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Uniqueness of Meromorphic functions concerning k-th derivatives and difference operators

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Abstract In this paper, we continue to study the sharing value problems for higher order derivatives of meromorphic functions with its linear difference and q-difference operators. Some of our results generalize and improve the results of Meng–Liu (J. Appl. Math. and Informatics, 37(2019), 133–148) to a large extent.

1 Introduction

Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, $f - a$ and $g - a$ have the same set of zeros with the same multiplicities, we say that f and g share the value a CM (counting multiplicities), and if we do not consider the multiplicities into account, then f and g are said to share the value a IM (ignoring multiplicities). We assume that the readers are familiar with the standard notations and symbols such as $T(r, f)$, $N(r, a; f)$ ($\overline{N}(r, a; f)$), $m(r, f)$ etc. of Nevanlinna's value distribution theory (see [11]).

In 2001, Lahiri ([19], [17]) introduced the definition of weighted sharing which plays a key role in uniqueness theory as far as relaxation of sharing is concerned. In the following, we explain the notion.

Definition 1.1. [19] Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a, f)$ the set of all a-points of f, where an a point of multiplicity m is counted m times if $m \leq k$, and $k + 1$ times if $m > k$. If $E_k(a, f) = E_k(a, g)$, we say that f, g share the value a with weight k.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly, if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2. [16] For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $N(r, a; f \mid= 1)$, the counting function of simple a-points of f. For a positive integer m, we denote by $N(r, a; f \mid\leq m)$ ($N(r, a; f \mid\geq m)$) the counting function of those a-point of f whose multiplicities are not greater (less) than m, where each a-point is counted according to its multiplicity.

$\overline{N}(r, a; f \mid\leq m)$ ($\overline{N}(r, a; f \mid\geq m)$) are defined similarly except that in counting the a-points of f we ignore the multiplicity. Also $N(r, a; f \mid< m)$, $N(r, a; f \mid> m)$, $\overline{N}(r, a; f \mid< m)$ and $\overline{N}(r, a; f \mid> m)$ are defined similarly.

Definition 1.3. [19] We denote by $N_2(r, a; f)$ the sum $\overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2)$.

Definition 1.4. [19] Let f and g share a value a IM. We denote by $\overline{N}_*(r, a; f, g)$ the counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Let c be a nonzero complex constant, and let $f(z)$ be a meromorphic function. The shift operator of $f(z)$ is denoted by $f(z + c)$. Also, we use the notations $\Delta_c f$ and $\Delta_c^k f$ to denote the difference and k-th order difference operators of f, which are respectively defined as

$$\Delta_c f = f(z + c) - f(z), \ \ \Delta_c^k f(z) = \Delta_c(\Delta_c^{k-1} f(z)), \ \ k \in \mathbb{N}, \ k \geq 2.$$

We note that $\Delta_c f$ and $\Delta_c^k f$ are nothing but linear combination of different shift operators. So for generalization of those operators, it is reasonable to introduce the linear difference operators $L(z, f)$ as follows:

$$L(z,f)=\sum_{j=0}^{p} a_j f(z+c_j), \tag{1.1}$$

where $p \in \mathbb{N} \cup \{0\}$ and a_j and c_j's are complex constants with at-least one a_j's are non-zero.

For a non-zero complex constant q and a meromorphic function f, the q-shift and q-difference operators are defined, respectively by $f(qz)$ and $\Delta_q f = f(qz) - f(z)$. Here also we generalize these operators as follows:

$$L_q(z,f)=\sum_{j=0}^{r} b_j f(q_j z+d_j), \tag{1.2}$$

where r is a non-negative integer, and q_j, b_j, d_j's are complex constants with at-least one of b_j is non-zero.

It was Rubel–Yang [30] who first initiated the problem of uniqueness of meromorphic functions sharing two values, and obtained the following result.

Theorem 1.5. *[30] Let f be a non-constant entire function. If f shares two distinct finite values CM with f', then $f \equiv f'$.*

Mues–Steinmetz [26] improved the above result by relaxing the nature of sharing two values from CM to IM. After that Mues–Steinmetz [27], and Gundersen [9] improved Theorem A to non-constant meromorphic functions.

Recently, the difference analogue of classical Nevanlinna theory for meromorphic functions of finite order was established by Halburd–Korhonen [12, 13], Chiang–Feng [8], independently, and developed by Halburd–Korhonen–Tohge [14] for hyper order strictly less than 1. After that, there has been an increasing interest in studying the uniqueness problems of meromorphic functions related to their shift or difference operators (see [7, 15, 21, 23, 28, 4, 5, 6, 22, 34]).

