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# Nonlinear parabolic problem with lower order terms in Musielak-Sobolev spaces without sign condition and with Measure data

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**Abstract** We prove an entropy solutions for a class of strongly nonlinear parabolic problems with lower order terms in Musielak-Sobolev spaces, without sign condition on the nonlinearities and with measure data.

## 1 Introduction

Let $Q$ be the cylinder $\Omega \times (0, T)$, $T > 0$, $\Omega$ is a bounded domain of $\mathbb{R}^N$ with the segment property, In this work, we consider the following boundary value problem:

$$\begin{cases} \frac{\partial b(x,u)}{\partial t} + A(u) + \operatorname{div}(\Phi(x,t,u)) + g(x,t,u,\nabla u) = \mu & \text{in } Q \\ u(x,t) = 0 & \text{on } \partial\Omega \times [0,T] \\ b(x,u)|_{t=0} = b(x, u_0) & \text{on } \Omega \end{cases} \tag{1.1}$$

where $b : \Omega \times R \longrightarrow R$ be a Carathedory function (see the hypothesis (6.1) and (6.2), the operator of Leray-Lions type $A(u) = -\operatorname{div}(a(x,t,u,\nabla u))$ satisfies the classical Leray Lions assumptions of Musielak type (see assumptions ((6.3)-(6.5)), the term $\Phi(x,t,u)$ is a Cratheodory function assumed to be continuous on $u$ and satisfy only the growth condition $\Phi(x,t,u) \leq c(x,t)\bar{\varphi}^1\varphi(x, \alpha_0 u)$ , the nonlinearities $g$ satisfying the growth condition (see (6.6)) and the datum $\mu$ is assumed to be in $L^1(Q) + W^{-1,x}E_\psi(Q)$.

Under these assumptions, the above problem does not admit, in general, a weak solution since the field $a(x,t,u,\nabla u)$ does not belong to $\left(L^1_{loc}(Q)\right)^N$ in general. To overcome this difficulty we use in this paper the framework of entropy solutions.

The study of the nonlinear partial differential equations in Musielak-sobolev spaces is strongly motivated by the problems related to non-Newtonian fluids of strongly inhomogeneous behavior with a high ability of increasing their viscosity under a different stimulus, like the shear rate, magnetic or electric field (see for examples [22] and [23]).

Several work have treated the same problem ,we can't recite all examples but i will just choose some of them, for instance :

In the setting of classical Sobolev spaces, where $A$ is a Leary-Lions operator defined on $L^p\left(0, T; W^{1,p}(\Omega)\right)$ Porretta [27] proved the existence of solutions for the problem (1.1), where $g$ is a nonlinearity with natural growth condition and which satisfies the classical sign condition $g(x,t,s,\xi)s \geq 0$, the same problems of (1.1) have been studied by L. Boccardo and T. Gallouët in [8] where they proved the existence of solutions of (1.1) where $b(x,u) \equiv u$.

In the variable exponent case, in [3] the authors have studied the problem (1.1) where $b(x,u) = b(u)$ and $F \equiv 0$.

In the setting of Orlicz spaces where $\Phi \equiv \operatorname{div}(F) \equiv 0$, the existence of entropy solutions for parabolic problems of the form (1.1) has been proved by A. Elmahi and D. Meskine in [20] where $f$ belongs to $L^1(Q)$ and $g$ be a carathéodory function satisfying

$$|g(x,t,s,\xi)| \leq b(|s|)(c(x,t) + M(|\xi|)),$$
$$g(x,t,s,\xi)s \geq 0.$$

Recently, in the framework of Musielak spaces, Agnieszka, Swierczewska and Gwiazda in [32] studied the existence of weak solutions of problem (1.1) in the case where $\Phi \equiv g \equiv 0$ and $f \in L^\infty(Q)$, M.S.B. Elemine Vall and all in [18] have proved the existence of entropy solutions of (1.1) in the case where $b(x,u) = b(u)$, $g(x,t,s,\xi) = -\operatorname{div}(\Theta(x,t,u))$ where $\Theta$ a Carathéodory function does not satisfy any growth condition and $F \equiv 0$, also in [23] proved the existence of renormalized solutions of (1.1) where $a = a(x,\xi)$ and $\Phi \equiv g \equiv 0$ with the right hand side $f \in L^1(Q)$.

A large number of research deals with existence solutions of elliptic and parabolic problems under different assumptions in order to get a classical results see [5, 10, 11, 12, 13, 14, 15, 16, 17] for more details.

