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# An improvement for oscillation of first order difference equations

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**Abstract** In this article, we examine the first-order retarded difference equation

$$\Delta v(n)+\sum_{i=1}^{m} p_i(n)v\left(\delta_i(n)\right)=0,\ \ n\geq 0,$$

where $(\delta_i(n))(i=1,\ldots,m)$ are sequences of positive real numbers such that $\delta_i(n)\leq n$ for $n\geq 0$, and $\lim_{n\to\infty}\delta_i(n)=\infty$ and $(p_i(n))(i=1,\ldots,m)$ are sequences of nonnegative real numbers, When the delay terms $\delta_i(n)$ are not necessarily monotone, we will give an oscillation criterion obtained for the differential equation, but not yet obtained for its discrete analogous. Finally, we give an example to illustrate our result.

## 1 Introduction

In this article, we present a new sufficient criterion for the oscillation of all solutions of the retarded difference equations.

$$\Delta v(n)+\sum_{i=1}^{m} p_i(n)v\left(\delta_i(n)\right)=0,\ \ n=0,1,\ldots, \tag{1.1}$$

where $(p_i(n))$ are sequences of nonnegative real numbers, $(\delta_i(n))$ are sequences of positive real numbers and are not necessarily monotone such that

$$\delta_i(n)\leq n \text{ for } n\geq 0, \text{ and } \lim_{n\to\infty}\delta_i(n)=\infty. \tag{1.2}$$

$\Delta$ symbolizes the forward difference operator such that $\Delta v(n)=v(n+1)-v(n)$. Describe

$$k=-\min_{n\geq 0,\ 1\leq i\leq m}\delta_i(n),\quad \text{(Obviously, } k \text{ is a positive integer).}$$

By a solution of the difference equation (1.1), it means a sequence of real numbers $(v(n))$ which satisfies (1.1) for all $n\geq 0$. It is obvious that, for every choice of real numbers $c_{-k}, c_{-k+1}, \ldots, c_{-1}, c_0$, there is a unique solution $(v(n))$ of (1.1) which satisfies the initial conditions $v(-k)=c_{-k},\ v(-k+1)=c_{-k+1}, \ldots,\ v(-1)=c_{-1},\ v(0)=c_0$.
A solution $(v(n))$ of the difference equation (1.1) is said to be oscillatory, if the terms $v(n)$ of the sequence are neither eventually positive nor eventually negative. In other situation, the solution is called nonoscillatory.
If $m=1$, Eq. (1.1) reduces to

$$\Delta v(n)+p(n)v\left(\delta(n)\right)=0,\ \ n\in\mathbb{N}_0. \tag{1.3}$$

The oscillatory behavior of solutions of equations of (1.1) and (1.3) have been the topic of numerous investigations. See [1-23] and the references which are cited here.

In 1998, Zhang and Tian [23], analyzed (1.3) and established that, if $(\delta(n))$ is not necessarily monotone and

$$\limsup_{n\to\infty} p(n) > 0 \text{ and } \liminf_{n\to\infty} \sum_{j=\delta(n)}^{n-1} p(j) > \frac{1}{e}, \tag{1.4}$$

then all solutions of (1.3) are oscillatory.

When $(\delta(n))$ is not necessarily monotone, in 2008, Chatzarakis et al. [3, 4] considered (1.3) and found out that, if one of the following conditions

$$\limsup_{n\to\infty} \sum_{j=\varphi(n)}^{n} p(j) > 1 \tag{1.5}$$

or

$$\limsup_{n\to\infty} \sum_{j=\delta(n)}^{n-1} p(j) < \infty \quad \text{and} \quad c := \liminf_{n\to\infty} \sum_{j=\delta(n)}^{n-1} p(j) > \frac{1}{e} \tag{1.6}$$

hold, then all solutions of (1.3) are oscillatory, where $\varphi(n) = \max_{0\le s\le n} \delta(s)$, $n \ge 0$.

In 2008, Chatzarakis, Koplatadze and Stavroulakis [3] and in 2008 and 2009, Chatzarakis, Philos and Stavroulakis [5, 6] established the following criteria.