As we know that the time-delay differential equation $f(x) = f(x-k)$, $k > 0$ plays an important roll in real analysis, and it has been rigorously studied. For complex variable counterpart, Liu-Dong [24] studied the complex differential-difference equation $f(z) = f(z+c)$, where c is a non-zero constant.

In 2018, Qi *et al.* [29] looked at this complex differential-difference equation from a different perspective. In fact, they considered the value sharing problem related to $f'(z)$ and $f(z+c)$, where c is a complex number, and obtained the following result.

Theorem 1.6. *[29] Let f be a non-constant meromorphic function of finite order, $n \geq 9$ be an integer. If $[f'(z)]^n$ and $f^n(z+c)$ share $a(\neq 0)$ and ∞ CM, then $f'(z) = tf(z+c)$, for a constant t that satisfies $t^n = 1$.*

In 2019, Meng–Liu [25] reduced the nature of sharing values from CM to finite weight and obtained the following results.

Theorem 1.7. *[25] Let f be a non-constant meromorphic function of finite order, $n \geq 10$ an integer. If $[f'(z)]^n$ and $f^n(z+c)$ share $(1,2)$ and $(\infty,0)$, then $f'(z) = tf(z+c)$, for a constant t that satisfies $t^n = 1$.*

Theorem 1.8. *[25] Let f be a non-constant meromorphic function of finite order, $n \geq 9$ an integer. If $[f'(z)]^n$ and $f^n(z+c)$ share $(1,2)$ and (∞,∞), then $f'(z) = tf(z+c)$, for a constant t that satisfies $t^n = 1$.*

Theorem 1.9. *[25] Let f be a non-constant meromorphic function of finite order, $n \geq 17$ an integer. If $[f'(z)]^n$ and $f^n(z+c)$ share $(1,0)$ and $(\infty,0)$, then $f'(z) = tf(z+c)$, for a constant t that satisfies $t^n = 1$.*

For further investigation of Theorems 1.6–1.9, we pose the following questions.

Question 1.1. Could we determine the relationship between the k-th derivative $f^{(k)}(z)$ and the linear difference polynomial $L(z,f)$ as defined in (1.1) of a meromorphic (or entire) function $f(z)$ under relax sharing hypothesis?

Question 1.2. Could we further reduce the lower bound of n in Theorems 1.7–1.9?

In this direction, we prove the following result.

Theorem 1.10. *Let f be a non-constant meromorphic function of finite order, $n,\ k$ are positive integers, and $L(z,f)$ are defined in (1.1). Suppose $[f^{(k)}]^n$ and $[L(z,f)]^n$ share $(1,l)$ and (∞,m), where $0\le l<\infty$ and $0\le m\le\infty$, and one of the following conditions holds:*

(i) $l\ge 2$, $m=0$ *and* $n\ge 8$;

(ii) $l\ge 2$, $m=\infty$ *and* $n\ge 7$;

(iii) $l=1$, $m=0$ *and* $n\ge 9$;

(iv) $l=0$, $m=0$ *and* $n\ge 12$.

Then $f^{(k)}(z)=tL(z,f)$, for a non-zero constant t that satisfies $t^n=1$.

We give the following example in the support of Theorem 1.10.

Example 1.11. Let $f(z)=e^{z/n}$, where n is a positive integer. Suppose $L(z,f)=f(z+c)+c_0f(z)$, where c_0 is a non-zero complex constant such that $c_0\neq 1/n$, and $c=n\log((1-c_0n)/n)$.Then one can easily verify that $(f')^n$ and $(L(z,f))^n$ satisfy all the conditions of Theorem 1.10. Here $f'(z)=tL(z,f)$, where t is a constant such that $t^n=1$.

Remark 1.12. Let us suppose that $c_j=jc$, $j=0,1,\ldots,p$ and $a_p(z)=\binom{p}{0}$, $a_{p-1}=-\binom{p}{1}$, $a_{p-2}=\binom{p}{2}$. Then from (1.1), it is easily seen that $L(z,f)=\Delta_c^p f$. Therefore, we obtain the following corollary from Theorem 1.10.