Our purpose in this note is to give an existence result of entropy solutions of the problem (1.1) in the setting of inhomogeneous Musielak- Orlicz-Sobolev spaces $W_0^{1,x}L_\varphi(Q)$ without $\Delta_2$ condition, losing the reflexivity of the spaces $L_\varphi(Q)$ and $W_0^1L\varphi(Q)$. The difficulty encountered during the proof of the existence of the solution is that the lower order term $g$ does not check the sign condition, no hypothesis of coercivity is assumed on $\Phi$ and the fact that the second term is a bounded measure.

This paper is organized as follows. In the second section we present some preliminaries results of Musielak Orlicz Sobolev spaces. The third section contains some important lemmas useful to prove our main results. In the fourth section we introduce some new approximations results in inhomogeneous Musielak-Orlicz-Sobolev spaces, and trace results. The fifth section consecrate to the compactness results used in this paper. We introduce in the final section some assumptions on $b(x,s), a(x,t,s,\xi), \Phi(x,t,s)$ and $g(x,t,s,\xi)$ for which our problem has a solution, and will be state and proved our main results.

## 2 Background

Here we give some definitions and properties that concern Musielak-Orlicz spaces (see [25]).

### 2.1 Musielak-Orlicz functions

Let $\Omega$ be an open subset of $\mathbb{R}^n$.

A Musielak-Orlicz function $\varphi$ is a real-valued function defined in $\Omega \times \mathbb{R}_+$ such that
a) $\varphi(x,t)$ is an N-function i.e. convex, nondecreasing, continuous, $\varphi(x,0) = 0, \varphi(x,t) > 0$ for all $t > 0$ and

$$\lim_{t\to 0} \sup_{x\in\Omega} \frac{\varphi(x,t)}{t} = 0, \qquad \lim_{t\to\infty} \inf_{x\in\Omega} \frac{\varphi(x,t)}{t} = 0.$$

b) $\varphi(\cdot,t)$ is a Lebesgue measurable function.

Now, let $\varphi_x(t) = \varphi(x,t)$ and let $\varphi_x^{-1}$ be the non-negative reciprocal function with respect to $t$, i.e the function that satisfies

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi\left(x, \varphi_x^{-1}(t)\right) = t.$$

The Musielak-orlicz function $\varphi$ is said to satisfy the $\Delta_2$ -condition if for some $k > 0$, and a non negative function $h$, integrable in $\Omega$, we have

$$\varphi(x,2t) \le k\varphi(x,t) + h(x) \text{ for all } x \in \Omega \text{ and } t \ge 0. \tag{2.1}$$

When 2.1 holds only for $t \ge t_0 > 0$, then $\varphi$ is said to satisfy the $\Delta_2$ -condition near infinity. Let $\varphi$ and $\gamma$ be two Musielak-orlicz functions, we say that $\varphi$ dominate $\gamma$ and we write $\gamma \prec \varphi$, near infinity (resp. globally) if there exist two positive constants $c$ and $t_0$ such that for almost all $x \in \Omega$

$$\gamma(x,t) \le \varphi(x,ct) \text{ for all } t \ge t_0, \quad (\text{ resp. for all } t \ge 0 \text{ i.e. } t_0 = 0)\,.$$

We say that $\gamma$ grows essentially less rapidly than $\varphi$ at 0 (resp. near infinity) and we write $\gamma \prec\prec \varphi$ if for every positive constant $c$ we have

$$\lim_{t\to 0}\left(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\right) = 0, \quad \left(\text{resp. } \lim_{t\to\infty}\left(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\right) = 0\right).$$

**Remark 2.1.** (see [1]) If $\gamma \prec\prec \varphi$ near infinity, then $\forall\varepsilon > 0$ there exists a nonnegative integrable function $h$, such that

$$\gamma(x,t) \leq \varphi(x,\varepsilon t) + h(x). \text{ for all } t \geq 0 \text{ and for a. e. } x \in \Omega. \tag{2.2}$$

