. *Let M be a Γ-nearring with* 1. *Then every* 3*-prime ideal of G is a $(c,2)$-absorbing ideal of G.*

Example 4.10. Let $M = \mathbb{Z}_6 = G$ with $\Gamma = \{\gamma_1, \gamma_2\}$ where γ_1, γ_2 are given by scheme 1: (0,1,0,0,0,0) and scheme 2: (0,0,1,0,0,0) (see p. 409, Pilz [27]), and $+$ given in Table 7:

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Table 7.

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Table 8.

$$x\gamma_1 y = \begin{cases} x, & \text{if } y = 1 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad x\gamma_2 y = \begin{cases} x, & \text{if } y = 2 \\ 0, & \text{otherwise} \end{cases}$$

Then G is an $M\Gamma$-group. Clearly, $I = \{0\}$ is 3-prime as well as $(c, 2)$-absorbing, but not c-prime. Here, $2\gamma_1 3 = 0 \in I$ whereas $3 \notin I$ and $2\Gamma G = \{0, 2\} \nsubseteq I$.

Example 4.11. Here $M = \{0, a, b, c\} = G$. We define binary operations $+$ given in Table 8 and $\Gamma = \{\gamma_1, \gamma_2\}$ as follows:

$$x\gamma_1 y = \begin{cases} y, & \text{if } x \in \{a, c\} \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad x\gamma_2 y = \begin{cases} x, & \text{if } y = c \\ a, & \text{if } x \in \{a, c\},\ y \in M \setminus \{c\} \\ 0, & \text{otherwise} \end{cases}$$

Then M is an $M\Gamma$-group over itself. Here $I = \{0\}$ is $(c, 2)$-absorbing but not c-prime, as $b\gamma_1 a = 0 \in I$, $a \notin I$, and $b\Gamma M = \{0, b\} \nsubseteq I$. Also, I is not 3-prime as $b\gamma M\gamma a = \{0\} \in I$, but $a \notin I$ and $b\Gamma M \nsubseteq I$.

Example 4.12. Here $M = \{0, a, b, c\} = G$. We define binary operations $+$ given in Table 8 and $\Gamma = \{\gamma_1, \gamma_2\}$ as follows:

$$x\gamma_1 y = \begin{cases} b, & \text{if } x, y \in \{a, c\} \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad x\gamma_2 y = \begin{cases} x, & \text{if } x = y \\ 0, & \text{otherwise} \end{cases}$$

Then M is an $M\Gamma$-group over itself. Here $I = \{0\}$ is not $(c, 2)$-absorbing, as $a\gamma_1 c\gamma_1 a = 0 \in I$ but $a\gamma_1 c = b \notin I$, $c\gamma_1 a = b \notin I$, and $a\gamma_1 a = b \notin I$. I is not c-prime, as $c\gamma_2 b = 0 \in I$ but $b \notin I$ and $c\Gamma M = \{0, b, c\} \nsubseteq I$. Also, I is not 3-prime, as $b\gamma M\gamma c = \{0\} \in I$ but $c \notin I$ and $b\Gamma M = \{0, b\} \nsubseteq I$.

Lemma 4.13. *If I is a 3-prime ideal of G, then $(I : \Gamma G)$ is a 3-prime ideal in M.*

Proof. Since $I \neq G$, $1 \notin (I : \Gamma G)$ and so $(I : \Gamma G)$ is proper. Suppose that $m\gamma M\gamma m_1 \subseteq (I : \Gamma G)$ and $m_1 \notin (I : \Gamma G)$. So $m_1\Gamma G \nsubseteq I$ implies there exists $g \in G$, $\gamma_1 \in \Gamma$ with $m_1\gamma_1 g \notin I$. Now, $m\gamma M\gamma m_1 \subseteq (I : \Gamma G)$ implies $m\gamma M\gamma m_1\Gamma G \subseteq I$. Hence, $m\gamma M\gamma(m_1\gamma_2 g_2) \subseteq I$ for all $\gamma_2 \in \Gamma$ and $g_2 \in G$. Since I is 3-prime, we have $m\Gamma G \subseteq I$. This shows that $m \in (I : \Gamma G)$. Therefore, $(I : \Gamma G)$ is a 3-prime ideal of M. □