Corollary 1.13. *Let f be a non-constant meromorphic function of finite order, $n,\ k$ are positive integers, and $L(z,f)$ are defined in (1.1). Suppose $[f^{(k)}]^n$ and $[\Delta_c^p f]^n$ share $(1,l)$ and (∞,m), where $0\le l<\infty$ and $0\le m\le\infty$, and one of the following conditions holds:*

(i) $l\ge 2$, $m=0$ *and* $n\ge 8$;

(ii) $l\ge 2$, $m=\infty$ *and* $n\ge 7$;

(iii) $l=1$, $m=0$ *and* $n\ge 9$;

(iv) $l=0$, $m=0$ *and* $n\ge 12$.

Then $f^{(k)}(z)=t\Delta_c^p f$, for a non-zero constant t that satisfies $t^n=1$.

For entire function we prove the following result which is an improvement of Corollary 1.8 of [25].

Theorem 1.14. *Let f be a non-constant entire function of finite order, $n,\ k$ are positive integers, and $L(z,f)$ are defined in (1.1). Suppose $[f^{(k)}]^n$ and $[L(z,f)]^n$ share $(1,l)$, and one of the following conditions holds:*

(i) $l\ge 1$ *and* $n\ge 5$;

(ii) $l=0$ *and* $n\ge 8$;

Then $f^{(k)}(z)=tL(z,f)$, for a non-zero constant t that satisfies $t^n=1$.

In the same paper, Meng–Liu [25] also obtained the following results by replacing $f(z+c)$ with q-shift operator $f(qz)$.

Theorem 1.15. *[25] Let f be a non-constant meromorphic function of zero order, $n\ge 10$ an integer. If $[f'(z)]^n$ and $f^n(qz)$ share $(1,2)$ and $(\infty,0)$, then $f'(z)=tf(qz)$, for a constant t that satisfies $t^n=1$.*

Theorem 1.16. *[25] Let f be a non-constant meromorphic function of zero order, $n \geq 9$ an integer. If $[f'(z)]^n$ and $f^n(qz)$ share $(1,2)$ and (∞,∞), then $f'(z) = tf(qz)$, for a constant t that satisfies $t^n = 1$.*

Theorem 1.17. *[25] Let f be a non-constant meromorphic function of zero order, $n \geq 17$ an integer. If $[f'(z)]^n$ and $f^n(qz)$ share $(1,0)$ and $(\infty,0)$, then $f'(z) = tf(qz)$, for a constant t that satisfies $t^n = 1$.*

For the generalizations and improvements of Theorems 1.15–1.17 to a large extent, we obtain the following result.

Theorem 1.18. *Let f be a non-constant meromorphic function of zero order, $n,\ k$ are positive integers, and $L_q(z,f)$ are defined in (1.2). Suppose $[f^{(k)}]^n$ and $[L_q(z,f)]^n$ share $(1,l)$ and (∞,m), where $0 \leq l < \infty$ and $0 \leq m \leq \infty$, and one of the following conditions holds:*

(i) $l \geq 2$, $m = 0$ and $n \geq 8$;

(ii) $l \geq 2$, $m = \infty$ and $n \geq 7$;

(iii) $l = 1$, $m = 0$ and $n \geq 9$;

(iv) $l = 0$, $m = 0$ and $n \geq 12$.

Then $f^{(k)} = tL_q(z,f)$, for a non-zero constant t that satisfies $t^n = 1$.

In 2018, Qi *et al.* [29] also proved the following result.

Theorem 1.19. *[29] Let f be a meromorphic function of finite order. Suppose that f' and $\Delta_c f$ share a_1, a_2, a_3, a_4 IM, where a_1, a_2, a_3, a_4 are four distinct finite values. Then, $f'(z) \equiv \Delta_c f$.*

We prove the following uniqueness theorem about the k-th derivative $f^{(k)}$ and linear difference polynomial $L(z,f)$ of a meromorphic function f, which is an extension of Theorem 1.19.

Theorem 1.20. *Let f be a meromorphic function of finite order. Suppose that $f^{(k)}$ and $L(z,f)$ share a_1, a_2, a_3, a_4 IM, where a_1, a_2, a_3, a_4 are four distinct finite values. Then,*

$$f^{(k)}(z) \equiv L(z,f).$$

2 Key Lemmas

In this section, we present some lemmas which will be needed in the sequel. Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We also denote by H, the following function

$$H = \Big(\frac{F''}{F'} - \frac{2F'}{F-1}\Big) - \Big(\frac{G''}{G'} - \frac{2G'}{G-1}\Big). \tag{2.1}$$