## 2.2 Musielak-Orlicz-Sobolev spaces

For a Musielak-Orlicz function $\varphi$ and a measurable function $u : \Omega \longrightarrow \mathbb{R}$, we define the functional

$$\rho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x, |u(x)|)dx.$$

The set $K_\varphi(\Omega) = \{u : \Omega \longrightarrow \mathbb{R} \text{ measurable } / \rho_{\varphi,\Omega}(u) < \infty\}$ is called the Musielak-Orlicz class (or generalized Orlicz class). The Musielak-Orlicz space (the generalized Orlicz spaces) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently

$$L_\varphi(\Omega) = \left\{u : \Omega \longrightarrow \mathbb{R} \text{ measurable } / \rho_{\varphi,\Omega}\left(\frac{u}{\lambda}\right) < \infty, \text{ for some } \lambda > 0\right\}.$$

For a Musielak-Orlicz function $\varphi$ we put: $\psi(x,s) = \sup_{t>0}\{st - \varphi(x,t)\}$, $\psi$ is the Musielak-Orlicz function complementary to $\varphi$ (or conjugate of $\varphi$ ) in the sens of Young with respect to the variable $s$ in the space $L_\varphi(\Omega)$ we define the following two norms:

$$\|u\|_{\varphi,\Omega} = \inf\left\{\lambda > 0 / \int_\Omega \varphi\left(x, \frac{|u(x)|}{\lambda}\right) dx \leq 1\right\},$$

which is called the Luxemburg norm and the so-called Orlicz norm by:

$$|||u|||_{\varphi,\Omega} = \sup_{\|v\|_\psi \leq 1} \int_\Omega |u(x)v(x)|dx,$$

where $\psi$ is the Musielak Orlicz function complementary to $\varphi$. These two norms are equivalent (see [25])

We will also use the space $E_\varphi(\Omega)$ defined by

$$E_\varphi(\Omega) = \left\{u : \Omega \longrightarrow \mathbb{R} \text{ measurable } / \rho_{\varphi,\Omega}\left(\frac{u}{\lambda}\right) < \infty, \text{ for all } \lambda > 0\right\}.$$

A Musielak function $\varphi$ is called locally integrable on $\Omega$ if $\rho_\varphi(t\chi_D) < \infty$ for all $t > 0$ and all measurable $D \subset \Omega$ with meas $(D) < \infty$ Let $\varphi$ a Musielak function which is locally integrable. Then $E_\varphi(\Omega)$ is separable (see [25], Theorem 7.10) .

We say that sequence of functions $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $\lambda > 0$ such that

$$\lim_{n\to\infty} \rho_{\varphi,\Omega}\left(\frac{u_n - u}{\lambda}\right) = 0.$$

For any fixed nonnegative integer $m$ we define

$$W^m L_\varphi(\Omega) = \{u \in L_\varphi(\Omega) : \forall|\alpha| \leq m, D^\alpha u \in L_\varphi(\Omega)\},$$

and

$$W^m E_\varphi(\Omega) = \{u \in E_\varphi(\Omega) : \forall|\alpha| \leq m, D^\alpha u \in E_\varphi(\Omega)\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ with nonnegative integers $\alpha_i$, $|\alpha| = |\alpha_1| + \ldots + |\alpha_n|$ and $D^\alpha u$ denote the distributional derivatives.
The space $W^m L_\varphi(\Omega)$ is called the Musielak Orlicz Sobolev space.

Let

$$\bar{\rho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\leq m} \rho_{\varphi,\Omega}(D^\alpha u) \text{ and } \|u\|^m_{\varphi,\Omega} = \inf\left\{\lambda > 0 : \bar{\rho}_{\varphi,\Omega}\left(\frac{u}{\lambda}\right) \leq 1\right\}$$

for $u \in W^m L_\varphi(\Omega)$.
These functionals are a convex modular and a norm on $W^m L_\varphi(\Omega)$, respectively, and the pair $\left(W^m L_\varphi(\Omega), \|\|_{\varphi,\Omega}^m\right)$ is a Banach space if $\varphi$ satisfies the following condition (see[25]):

$$\text{there exist a constant } c_0 > 0 \text{ such that } \inf_{x\in\Omega} \varphi(x,1) \geq c_0. \tag{2.3}$$

The space $W^m L_\varphi(\Omega)$ will always be identified to a subspace of the product $\prod_{|\alpha|\leq m} L_\varphi(\Omega) = \Pi L_\varphi$, this subspace is $\sigma\left(\Pi L_\varphi, \Pi E_\psi\right)$ closed.

The space $W_0^m L_\varphi(\Omega)$ is defined as the $\sigma\left(\Pi L_\varphi, \Pi E_\psi\right)$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$. and the space $W_0^m E_\varphi(\Omega)$ as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$.