**Theorem 1.1.** $(I)$ *Suppose that* $0 < c \le \frac{1}{e}$. *Then either one of the conditions:*

$$\limsup_{n\to\infty} \sum_{j=\varphi(n)}^{n} p(j) > 1 - \left(1 - \sqrt{1-c}\right)^2, \tag{1.7}$$

*or*

$$\limsup_{n\to\infty} \sum_{j=\varphi(n)}^{n} p(j) > 1 - \frac{1}{2}\left(1 - c - \sqrt{1-2c}\right), \tag{1.8}$$

*or*

$$\limsup_{n\to\infty} \sum_{j=\varphi(n)}^{n} p(j) > 1 - \frac{1}{2}\left(1 - c - \sqrt{1-2c-c^2}\right) \tag{1.9}$$

*implies that all solutions of* (1.3) *are oscillatory.*
$(II)$ *If* $0 < c \le \frac{1}{e}$ *and also,* $p(n) \ge 1 - \sqrt{1-c}$ *for all large* $n$*, and*

$$\limsup_{n\to\infty} \sum_{j=\varphi(n)}^{n} p(j) > 1 - c\frac{1-\sqrt{1-c}}{\sqrt{1-c}}, \tag{1.10}$$

*or, if* $0 < c \le 6 - 4\sqrt{2}$ *and also,* $p(n) \ge \frac{c}{2}$ *for all large* $n$*, and*

$$\limsup_{n\to\infty} \sum_{j=\varphi(n)}^{n} p(j) > 1 - \frac{1}{4}\left(2 - 3c - \sqrt{4-12c+c^2}\right), \tag{1.11}$$

*then all solutions of* (1.3) *are oscillatory.*

We remark that
$(i)$ When $0 < c \le \frac{1}{e}$, it is easy to affirm that

$$\frac{1}{2}\left(1-c-\sqrt{1-2c-c^2}\right) > c\frac{1-\sqrt{1-c}}{\sqrt{1-c}} > \frac{1}{2}\left(1-c-\sqrt{1-2c}\right) > \left(1-\sqrt{1-c}\right)^2,$$

and therefore condition (1.9) is weaker than conditions (1.7), (1.8) and (1.10).
$(ii)$ When $0 < c \le 6 - 4\sqrt{2}$, it is easy to obtain that

$$\frac{1}{4}\left(2-3c-\sqrt{4-12c+c^2}\right) > \frac{1}{2}\left(1-c-\sqrt{1-2c-c^2}\right),$$

and therefore in this case, condition (1.11) is better than condition (1.9).

Now, we consider the equation (1.1). In 2006, Berezansky and Braverman [1] found out the following result for (1.1). If $(\delta_i(n))$ are not necessarily monotone, and

$$\limsup_{n\to\infty}\sum_{i=1}^{m}p_i(n)>0 \ \text{ and } \ \liminf_{n\to\infty}\sum_{j=\delta(n)}^{n-1}\sum_{i=1}^{m}p_i(j)>\frac{1}{e}, \tag{1.12}$$

where $\delta(n)=\max_{1\le i\le m}\delta_i(n)$, then every solution of (1.1) is oscillatory.

In 2013, Chatzarakis et al. [7], studied (1.1) and proved that, if $(\delta_i(n))$ are nondecreasing and

$$\limsup_{n\to\infty}\sum_{j=\delta(n)}^{n}\sum_{i=1}^{m}p_i(j)>1, \tag{1.13}$$

where $\delta(n)=\max_{1\le i\le m}\delta_i(n)$, then every solution of (1.1) is oscillatory.

Set

$$\varphi_i(n):=\max_{s\le n}\delta_i(s), \quad n\ge 0 \text{ and } \varphi(n)=\max_{1\le i\le m}\varphi_i(n), \tag{1.14}$$

and

$$\alpha:=\liminf_{n\to\infty}\sum_{j=\delta(n)}^{n-1}\sum_{i=1}^{m}p_i(j). \tag{1.15}$$

Clearly, $(\varphi_i(n))$ are nondecreasing, and $\delta_i(n)\le\varphi_i(n)\le\varphi(n)$ for all $n\ge 0$ and $1\le i\le m$.