Theorem 4.14. *Let M be a Γ-nearring with 1, G be monogenic and I be an ideal of G. If $(I : \Gamma G)$ is a $(3, 2)$-absorbing ideal in M, then I is a $(3, 2)$-absorbing ideal of G.*

Proof. Let $(I : \Gamma G)$ be a $(3, 2)$-absorbing ideal of M. To prove I is a $(3, 2)$-absorbing ideal of G, take $m \in M$, $\gamma \in \Gamma$ and $g \in G$ with $m\gamma M\gamma g \subseteq I$. Since G is monogenic, we have $G = M\gamma g_1$ for some $g_1 \in G$. Now, $g = m_1\gamma g_1 \in I$, for some $m_1 \in M$. Then, $m\gamma M\gamma m_1\gamma g_1 = m\gamma M\gamma g \subseteq I$ implies $m\gamma M\gamma m_1 \subseteq (I : \gamma g_1)$. Hence, by Note 3.3, we have $m\gamma M\gamma m_1 \subseteq (I : \Gamma G)$. Since $(I : \Gamma G)$ is $(3, 2)$-absorbing, we have $m\Gamma M \subseteq (I : \Gamma G)$ or $m_1 \in (I : \Gamma G)$ or $M\gamma m_1 \subseteq (I : \Gamma G)$. That is, $m\Gamma M\Gamma G \subseteq I$ or $m_1 \in (I : \Gamma G)$ or $M\gamma m_1 \subseteq (I : \Gamma G)$. Since $1 \in M$, we get $m\Gamma G = m\Gamma 1\Gamma G \subseteq I$. Therefore, I is $(3, 2)$-absorbing. □

Theorem 4.15. *Let G be locally monogenic over a distributive Γ-nearring M. If I is a $(3, 2)$-absorbing ideal of G, then $(I : \Gamma G)$ is a $(3, 2)$-absorbing ideal of M.*