Let $W_0^m L_\varphi(\Omega)$ be the $\sigma\left(\Pi L_\varphi, \Pi E_\psi\right)$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$, the following spaces of distributions will also be used:

$$W^{-m} L_\psi(\Omega) = \left\{ f \in D'(\Omega); f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_\psi(\Omega) \right\},$$

and

$$W^{-m} E_\psi(\Omega) = \left\{ f \in D'(\Omega); f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in E_\psi(\Omega) \right\}.$$

We say that a sequence of functions $u_n \in W^m L_\varphi(\Omega)$ is modular convergent to $u \in W^m L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \bar{\rho}_{\varphi,\Omega}\left(\frac{u_n - u}{k}\right) = 0.$$

For $\varphi$ and her complementary function $\psi$, the following inequality is called the Young inequality (see[25]):

$$ts \leq \varphi(x,t) + \psi(x,s), \quad \forall t, s \geq 0, x \in \Omega, \tag{2.4}$$

this inequality implies that

$$\|u\|_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) + 1. \tag{2.5}$$

In $L_\varphi(\Omega)$ we have the relation between the norm and the modular

$$\|u\|_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) \text{ if } \|u\|_{\varphi,\Omega} > 1, \tag{2.6}$$

$$\|u\|_{\varphi,\Omega} \geq \rho_{\varphi,\Omega}(u) \text{ if } \|u\|_{\varphi,\Omega} \leq 1. \tag{2.7}$$

For two complementary Musielak Orlicz functions $\varphi$ and $\psi$, let $u \in L_\varphi(\Omega)$ and $v \in L_\psi(\Omega)$, then we have the Holder inequality (see[25]):

$$\left| \int_\Omega u(x)v(x)dx \right| \leq \|u\|_{\varphi,\Omega} \||v\||_{\psi,\Omega}. \tag{2.8}$$

## 2.3 Inhomogeneous Musielak-Orlicz-Sobolev spaces

Let $\Omega$ a bounded open subset of $\mathbb{R}^N$ and let $Q = \Omega \times ]0,T[$ with some given $T > 0$. Let $\varphi$ and $\psi$ be two complementary Musielak-Orlicz functions. For each $\alpha \in \mathbb{N}^N$ denote by $D_x^\alpha$ the distributional derivative on $Q$ of order $\alpha$ with respect to the variable $x \in \mathbb{R}^N$. The inhomogeneous Musielak-Orlicz-Sobolev spaces of order 1 are defined as follows.

$$W^{1,x} L_\varphi(Q) = \{u \in L_\varphi(Q) : \forall |\alpha| \leq 1 D_x^\alpha u \in L_\varphi(Q)\}$$

et

$$W^{1,x} E_\varphi(Q) = \{u \in E_\varphi(Q) : \forall |\alpha| \leq 1 D_x^\alpha u \in E_\varphi(Q)\}.$$

This second space is a subspace of the first one, and both are Banach spaces under the norm

$$\|u\| = \sum_{|\alpha| \leq 1} \|D_x^\alpha u\|_{\varphi,Q}$$

These spaces constitute a complementary system since $\Omega$ satisfies the segment property.These spaces are considered as subspaces of the product space $\Pi L_\varphi(Q)$ which has $(N+1)$ copies.

We shall also consider the weak topologies $\sigma(\Pi L_\varphi, \Pi E_\psi)$ and $\sigma(\Pi L_\varphi, \Pi L_\psi)$ If $u \in W^{1,x}L_\varphi(Q)$ then the function $t \to u(t) = u(\cdot,t)$ is defined on $[0,T]$ with values in $W^1L_\varphi(\Omega)$. If $u \in W^{1,x}E_\varphi(Q)$, then $u \in W^1E_\varphi(\Omega)$ and it is strongly measurable. Furthermore, the imbedding $W^{1,x}E_\varphi(Q) \subset L^1\left(0,T,W^1E_\varphi(\Omega)\right)$ holds. The space $W^{1,x}L_\varphi(Q)$ is not in general separable, for $u \in W^{1,x}L_\varphi(Q)$ we cannot conclude that the function $u(t)$ is measurable on $[0,T]$.