Lemma 2.1. *[19] Let $F,\ G$ be two non-constant meromorphic functions such that they share $(1,1)$ and $H \not\equiv 0$. Then*

$$N(r,1;F \mid= 1) = N(r,1;G \mid= 1) \leq N(r,H) + S(r,F) + S(r,G).$$

Lemma 2.2. *[3] Let $F,\ G$ be two non-constant meromorphic functions sharing $(1,t)$, where $0 \leq t < \infty$. Then*

$$\begin{aligned} &\overline{N}(r,1;F) + \overline{N}(r,1;G) - N_E^{1)}(r,1;F) + \left(t - \frac{1}{2}\right)\overline{N}_*(r,1;F,G) \\ \leq\ & \frac{1}{2}(N(r,1;F) + N(r,1;G)). \end{aligned}$$

Lemma 2.3. *[20] Suppose $F,\ G$ share $(1,0)$, $(\infty,0)$. If $H \not\equiv 0$, then,*

$$\begin{aligned} N(r,H) \quad &\leq N(r,0;F \mid\geq 2) + N(r,0;G \mid\geq 2) + \overline{N}_*(r,1;F,G) \\ &+\overline{N}_*(r,\infty;F,G) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,F) + S(r,G), \end{aligned}$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$, and $\overline{N}_0(r,0;G')$ is similarly defined.

Lemma 2.4. *[31] Let f be a non-constant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + \ldots + a_n f^n$, where $a_0, a_1, a_2, \ldots, a_n$ are constants and $a_n \neq 0$. Then*

$$T(r, P(f)) = nT(r, f) + O(1).$$

Lemma 2.5. *[18] If $N\left(r, 0; f^{(k)} \mid f \neq 0\right)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity then*

$$N\left(r, 0; f^{(k)} \mid f \neq 0\right) \leq k\overline{N}(r, \infty; f) + N\left(r, 0; f \mid < k\right) + k\overline{N}\left(r, 0; f \mid \geq k\right) + S(r, f).$$

Lemma 2.6. *[33] Let F and G be two non-constant meromorphic functions such that they share $(1, 0)$, and $H \not\equiv 0$, then*

$$N_E^{1)}(r, 1; F) \leq N(r, \infty; H) + S(r, F) + S(r, G).$$

Similar inequality holds for G also.

Lemma 2.7. *[1] If F, G be two non-constant meromorphic functions such that they share $(1, 1)$. Then*

$$\begin{aligned} & 2\overline{N}_L(r, 1; F) + 2\overline{N}_L(r, 1; G) + \overline{N}_E^{(2}(r, 1; F) - \overline{N}_{F>2}(r, 1; G) \\ \leq \; & N(r, 1; G) - \overline{N}(r, 1; G). \end{aligned}$$

Lemma 2.8. *[2] If two non-constant meromorphic functions F, G share $(1, 1)$, then*

$$\overline{N}_{F>2}(r, 1; G) \leq \frac{1}{2}(\overline{N}(r, 0; F) + \overline{N}(r, \infty; F) - N_0(r, 0; F')) + S(r, F),$$

where $N_0(r, 0; F'))$ is the counting function of those zeros of F' which are not the zeros of $F(F - 1)$.

Lemma 2.9. *[2] Let F and G be two non-constant meromorphic functions sharing $(1, 0)$. Then*

$$\begin{aligned} & \overline{N}_L(r, 1; F) + 2\overline{N}_L(r, 1; G) + \overline{N}_E^{(2}(r, 1; F) - \overline{N}_{F>1}(r, 1; G) - \overline{N}_{G>1}(r, 1; F) \\ & \leq N(r, 1; G) - \overline{N}(r, 1; G). \end{aligned}$$

Lemma 2.10. *[2] If F and G share $(1, 0)$, then*

$$\overline{N}_L(r, 1; F) \leq \overline{N}(r, 0; F) + \overline{N}(r, \infty; F) + S(r, F)$$

$$\overline{N}_{F>1}(r, 1; G) \leq \overline{N}(r, 0; F) + \overline{N}(r, \infty; F) - N_0(r, 0; F') + S(r, F).$$

Similar inequalities hold for G also.