In 2015, Braverman et al. [2], considered the equation (1.1) and proved that, if $(\delta_i(n))$ are not necessarily monotone and

$$\limsup_{n\to\infty}\sum_{j=\varphi(n)}^{n}\sum_{i=1}^{m}p_i(j)>1, \tag{1.16}$$

or

$$\limsup_{n\to\infty}\sum_{j=\varphi(n)}^{n}\sum_{i=1}^{m}p_i(j)>1-\frac{1}{2}\left(1-\alpha-\sqrt{1-2\alpha-\alpha^2}\right), \tag{1.17}$$

where $\varphi(n)$ is described by (1.14), then every solution of (1.1) is oscillatory.

In 2020, Kılıç and Öcalan [15], examined the equation (1.1) and established that, if $(\delta_i(n))(i=1,\dots,m)$ are not necessarily monotone and

$$\liminf_{n\to\infty}\sum_{j=\delta(n)}^{n-1}\sum_{i=1}^{m}p_i(j)=\liminf_{n\to\infty}\sum_{j=\varphi(n)}^{n-1}\sum_{i=1}^{m}p_i(j)>\frac{1}{e}, \tag{1.18}$$

where $\varphi(n)$ is described by (1.14), then every solution of (1.1) is oscillatory.

There is a huge concern for Eq. (1.1), due to the fact that retarded difference equation (1.1) which symbolizes a discrete analogue of the differential equation with delay

$$y'(t)+\sum_{i=1}^{m}p_i(t)y\left(\delta_i(t)\right)=0, \quad t\ge t_0, \tag{1.19}$$

where $\delta_i(t)\le t$ and $\lim_{t\to\infty}\delta_i(t)=\infty$ for $i=1,\dots,m$.

In the literature, the first method of proving many lemmas and theorems for oscillation in difference equations, which are discrete analogs of differential equations, is the method of calculating with direct sequences. However, in some cases, this proof technique is not valid. In cases where the first method cannot be used, the second method is the use of continuous functions in the interval of $n\le t<n+1$, where $n\in\mathbb{N}$, $t\in\mathbb{R}$. (See [6] and [11] ).

In 2017, Chatzarakis and Péics [9] (See also [16]) established that if $\delta_i(t)$ are not necessarily monotone and

$$\limsup_{t\to\infty}\int_{\varphi(t)}^{t}\sum_{i=1}^{m}p_i(s)ds>\frac{1+\ln\lambda_1}{\lambda_1}-\frac{1}{2}\left(1-\alpha_1-\sqrt{1-2\alpha_1-\alpha_1^2}\right), \tag{1.20}$$

where $\varphi_i(t) := \sup_{s \leq t} \delta_i(s)$, $t \geq 0$ and $\varphi(t) = \max_{1 \leq i \leq m} \varphi_i(t)$ and $\lambda_1 \in [1, e]$ is the smaller root of the equation $\lambda = e^{\alpha_1 \lambda}$ with $\alpha_1 = \liminf_{t \to \infty} \int\limits_{\delta(t)}^{t} \sum\limits_{i=1}^{m} p_i(s) ds$ (where $\delta(t) = \max_{1 \leq i \leq m} \delta_i(t)$), then all solutions of (1.19) oscillate.

A significant question originates whether there is a discrete analogue of condition (1.20). Clearly, the two methods mentioned above failed to answer this question. Therefore, our purpose in this article is to present a positive answer to this question by using a different proof technique.

## 2 Main Results

We established a new sufficient condition for the oscillatory behaviour of all solutions of (1.1), under the assumption that the delay terms $(\delta_i(n))$ are not necessarily monotone.

The next lemmas were given in [10], which they will be a key role in our main result.

**Lemma 2.1.** *Suppose that* (1.1) *is satisfied and* $\alpha$ *is defined by* (1.15) *with* $0 \leq \alpha \leq \frac{1}{e}$, *and* $(v(n))$ *is an eventually positive solution of* (1.1). *Then we have*

$$\liminf_{n \to \infty} \frac{v(\varphi(n))}{v(n)} \geq \lambda_0, \tag{2.1}$$

*where* $\varphi(n)$ *is defined by* (1.14) *and* $\lambda_0 \in [1, e]$ *is the smaller root of the equation* $\lambda = e^{\alpha \lambda}$.