However, the scalar function $t \to \|u(t)\|_{\varphi,\Omega}$ is in $L^1(0,T)$. The space $W_0^{1,x}E_\varphi(Q)$ is defined as the norm closure of $\mathcal{D}(Q)$ in $W^{1,x}E_\varphi(Q)$. We can easily show as in [21] that when $\Omega$ has the segment property, then each element $u$ of the closure of $\mathcal{D}(Q)$ with respect of the weak $*$ topology $\sigma(\Pi L_\varphi, \Pi E_\psi)$ is a limit in $W^{1,x}L_\varphi(Q)$ of some subsequence $(v_j) \in \mathcal{D}(Q)$ for the modular convergence, i.e. there exists $\lambda > 0$ such that for all $|\alpha| \leq 1$

$$\int_Q \varphi\left(x, \left(\frac{D_x^\alpha v_j - D_x^\alpha u}{\lambda}\right)\right) dxdt \to 0 \text{ as } j \to \infty,$$

this implies that $(v_j)$ converges to $u$ in $W^{1,x}L_\varphi(Q)$ for the weak topology $\sigma(\Pi L_\varphi, \Pi L_\psi)$ Consequently

$$\overline{\mathcal{D}(Q)}^{\sigma(\Pi L_\varphi, \Pi E_\psi)} = \overline{\mathcal{D}(Q)}^{\sigma(\Pi L_\varphi, \Pi L_\psi)}$$

The space of functions satisfying such a property will be denoted by $W_0^{1,x}L_\psi(Q)$ Furthermore, $W_0^{1,x}E_\varphi(Q) = W_0^{1,x}L_\varphi(Q) \cap \Pi E_\varphi(Q)$. Thus, both sides of the last inequality are equivalent norms on $W_0^{1,x}L_\varphi(Q)$. We then have the following complementary system:

$$\begin{pmatrix} W_0^{1,x}L_\varphi(Q) & F \\ W_0^{1,x}E_\varphi(Q) & F_0 \end{pmatrix}$$

where $F$ states for the dual space of $W_0^{1,x}E_\varphi(Q)$. and can be defined, except for an isomorphism, as the quotient of $\Pi L_\psi$ by the polar set $W_0^{1,x}E_\varphi(Q)^\perp$. It will be denoted by $F = W^{-1,x}L_\psi(Q)$, where

$$W^{-1,x}L_\psi(Q) = \left\{ f = \sum_{|\alpha| \leq 1} D_x^\alpha f_\alpha : f_\alpha \in L_\psi(Q) \right\}$$

This space will be equipped with the usual quotient norm

$$\|f\| = \inf \sum_{|\alpha| \leq 1} \|f_\alpha\|_{\psi,Q}$$

where the infimum is taken over all possible decompositions

$$f = \sum_{|\alpha| \leq 1} D_x^\alpha f_\alpha, \quad f_\alpha \in L_\psi(Q)$$

The space $F_0$ is then given by

$$F_0 = \left\{ f = \sum_{|\alpha| \leq 1} D_x^\alpha f_\alpha : f_\alpha \in E_\psi(Q) \right\}$$

and is denoted by $F_0 = W^{-1,x}E_\psi(Q)$ .

## 3 Truncation Operator

$T_k, k > 0$, denotes the truncation function at level $k$ defined on $\mathbb{R}$ by $T_k(r) = \max(-k, \min(k, r))$. The following abstract lemmas will be applied to the truncation operators.

**Definition 3.1.** A Musielak function $\varphi$ satisfies the log-Hölder continuity condition on $\Omega$ if there exists a constant $A > 0$ such that

$$\frac{\varphi(x,t)}{\varphi(y,t)} \leq t^{\left(\frac{A}{\log\left(\frac{1}{|x-y|}\right)}\right)} \tag{3.1}$$

for all $t \geq 1$ and for all $x, y \in \Omega$ with $|x-y| \leq \frac{1}{2}$

**Lemma 3.2.** *[1] Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^N (N \geq 2)$ and let $\varphi$ be a Musielak function satisfying the log-Hölder continuity such that*

$$\bar{\varphi}(x,1) \leq c_1 \quad \text{a.e in } \Omega \text{ for some } c_1 > 0 \tag{3.2}$$

*Then $\mathfrak{D}(\Omega)$ is dense in $L_\varphi(\Omega)$ and in $W_0^1 L_\varphi(\Omega)$ for the modular convergence.*

**Remark 3.3.** Note that if $\lim_{t\to\infty} \inf_{x\in\Omega} \frac{\varphi(x,t)}{t} = \infty$, then (3.2) holds (see [1]).