Proof. Suppose that I is $(3,2)$-absorbing in G. To show $(I:\Gamma G)$ is $(3,2)$-absorbing in M, let $m, m_1 \in M$, $\gamma \in \Gamma$ such that $m\gamma M\gamma m_1 \subseteq (I:\Gamma G)$. Assume that $m_1 \notin (I:\Gamma G)$ and $m\gamma m_1 \nsubseteq (I:\Gamma G)$. In this case, we show that $m\Gamma M \subseteq (I:\Gamma G)$. Since $m_1\Gamma G \nsubseteq I$ and $M\gamma m_1\Gamma G \nsubseteq I$, we have $m_1\gamma_1 g_1 \notin I$ and $m_2\gamma m_1\gamma_2 g_2 \notin I$, for some $g_1, g_2 \in G$ and $\gamma_1, \gamma_2 \in \Gamma$. Further, since $m\gamma M\gamma m_1 \subseteq (I:\Gamma G)$, we have $m\gamma M\gamma m_1\Gamma G \subseteq I$. In particular, $m\gamma m_2\gamma m_1\gamma_1 g_1, m\gamma m_2\gamma m_1\gamma_2 g_2 \in I$, and since I is an additive subgroup of G, it follows that $m\gamma m_2\gamma m_1\gamma_1 g_1 + m\gamma m_2\gamma m_1\gamma_2 g_2 \in I$. Now, by Lemma 3.25, $m\gamma m_2\gamma(m_1\gamma_1 g_1 + m_1\gamma_2 g_2) \in I$ for all $m_2 \in M$. Since I is $(3,2)$-absorbing in G, it follows that $m\Gamma G \subseteq I$ or $m_1\gamma_1 g_1 + m_1\gamma_2 g_2 \in I$ or $M\gamma(m_1\gamma_1 g_1 + m_1\gamma_2 g_2) \subseteq I$.
Case (i): If $m\Gamma G \subseteq I$, then $m\gamma g \in I$ for all $\gamma \in \Gamma$ and $g \in G$. Now, $m'\alpha m\beta g = m'\alpha(0 + m\beta g) - m'\alpha 0 \in I$ for all $m' \in M$. Since I is symmetric, we have $m\alpha m'\beta g \in I$ for all $m' \in M$, $\alpha, \beta \in \Gamma$ and $g \in G$. Hence, $m\Gamma M\Gamma G \subseteq I$ implies $m\Gamma M \subseteq (I:\Gamma G)$.
Case (ii): If $M\gamma(m_1\gamma_1 g_1 + m_1\gamma_2 g_2) \subseteq I$, then we have $m'\gamma(m_1\gamma_1 g_1 + m_1\gamma_2 g_2) \in I$ for all $m' \in M$. In particular, $m_2\gamma(m_1\gamma_1 g_1 + m_1\gamma_2 g_2) \in I$, implies $m_2\gamma m_1\gamma_1 g_1 + m_2\gamma m_1\gamma_2 g_2) \in I$. If $m_2\gamma m_1\gamma_2 g_2 \in I$, then $m_2\gamma m_1\gamma_2 g_2 = -m_2\gamma m_1\gamma_1 g_1 + m_2\gamma m_1\gamma_1 g_1 + m_2\gamma m_1\gamma_2 g_2 \in I$, a contradiction. Therefore, $m_2\gamma m_1\gamma_1 g_1 \notin I$, implies $M\gamma m_1\gamma_1 g_1 \nsubseteq I$. Also, $m_1\gamma_1 g_1 \notin I$. But $m\gamma M\gamma(m_1\gamma_1 g_1) \subseteq I$. Since I is a $(3,2)$-absorbing ideal of G, we have $m\Gamma G \subseteq I$. Hence by Case (i), we get $m\Gamma M \subseteq (I:\Gamma G)$.
Case (iii): Let $m_1\gamma_1 g_1 + m_1\gamma_2 g_2 \in I$. If $m_1\gamma_2 g_2 \in I$, we have $m_1\gamma_1 g_1 = m_1\gamma_1 g_1 + m_1\gamma_2 g_2 - m_1\gamma_2 g_2 \in I$, a contradiction. Therefore, $m_1\gamma_2 g_2 \notin I$. Also, $m_2\gamma m_1\gamma_2 g_2 \notin I$ implies $M\gamma m_1\gamma_2 g_2 \nsubseteq I$. But $m\gamma M\gamma m_1\gamma_2 g_2 \subseteq I$. Since I is a $(3,2)$-absorbing ideal in G, we have $m\Gamma G \subseteq I$. Hence by Case (i), we get $m\Gamma M \subseteq (I:\Gamma G)$.
Therefore, $(I:\Gamma G)$ is a $(3,2)$-absorbing ideal of M. □

5 Conclusion

We have defined the concepts $(i,2)$-absorbing, $i \in \{c,3\}$ ideals of G, as a generalization of i-prime ideal of G. We have proved significant properties and exhibited examples which indicate the classes are different from the existing classes of prime ideals. Further, one can be extend the concept to study the corresponding radicals and their properties of $(i,2)$-absorbing ideals.

References

[1] A. Badawi, On 2-Absorbing ideals of commutative rings, *Bull. Austral. Math. Soc.*, **75(3)**, 417–429 (2007). doi: 10.1017/S0004972700039344

[2] S. Bhavanari, *Contributions to Nearring Theory*, Nagarjuana University, (1984).

[3] S. Bhavanari, and G. Koteswara Rao, On a class of modules and N-groups, *J.Indian Math Soc.*, **59**, 39–44 (1993).

[4] S. Bhavanari, Lokeswara Rao and S.P. Kuncham, A note on Primeness in Nearrings and Matrix Nearrings, *Indian J Pure and Appl Math*, **27**, 227–234 (1996).

[5] S. Bhavanari, and S.P. Kuncham, *Nearrings, Fuzzy ideals and Graph theory*, CRC press(Taylor and Francis, U.K, U.S.A), (2013). ISBN 13: 9781439873106.