Lemma 2.11. *[33] Let F and G be two non-constant meromorphic functions such that they share $(1, 0)$ and $H \not\equiv 0$. Then*

$$N_E^{1)}(r, 1; F) \leq N(r, \infty; H) + S(r, F) + S(r, H).$$

Lemma 2.12. *[32] Let f and g be two distinct non-constant rational functions and let a_1, a_2, a_3, a_4 be four distinct values. If f and g share a_1, a_2, a_3, a_4 IM, then $f(z) = g(z)$.*

Lemma 2.13. *[10] Suppose f and g are two distinct non-constant meromorphic functions, and $a_1, a_2, a_3, a_4 \in \mathbb{C} \cup \{\infty\}$ are four distinct values. If f and g share a_1, a_2, a_3, a_4 IM, then*

(i) $T(r, f) = T(r, g) + O(\log(rT(r, f)))$, *as $r \notin E$ and $r \to \infty$,*

(ii) $2T(r, f) = \sum_{j=1}^{4} \overline{N}\left(r, \dfrac{1}{f - a_j}\right) + O(\log(rT(r, f)))$, *as $r \notin E$ and $r \to \infty$, where $E \subset (1, \infty)$ is of finite linear measure.*

3 Proof of the theorems

We prove only Theorems 1.10 and 1.20 as the proof of the rest of the theorems are very much similar to the proof of Theorem 1.10.

***Proof of Theorem* 1.10.** **Case 1:** Suppose $H \not\equiv 0$.

Let $F = (L(z,f))^n$ and $G = (f^{(k)})^n$.

Keeping in view of Lemma 2.4, we get by applying Second fundamental theorem of Nevanlinna on F and G that

$$\begin{aligned} & n(T(r,L(z,f)) + T(r,f^{(k)})) \\ \leq\; & \overline{N}(r,0;F) + \overline{N}(r,1;F) + \overline{N}(r,\infty;F) + \overline{N}(r,0;G) + \overline{N}(r,1;G) \\ & + \overline{N}(r,\infty;G) - \overline{N}_0(r,0;F') - \overline{N}_0(r,0;G') + S(r,F) + S(r,G), \end{aligned} \tag{3.1}$$

where $\overline{N}_0(r,0;F')$ and $\overline{N}_0(r,0;G')$ are defined as in Lemma 2.3.

(i). Suppose $l \geq 2$ and $m = 0$.

Then using Lemmas 2.1, 2.2 and 2.3 in (3.1) we obtain

$$\begin{aligned} & \frac{n}{2}(T(r,L(z,f)) + T(r,(f^{(k)}))) \\ \leq\; & N_2(r,0;F) + N_2(r,0;G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) \\ & + \overline{N}_*(r,\infty;F,G) - \left(l - \frac{3}{2}\right)\overline{N}_*(r,1;F,G) + S(r,F) + S(r,G) \\ \leq\; & 2\overline{N}(r,0;L(z,f)) + 2\overline{N}(r,0;f^{(k)}) + \overline{N}(r,\infty;L(z,f)) \\ & + \overline{N}(r,\infty;f^{(k)}) + \frac{1}{2}(\overline{N}(r,\infty;L(z,f)) + \overline{N}(r,\infty;f^{(k)}) \\ & + S(r,F) + S(r,G) \\ \leq\; & \frac{7}{2}(T(r,L(z,f)) + T(r,f^{(k)}) + S(r,F) + S(r,G). \end{aligned}$$

This implies that

$$(n-7)(T(r,L(z,f)) + f^{(k)}) \leq S(r,L(z,f)) + S(r,f^{(k)}), \tag{3.2}$$

which contradict to the fact that $n \geq 8$.

(ii). Suppose $l \geq 2$ and $m = \infty$. Then using Lemmas 2.1, 2.2 and 2.3 in (3.1) we obtain

$$\begin{aligned} & \frac{n}{2}(T(r,L(z,f)) + T(r,f^{(k)})) \\ \leq\; & N_2(r,0;F) + N_2(r,0;G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) \\ & - \left(l - \frac{3}{2}\right)\overline{N}_*(r,1;F,G) + S(r,F) + S(r,G) \\ \leq\; & 2\overline{N}(r,0;L(z,f)) + 2\overline{N}(r,0;f^{(k)}) + \overline{N}(r,\infty;L(z,f)) \\ & + \overline{N}(r,\infty;f^{(k)}) + S(r,F) + S(r,G) \\ \leq\; & 3(T(r,L(z,f)) + T(r,f^{(k)})) + S(r,F) + S(r,G). \end{aligned}$$

This implies that

$$(n-6)(T(r,L(z,f)) + T(r,f^{(k)})) \leq S(r,L(z,f)) + S(r,f^{(k)}),$$

which contradict to the fact that $n \geq 7$.

(iii). Suppose $l = 1$ and $m = 0$.