**Example 3.4.** Let $p \in P(\Omega)$ a bounded variable exponent on $\Omega$, such that there exists a constant $A > 0$ such that for all points $x, y \in \Omega$ with $|x-y| < \frac{1}{2}$, we have the inequality

$$|p(x) - p(y)| \leq \frac{A}{\log\left(\frac{1}{|x-y|}\right)}$$

We can verify that the Musielak function defined by $\varphi(x,t) = t^{p(x)} \log(1+t)$ satisfies the conditions of Lemma 3.2.

**Proof :** (see [1]).

**Lemma 3.5.** *[1] (Poincare's inequality: Integral form) Let $\Omega$ be a bounded Lipschitz domain of $R^N (N \geq 2)$ and let $\varphi$ be a Musielak function satisfying the conditions of Lemma 3.2. Then there exists positive constants $\beta, \eta$ and $\lambda$ depending only on $\Omega$ and $\varphi$ such that*

$$\int_\Omega \varphi(x, |v|)dx \leq \beta + \eta \int_\Omega \varphi(x, \lambda|\nabla v|)dx \text{ for all } v \in W_0^1 L_\varphi(\Omega). \tag{3.3}$$

**Lemma 3.6.** *[1] (Poincare's inequality) Let $\Omega$ be a bounded Lipchitz domain of $\mathbb{R}^N (N \geq 2)$ and let $\varphi$ be a Musielak function satisfying the same conditions of Lemma 3.5. Then there exists a constant $C > 0$ such that*

$$\|v\|_\varphi \leq C\|\nabla v\|_\varphi \quad \forall v \in W_0^1 L_\varphi(\Omega).$$

**Lemma 3.7.** *[27]. Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let $\varphi$ be a Musielak- Orlicz function and let $u \in W_0^1 L_\varphi(\Omega)$. Then $F(u) \in W_0^1 L_\varphi(\Omega)$. Moreover, if the set $D$ of discontinuity points of $F'$ is finite, we have*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} \text{ a.e in } \{x \in \Omega : u(x) \in D\}. \\ 0 \quad \text{a.e in } \{x \in \Omega : u(x) \notin D\}. \end{cases}$$

**Lemma 3.8** (Poincaré inequality). *[1]Let $\varphi$ a Musielak Orlicz function which satisfies the assumptions of lemma 3.2, suppose that $\varphi(x,t)$ decreases with respect of one of coordinate of $x$. Then, that exists a constant $c > 0$ depends only of $\Omega$ such that*

$$\int_\Omega \varphi(x, |u(x)|)dx \leq \int_\Omega \varphi(x, c|\nabla u(x)|)dx, \quad \forall u \in W_0^1 L_\varphi(\Omega). \tag{3.4}$$

**Lemma 3.9.** *[7] Let $u_n, u \in L_\varphi(\Omega)$. If $u_n \to u$ with respect to the modular convergence, then $u_n \to u$ for $\sigma(L_\varphi(\Omega), L_\psi(\Omega))$.*

**Lemma 3.10** (The Nemytskii Operator). *[2]. Let $\Omega$ be an open subset of $\mathbb{R}^N$ with finite measure and let $\varphi$ and $\psi$ be two Musielak-Orlicz functions. Let $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Carathéodory function such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$ :*

$$|f(x,s)| \leq c(x) + k_1 \psi_x^{-1} \varphi(x, k_2|s|). \tag{3.5}$$

*where $k_1$ and $k_2$ are real positives constants and $c(.) \in E_\psi(\Omega)$.*

*Then the Nemytskii Operator $N_f$ defined by $N_f(u)(x) = f(x, u(x))$ is continuous from*

$$\mathcal{P}\Big(E_\varphi(\Omega), \frac{1}{k_2}\Big) = \Big\{u \in L_\varphi(\Omega) : d(u, E_\varphi(\Omega)) < \frac{1}{k_2}\Big\}.$$

*into $L_\psi(\Omega)$.*

*Furthermore if $c(\cdot) \in E_\gamma(\Omega)$ and $\gamma \prec\prec \psi$ then $N_f$ is strongly continuous from $\mathcal{P}\Big(E_\varphi(\Omega), \frac{1}{k_2}\Big)$ to $E_\gamma(\Omega)$.*

**Lemma 3.11.** *Assume that (6.3)-(6.5) are satisfies and let $(z_n)_n$ be a sequence in $W_0^1 L_\varphi(\Omega)$ such that*