[6] S. Bhavanari, The f-prime radical in Γ-nearrings, *South-East Asian Bulletin of Mathematics*, **23**, 507–511 (1999).

[7] S. Bhavanari, A Note on Γ-nearrings. *Indian J. Mathematics (B.N. Prasad Birth Centenary commemoration volume)*, **41**, 427–433 (1999).

[8] S. Bhavanari, Modules over Gamma Nearrings, *Acharya Nagarjuna International Journal of Mathematics and Information Technology*, **1 (2)** 109–120 (2004).

[9] S. Bhavanari and Y.V. Reddy, A Note on N-groups, *Indian J. Pure & Appl. Math.*, **19** 842–845 (1988).

[10] S. Bhavanari, S.P. Kuncham, V.R. Paruchuri and M. Bhavanari, A note on dimensions in N-groups, *Italian Journal of Pure and Applied Mathematics*, **44**, 649–657 (2020).

[11] S. Bhavanari, V.R. Paruchuri, S.P. Kuncham and M. Bhavanari, Prime ideals of $M\Gamma$-groups, *Advances in Mathematics: Scientific Journal*, **9(1)**, 437–444 (2020).

[12] G.L. Booth, A Note on Gamma nearrings. Stud, *Sci. Math. Hungarica*, **23**, 471–475, (1988).

[13] G.L. Booth and N J Groenewald, Equiprime Gamma nearrings, *Quaest. Math.*, **14**, 411–417 (1991).

[14] A.Y. Darani and F. Soheilnia, 2-absorbing and weakly 2-absorbing submodules, *Thai J. Math*, **9(3)**, 577–584 (2011).

[15] N.J. Groenewald and Bac T. Ngnyen, On 2-absorbing modules over Non-commutative rings, *Int. Electron. J. Algebra*, **25**, 212–223 (2019). doi: 10.24330/ieja.504155

[16] N.J. Groenewald, Weakly prime and weakly completely prime ideals of non-commutative rings, *Int. Electron. J. Algebra*, **28**, 43–60 (2020). doi: 10.24330/ieja.768127

[17] N. Hamsa, S.P. Kuncham, and B.S. Kedukodi, $\Theta\Gamma$-N-groups, *Mat. Vesnik*, **70(1)**, 64–78 (2018).

[18] W.M.L. Holcombe, Primitive nearrings, *PhD diss., University of Leeds*, (1970).

[19] S. Juglal, N.J. Groenewald, and K.S.E Lee, Different prime $R-$ideals, *Algebra Colloquium*, **17(1)**, 887–904 (2010). doi: 10.1142/S1005386710000830

[20] W.F. Ke and J.H. Meyer, Matrix nearrings and 0-primitivity, *Monatsh Math*,**165**, 353–363 (2012). doi: 10.1007/s00605-010-0267-z

[21] W.A. Khan, A. Muhammad, A. Taouti, and J. Maki, Almost prime ideal in gamma nearring, *Eur. J. Pure Appl. Math.*, **11(2)**, 449–456 (2018).

[22] S.P. Kuncham, and B.S. Kedukodi, P.K. Harikrishnan and S. Bhavanari (Editors), *Nearrings, Nearfields and Related Topics*, World Scientific (Singapore), (2017). ISBN: 978-981-3207-35-6.

[23] J. Lambek, *L*ectures on Rings and Modules **Vol. 283**, American Mathematical Soc. (2009).

[24] J.D.P. Meldrum, *Nearrings and their links with groups* **No. 134**, Pitman Adv. publications programs, (1985).

[25] N. Nobusawa, *On a generalization of ring theory*, Osaka J. Math., **1**, 81–89 (1964).

[26] S. Patlertsin and S. Pianskool, Proc. of the 22nd annual meeting in mathematics (AMM 2017), Chiang Mai University, Thailand, 1–10 (2017).

[27] G. Pilz, Near-rings. *North Holland*, (1983).

[28] C. Selvaraj, R. George, and G.L. Booth, On strongly equiprime gamma nearrings. *Bull. Of the Institute of Mathematics*, Academic Sinica (New Series) **4 (1)**, 35–46 (2009).

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