Using Lemmas 2.1, 2.3, 2.7 and 2.8, we obtain

$$\begin{aligned}
& \overline{N}(r,1;F) \\
\leq\ & N(r,1;F\mid=1)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F) \\
\leq\ & \overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2)+\overline{N}_*(r,1;F,G)+\overline{N}_*(r,\infty;F,G) \\
& +\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G') \\
& +S(r,F)+S(r,G) \\
\leq\ & \overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2)+\overline{N}_*(r,\infty;F,G)+2\overline{N}_L(r,1;F) \\
& +2\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G') \\
& +S(r,F)+S(r,G) \\
\leq\ & \overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2)+\overline{N}_*(r,\infty;F,G)+N(r,1;G) \\
& -\overline{N}(r,1;G)+\overline{N}_{F>2}(r,1;G)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G') \\
& +S(r,F)+S(r,G) \\
\leq\ & \overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2)+\overline{N}_*(r,\infty;F,G)+N(r,0;G\mid G\neq 0) \\
& +\frac{1}{2}\overline{N}(r,0;F)+\frac{1}{2}\overline{N}(r,\infty;F)+\overline{N}_0(r,0;F')+S(r,F)+S(r,G) \\
\leq\ & \overline{N}(r,0;F\mid\geq 2)+N_2(r,0;G)+\overline{N}_*(r,\infty;F,G)+\overline{N}(r,\infty;G) \\
& +\frac{1}{2}\overline{N}(r,0;F)+\frac{1}{2}\overline{N}(r,\infty;F)+\overline{N}_0(r,0;F')+S(r,F)+S(r,G).
\end{aligned} \tag{3.3}$$

Similarly, we obtain

$$\begin{aligned}
& \overline{N}(r,1;G) \\
\leq\ & \overline{N}(r,0;G\mid\geq 2)+N_2(r,0;F)+\overline{N}_*(r,\infty;F,G)+\overline{N}(r,\infty;F) \\
& +\frac{1}{2}\overline{N}(r,0;G)+\frac{1}{2}\overline{N}(r,\infty;G)+\overline{N}_0(r,0;G')+S(r,F)+S(r,G).
\end{aligned} \tag{3.4}$$

Putting the values of $\overline{N}(r,1;F)$ and $\overline{N}(r,1;G)$ from (3.3) and (3.4) to (3.1), a simple calculation reduces to

$$\begin{aligned}
& n(T(r,L(z,f))+T(r,f^{(k)})) \\
\leq\ & 2N_2(r,0;F)+2N_2(r,0;G)+\frac{1}{2}(\overline{N}(r,0;F)+\overline{N}(r,0;G)) \\
& +\frac{7}{2}(\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G))+S(r,F)+S(r,G) \\
\leq\ & \frac{9}{2}(\overline{N}(r,0;L(z,f))+\overline{N}(r,0;f^{(k)})+\frac{7}{2}\overline{N}(r,\infty;L(z,f)) \\
& +\frac{7}{2}\overline{N}(r,\infty;f^{(k)})+S(r,F)+S(r,G) \\
\leq\ & 8(T(r,L(z,f)+T(r,f^{(k)}))+S(r,L(z,f))+S(r,f^{(k)}),
\end{aligned}$$

which is a contradiction since $n\geq 9$.

(iv). Suppose $l=0$ and $m=0$. Using Lemmas 2.11, 2.3, 2.5, 2.9 and 2.10, we obtain

$$\begin{aligned}
&\overline{N}(r,1;F)\\
\leq\ & N_E^{1)}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)\\
\leq\ & \overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2)+\overline{N}_*(r,1;F,G)+\overline{N}_*(r,\infty;F,G)\\
&+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')\\
&+S(r,F)+S(r,G)\\
\leq\ & \overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2)+\overline{N}_*(r,\infty;F,G)+2\overline{N}_L(r,1;F)\\
&+\overline{2}N_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')\\
&+S(r,F)+S(r,G)\\
\leq\ & \overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2)+\overline{N}_*(r,\infty;F,G)+\overline{N}_L(r,1;F)\\
&+\overline{N}_{F>1}(r,1;G)+\overline{N}_{G>1}(r,1;F)+N(r,1;G)-\overline{N}(r,1;G)+\overline{N}_0(r,0;F')\\
&+\overline{N}_0(r,0;G')+S(r,F)+S(r,G)\\
\leq\ & \overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2)+\overline{N}_*(r,\infty;F,G)+2\overline{N}(r,0;F)\\
&+2\overline{N}(r,\infty;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G)+N(r,1;G)-\overline{N}(r,1;G)\\
&+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,F)+S(r,G)\\
\leq\ & N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,0;F)+2\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\\
&+\overline{N}_*(r,\infty;F,G)+N(r,0;G'\mid G\neq 0)+\overline{N}_0(r,0;F')+S(r,F)+S(r,G)\\
\leq\ & N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,0;F)+\overline{N}(r,0;G)+2\overline{N}(r,\infty;F)\\
&+2\overline{N}(r,\infty;G)+\overline{N}_*(r,\infty;F,G)+\overline{N}_0(r,0;F')+S(r,F)+S(r,G).
\end{aligned}\tag{3.5}$$