*i) $z_n \to z$ in $W_0^1 L_\varphi(\Omega)$ for $\sigma(\Pi L_\varphi, \Pi E_\psi)$,*

*ii) $(a(\cdot, t, z_n, \nabla z_n))_n$ is bounded in $(L_\psi(\Omega))^N$,*

*iii) $\displaystyle\int_\Omega (a(x,t,z_n,\nabla z_n) - a(x,t,z_n,\nabla z \chi_s))(\nabla z_n - \nabla z \chi_s)\,dx \longrightarrow 0$ as $n, s \longrightarrow \infty$,*

*where $\chi_s$ is the characteristic function of*

$$\Omega_s = \{x \in \Omega : |\nabla z| \leq s\}.$$

*Then, we have $z_n \longrightarrow z$ for the modular convergence in $W_0^1 L_\varphi(\Omega)$.*

**Proof:** It is easily adapted from that given in [4].

## 4 Approximation and trace results

In this section, $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^N$ with the segment property and $I$ is a subinterval of $R$ (both possibly unbounded) and $Q = \Omega \times I$. It is easy to see that $Q$ also satisfies Lipschitz domain.

**Definition 4.1.** We say that $u_n \longrightarrow u$ in $W^{-1,x} L_\psi(Q) + L^1(Q)$ for the modular eonvergence if we can write

$$u_n = \sum_{|\alpha| \leq 1} D_x^\alpha u_n^\alpha + u_n^0 \quad \text{and} \quad u = \sum_{|\alpha| \leq 1} D_x^\alpha u^\alpha + u^0$$

with $u_n^\alpha \longrightarrow u^\alpha$ in $L_\psi(Q)$ for the modular convergence for all $|\alpha| \leq 1$, and $u_n^0 \longrightarrow u^0$ strongly in $L^1(Q)$

The following approximation theorem, plays a fundamental role when the existence of solutions for parabolic problems is proved.

**Theorem 4.2.** *Let $\varphi$ be an Musielak-Orlicz function satisfies the assumption (3.1).*

*If $u \in W^{1,x} L_\varphi(Q) \cap L^1(Q)$ (resp. $W_0^{1,x} L_\varphi(Q) \cap L^1(Q)\big)$ and $\frac{\partial u}{\partial t} \in W^{-1,x} L_\psi(Q) + L^1(Q)$, then there exists a sequence $(v_j)$ in $\mathcal{D}(\bar{Q})$ (resp. $\mathcal{D}(\bar{I}, \mathcal{D}(Q))$ ) such that $v_j \to u$ in $W^{1,x} L_\varphi(Q)$ and*

$$\frac{\partial v_j}{\partial t} \to \frac{\partial u}{\partial t} \text{ in } W^{-1,x} L_\psi(Q) + L^1(Q)$$

*for the modular convergence.*

*Proof.*
Let $u \in W^{1,x}L_\varphi(Q) \cap L^1(Q)$ and $\frac{\partial u}{\partial t} \in W^{-1,x}L_\psi(Q) + L^1(Q)$, then for any $\epsilon > 0$. Writing $\frac{\partial u}{\partial t} = \sum_{|\alpha|\leq 1} D_x^\alpha u^\alpha + u^0$, where $u^\alpha \in L_\psi(Q)$ for all $|\alpha| \leq 1$ and $u^0 \in L^1(Q)$, we will show that there exits $\lambda > 0$ (depending Only on $u$ and $N$ and there exists $v \in \mathcal{D}(\bar{Q})$ for which we can write $\frac{\partial u}{\partial t} = \sum_{|\alpha|\leq 1} D_x^\alpha v^\alpha + v^0$ with $v^\alpha, v^0 \in \mathcal{D}(\bar{Q})$ such that

$$\int_Q \varphi\left(x, \frac{D_x^\alpha v - D_x^\alpha u}{\lambda}\right) dxdt \leq \epsilon, \forall |\alpha| \leq 1 \tag{4.1}$$

$$\|v - u\|_{L^1(Q)} \leq \epsilon \tag{4.2}$$

$$\left\|v^0 - u^0\right\|_{L^1(Q)} \leq \epsilon \tag{4.3}$$

$$\int_Q \psi\left(x, \frac{v^\alpha - u^\alpha}{\lambda}\right) dxdt \leq \epsilon, \quad \forall |\alpha| \leq 1 \tag{4.4}$$