**Lemma 2.2.** *Suppose that* (1.1) *is satisfied and* $\varphi(n)$ *is defined by* (1.14), $0 < \alpha \leq \frac{1}{e}$ *and* $v(n)$ *is an eventually positive solution of* (1.1). *Then*

$$\liminf_{n \to \infty} \frac{v(n+1)}{v(\varphi(n))} \geq \frac{1}{2}\left(1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}\right).$$

Now, we are ready for the main result.

**Theorem 2.3.** *Suppose that* (1.1) *holds and* $\alpha$ *is defined by* (1.15) *with* $0 \leq \alpha \leq \frac{1}{e}$. *If*

$$\limsup_{n \to \infty} \sum_{j=\varphi(n)}^{n} \sum_{i=1}^{m} p_i(j) > \frac{1 + \ln \lambda_0}{\lambda_0} - \frac{1}{2}\left(1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}\right), \tag{2.2}$$

*where* $\varphi(n)$ *is defined by* (1.14), *then all solutions of* (1.1) *oscillate.*

*Proof.* Assume, for the sake of contradiction, that there exists an eventually positive solution $v(n)$ of (1.1). If $v(n)$ is an eventually negative solution of (1.1), the proof of the theorem can be done similarly. Then there exists $n_1 > n_0$ such that $v(n)$, $v(\delta_i(n))$, $v(\varphi(n)) > 0$, for all $n \geq n_1$. Thus, from (1.1) we have

$$\Delta v(n) = -\sum_{i=1}^{m} p_i(n) v(\delta_i(n)) \leq 0, \quad \text{for all } n \geq n_1,$$

and it means that $(v(n))$ is an eventually nonincreasing sequence of positive numbers. On the other hand, since $\delta_i(n) \leq \varphi_i(n) \leq \varphi(n)$ for all $n \geq 0$, we have

$$\Delta v(n) + \sum_{i=1}^{m} p_i(n) v(\varphi(n)) \leq 0, \ \text{ for all } n \geq n_1. \tag{2.3}$$

By Lemma 2.1, inequality (2.1) is fulfilled. Therefore

$$\frac{v(\varphi(n))}{v(n)} > \lambda_0 - \varepsilon, \ \text{ for all } n \geq n_2 \geq n_1, \tag{2.4}$$

where $\varepsilon$ is an arbitrary real number with $0 < \varepsilon < \lambda_0$. Since the function

$$f(s) = \frac{v(\varphi(n))}{v[s]}, \ \ s \in \mathbb{R},$$

where $[\cdot]$ denotes the greatest integer function, is nondecreasing, we can find $t_* \leq t^* \in [\varphi(n), n]$, $t_*, t^* \in \mathbb{R}$ such that $[t^*] - [t_*] = 0$ and

$$\frac{v(\varphi(n))}{v\,[t^*]} \geq \lambda_0 - \varepsilon, \quad \text{for all } n \geq n_2, \tag{2.5}$$

and

$$\frac{v(\varphi(n))}{v\,[t_*]} \leq \lambda_0 - \varepsilon, \quad \text{for all } n \geq n_2. \tag{2.6}$$

We note that if $t_* = t^* = \varphi(n)$, then we get $\lambda_0 = 1$ and we return to the condition (1.16). Thus, we assume that $t_* \leq t^* \in (\varphi(n), n]$.

Summing up (1.1) from $[t^*]$ to $n$ and using the fact that the function $(v(n))$ is nonincreasing, we have

$$v(n+1) - v\,[t^*] + v(\varphi(n)) \sum_{i=1}^{m} \sum_{j=[t^*]}^{n} p_i(j) \leq 0. \tag{2.7}$$

Hence, we have

$$\sum_{i=1}^{m} \sum_{j=[t^*]}^{n} p_i(j) \leq \frac{v\,[t^*]}{v(\varphi(n))} - \frac{v(n+1)}{v(\varphi(n))},$$

which, in view of (2.5), we obtain

$$\sum_{i=1}^{m} \sum_{j=[t^*]}^{n} p_i(j) \leq \frac{1}{\lambda_0 - \varepsilon} - \frac{v(n+1)}{v(\varphi(n))}. \tag{2.8}$$