Similarly, we can obtain

$$\begin{aligned}
&\overline{N}(r.1;G)\\
\leq\ & N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,0;F)+\overline{N}(r,0;G)+2\overline{N}(r,\infty;F)\\
&+2\overline{N}(r,\infty;G)+\overline{N}_*(r,\infty;F,G)+\overline{N}_0(r,0;G')+S(r,F)+S(r,G).
\end{aligned}\tag{3.6}$$

Using (3.5) and (3.5), (3.1) reduces to

$$\begin{aligned}
&n(T(r,L(z,f))+T(r,f^{(k)}))\\
\leq\ & 2(N_2(r,0;F)+N_2(r,0;G))+3(\overline{N}(r,0;F)+\overline{N}(r,0;G))\\
&+2\overline{N}_*(r,\infty;F,G)+3(\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G))\\
&+S(r,F)+S(r,G)\\
\leq\ & 7(\overline{N}(r,0;L(z,f))+\overline{N}(r,0;f^{(k)}))+4\overline{N}(r,\infty;L(z,f))\\
&+4\overline{N}(r,\infty;f^{(k)})+S(r,L(z,f))+S(r,f^{(k)})\\
\leq\ & 11(T(r,L(z,f))+T(r,f^{(k)}))+S(r,L(z,f))+S(r,f^{(k)}).
\end{aligned}$$

This implies that

$$(n-11)(T(r,L(z,f))+T(r,f^{(k)}))\leq S(r,L(z,f))+S(r,f^{(k)}),$$

which is a contradiction since $n\geq 12$.

Case 2: Suppose $H\equiv 0$. Then by integration we get

$$F=\frac{AG+B}{CG+D},\tag{3.7}$$

where $A,\ B,\ C$ and D are complex constants such that $AD-BC\neq 0$.

From (3.7), it is easily seen that $T(r, L(z, f)) = T(r, f^{(k)}) + O(1)$.

Subcase 2.1: Suppose $AC \neq 0$. Then $F - A/C = -(AD - BC)/C(CG + D) \neq 0$. So F omits the value A/C. Therefore, by the second fundamental theorem, we get

$$T(r, F) \leq \overline{N}(r, \infty; F) + \overline{N}(r, 0; F) + \overline{N}\left(r, \frac{A}{C}; F\right) + S(r, F).$$

This implies that

$$\begin{aligned} nT(r, L(z, f)) &\leq \overline{N}(r, \infty; L(z, f)) + \overline{N}(r, 0; L(z, f)) + S(r, L(z, f)) \\ &\leq 2T(r, L(z, f)) + S(r, L(z, f), \end{aligned}$$

which is not possible in all cases.

Subcase 2.2: Suppose that $AC = 0$. Since $AD - BC \neq 0$, both A and C can not be simultaneously zero.

Subcase 2.2.1: Suppose $A \neq 0$ and $C = 0$. Then (3.7) becomes $F \equiv \alpha G + \beta$, where $\alpha = A/D$ and $\beta = B/D$.

If F has no 1-point, then by the second fundamental theorem of Nevanlinna, we have

$$T(r, F) \leq \overline{N}(r, 0; F) + \overline{N}(r, 1; F) + \overline{N}(r, \infty; F) + S(r, F)$$

or,

$$(n - 2)T(r, L(z, f)) \leq S(r, L(z, f)),$$

which is not possible in all cases.

Let F has some 1-point. Then $\alpha + \beta = 1$. If $\beta = 0$, then $\alpha = 1$ and then $F \equiv G$ which implies that

$$L(z, f) = tf^{(k)},$$

where t is a constant such that $t^n = 1$.

Let $\beta \neq 0$. Then applying the second main theorem of Nevanlinna to F, we obtain

$$\begin{aligned} nT(r, L(z, f)) &\leq \overline{N}(r, \infty; F) + \overline{N}(r, 0; F) + \overline{N}(r, \beta; F) + S(r, F) \\ &\leq 2T(r, L(z, f)) + T(r, f^{(k)}) + S(r, L(z, f)) \\ &\leq 3T(r, L(z, f)) + S(r, L(z, f)), \end{aligned}$$

which is not possible in all cases.

Subcase 2.2.2: Suppose $A = 0$ and $C \neq 0$. Then (3.7) becomes

$$F \equiv \frac{1}{\gamma G_1 + \delta},$$

where $\gamma = C/B$ and $\delta = D/B$.

If F has no 1-point, then applying the second fundamental theorem to F, we have

$$\begin{aligned} nT(r, L(z, f)) &\leq \overline{N}(r, \infty; F) + \overline{N}(r, 0; F) + \overline{N}(r, 1; F) + S(r, F) \\ &\leq 2T(r, L(z, f)) + S(r, L(z, f)), \end{aligned}$$

which is a contradiction.

Suppose that F has some 1-point. Then $\gamma + \delta = 1$.

Therefore, $F \equiv 1/(\gamma G + 1 - \gamma)$. Since $C \neq 0$, $\gamma \neq 0$, and so G omits the value $(\gamma - 1)/\gamma$.

By the second fundamental theorem of Nevanlinna, we have

$$T(r, G) \leq \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) + \overline{N}\left(r, -\frac{1 - \gamma}{\gamma}; G\right) + S(r, G).$$

$i.e.$,

$$(n - 2)T(r, f^{(k)}) \leq S(r, f^{(k)}),$$

which is a contradiction. This completes the proof of the theorem. □

***Proof of Theorem* 1.20.** If f is rational, the conclusion follows by Lemma 2.12. Assume that f is transcendental meromorphic function. Then $f^{(k)}$ must be transcendental also. Now we discuss the following two cases.

Case 1: Suppose that $f^{(k)}$ is transcendental and $L(z,f)$ is rational. Then from Lemma 2.13 (i), it follows that

$$T(r,f^{(k)}) = T(r,L(z,f)) + O(\log rT(r,f^{(k)})) = O(\log(rT(r,f^{(k)}))),$$

which is a contradiction.

Case 2: Suppose f and $L(z,f)$ are both transcendental.

Now keeping in view of Lemma 2.13 (ii), and applying the second fundamental theorem of Nevanlinna to $f^{(k)}$, we obtain

$$\begin{aligned} 3T(r,f^{(k)}) &\leq \overline{N}(r,f^{(k)}) + \sum_{j=1}^{4} \overline{N}\left(r,\frac{1}{f^{(k)}-a_j}\right) + S(r,f^{(k)}) \\ &\leq \overline{N}(r,f^{(k)}) + 2T(r,f^{(k)}) + S(r,f^{(k)}) \\ &\leq 3T(r,f^{(k)}) + S(r,f^{(k)}), \end{aligned}$$

which implies that

$$N(r,f^{(k)}) = \overline{N}(r,f^{(k)}) + S(r,f^{(k)}).$$

This implies that

$$N(r,f) + k\overline{N}(r,f) = N(r,f^{(k)}) = \overline{N}(r,f^{(k)}) + S(r,f^{(k)}) = \overline{N}(r,f) + S(r,f^{(k)}).$$

This shows that

$$N(r,f) = \overline{N}(r,f) = \overline{N}(r,f^{(k)}) = S(r,f^{(k)}). \tag{3.8}$$

Again from Lemma 2.13 (i), we have

$$T(r,f^{(k)}) = T(r,L(z,f)) + S(r,f^{(k)}).$$

Keeping in view of (3.8), the above equation, and applying the second main theorem to $f^{(k)}$, we obtain

$$\begin{aligned} 3T(r,f^{(k)}) &\leq \overline{N}(r,f^{(k)}) + \sum_{j=1}^{4} \overline{N}\left(r,\frac{1}{f^{(k)}-1}\right) + S(r,f^{(k)}) \\ &\leq \overline{N}(r,f^{(k)}) + N\left(r,\frac{1}{f^{(k)}-L(z,f)}\right) + S(r,f^{(k)}) \\ &\leq T(r,f^{(k)}) + T(r,L(z,f)) + S(r,f^{(k)}) \\ &\leq 2T(r,f^{(k)}) + S(r,f^{(k)}), \end{aligned}$$

which is a contradiction. This completes the proof of the theorem. □

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