Dividing (1.1) by $v(n)$, summing up from $\varphi(n)$ to $[t_*] - 1$, we have

$$\sum_{j=\varphi(n)}^{[t_*]-1} \frac{\Delta v(j)}{v(j)} + \sum_{i=1}^{m} \sum_{j=\varphi(n)}^{[t_*]-1} p_i(j) \frac{v(\varphi(j))}{v(j)} \leq 0.$$

Hence, in view of (2.4) and (2.6), and since $\ln \frac{v[t_*]}{v(\varphi(n))} \leq \sum\limits_{j=\varphi(n)}^{[t_*]-1} \frac{\Delta v(j)}{v(j)}$, we get

$$\ln \frac{v\,[t_*]}{v(\varphi(n))} + (\lambda_0 - \varepsilon) \sum_{i=1}^{m} \sum_{j=\varphi(n)}^{[t_*]-1} p_i(j) \leq 0,$$

or

$$\sum_{i=1}^{m} \sum_{j=\varphi(n)}^{[t_*]-1} p_i(j) \leq \frac{1}{\lambda_0 - \varepsilon} \ln (\lambda_0 - \varepsilon). \tag{2.9}$$

Combining the inequalities (2.8) and (2.9), we have

$$\sum_{i=1}^{m} \sum_{j=\varphi(n)}^{n} p_i(j) \leq \frac{1}{\lambda_0 - \varepsilon} + \frac{\ln (\lambda_0 - \varepsilon)}{\lambda_0 - \varepsilon} - \frac{v(n+1)}{v(\varphi(n))}. \tag{2.10}$$

(2.10) is provided for all real numbers $\varepsilon$ with $0 < \varepsilon < \lambda_0$. Hence, for $\varepsilon \to 0$ we have

$$\limsup_{n\to\infty} \sum_{i=1}^{m} \sum_{j=\varphi(n)}^{n} p_i(j) \leq \frac{1 + \ln \lambda_0}{\lambda_0} - \frac{v(n+1)}{v(\varphi(n))}.$$

Using Lemma 2.2, the last inequality gives

$$\limsup_{n\to\infty} \sum_{i=1}^{m} \sum_{j=\varphi(n)}^{n} p_i(j) \leq \frac{1 + \ln \lambda_0}{\lambda_0} - \frac{1}{2}\left(1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}\right),$$

then the last inequality contradicts to (2.2). The proof is completed. □

**Example 2.4.** We consider the delay difference equation

$$\Delta v(n) + p_1(n)v(n-1) + p_2(n)v(n-2) = 0, \quad n \geq 0, \tag{2.11}$$

where $p_1(2n) = 0.2,\ p_1(2n+1) = 0.3$ and $p_2(2n) = 0.1,\ p_2(2n+1) = 0.32$. If we observe that, we find

$$\alpha = \liminf_{n\to\infty} \sum_{j=n-1}^{n-1} \sum_{i=1}^{2} p_i(j) = 0,3 < \frac{1}{e},$$

which means that condition (1.18) is not applicable for this equation. On the other hand, we have $\lambda_0 = 1.6313$ and

$$\limsup_{n\to\infty} \sum_{j=n-1}^{n} \sum_{i=1}^{2} p_i(j) = 0.92 < 1 - \frac{1-(0.3)-\sqrt{1-(0.6)-(0.3)^2}}{2} \cong 0.928\,39$$

which means that condition (1.17) is not applicable for this equation. On the other hand, it can be easily seen that

$$\begin{aligned}\limsup_{n\to\infty} \sum_{j=n-1}^{n} \sum_{i=1}^{2} p_i(j) &= 0.92 > \frac{1+\ln 1.6313}{1.6313} - \frac{\left(1-(0.3)-\sqrt{1-(0.6)-(0.3)^2}\right)}{2} \\ &\cong 0.84139.\end{aligned}$$

Therefore, all criteria of Theorem 2.3 hold, and so every solution of (2.11) oscillates.

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