The equation (4.1) flows from a slight adaptation of the arguments [6], the equations (4.2)-(4.3) flows also from classical approximation results. For The equation (4.4) we know that $\mathcal{D}(\bar{Q})$ is dense in $L_\psi(Q)$ for the modular convergence. The case where $u \in W_0^{1,x}L_\varphi(Q) \cap L^1(Q)$ ) can be handled similarly without essential difficulty as it mentioned [6]. □

**Lemma 4.3.** *[26] Let $a < b \in \mathbb{R}$ and let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^N$. Then*

$$\left\{u \in W_0^{1,x}L_\varphi(\Omega \times ]a,b[) : \frac{\partial u}{\partial t} \in W^{-1,x}L_\psi(\Omega \times ]a,b[) + L^1(\Omega \times ]a,b[)\right\}$$

*is a subset of $C(]a,b\,[, L^1(\Omega))$.*

In order to deal with the time derivative, we introduce a time mollification of a function $u \in W_0^{1,x}L_\varphi(Q)$ Thus we define, for all $\mu > 0$ and all $(x,t) \in Q$

$$u_\mu(x,t) = \int_{-\infty}^{t} \tilde{u}(x,\sigma) \exp(\mu(\sigma - t)) d\sigma \tag{4.5}$$

where $\tilde{u}(x,t) = u(x,t)\chi_{[0,T]}(t)$ Throughout the paper the index ì always indicates this mollification.

**Lemma 4.4.** *[26] If $u \in L_\varphi(Q)$ then $u_\mu$ is measurable in $Q$ and $\frac{\partial u_\mu}{\partial t} = \mu(u - u_\mu)$ and if $u \in K_\varphi(Q)$ then*

$$\int_Q \varphi(x, u_\mu)\, dxdt \leq \int_Q \varphi(x,u) dxdt$$

**Lemma 4.5.** *[19]*

*(1) If $u \in L_\varphi(Q)$ then $u_\mu \longrightarrow u$ for the modular convergence in $L_\varphi(Q)$ as*

$$\mu \longrightarrow \infty.$$

*(2) If $u \in W_0^{1,x}L_\varphi(Q)$ then $u_\mu \longrightarrow u$ for the modular convergence in $W_0^{1,x}L_\varphi(Q)$*

*as $\mu \longrightarrow \infty$.*

**Remark 4.6.** If $u \in E_\varphi(Q)$, we can choose $\lambda$ arbitrary small since $D(Q)$ is (norm) dense in $E_\varphi(Q)$ Thus, for all $\lambda > 0$, we have

$$\int_Q \varphi\left(x, \frac{u_\mu - u}{\lambda}\right) dxdt \longrightarrow 0 \quad \text{as} \quad \mu \longrightarrow +\infty$$

and $u_\mu \longrightarrow u$ strongly in $E_\varphi(Q)$. Idem for $W^{1,x}E_\varphi(Q)$.

**Lemma 4.7.** *[19] If $u_n \longrightarrow u$ in $W_0^{1,x}L_\varphi(Q)$ strongly (resp., for the modular conver- gence ), then $(u_n)_\mu \longrightarrow u_\mu$ strongly (resp., for the modular convergence).*

## 5 Compactness results

For each $h > 0$, define the usual translated $\tau_h f$ of the function $f$ by $\tau_h f(t) = f(t+h)$.

If $f$ is defined on $[0, T]$ then $\tau_h f$ is defined on $[-h, T-h]$.

**Lemma 5.1.** *[26] Let $\varphi$ be a Musielak function and $\psi$ the complementary function of $\varphi$, we assume that there exists $c > 0$ such that $\psi(x, 1) \leq c$ a.e. in $\Omega$. Let $Y$ be a Banach space such that the following continuous embedding holds $L^1(\Omega) \subset Y$. Then for all $\varepsilon > 0$ and all $\lambda > 0$, there is $C_\varepsilon > 0$ such that for all $u \in W_0^{1,x} L_\varphi(Q)$ with $\frac{|\nabla u|}{\lambda} \in K_\varphi(Q)$, we have*

$$\|u\|_1 \leq \varepsilon\lambda \left( \int_Q \varphi\left(x, \frac{|\nabla u|}{\lambda}\right) dxdt + T \right) + C_\varepsilon \|u\|_{L^1(0,T,Y)}$$

The following lemma allows us to enlarge the space $Y$ whenever necessary.

**Lemma 5.2.** *[26] Let $\varphi$ be a Musielak function and $\psi$ the complementary function of $\varphi$, we assume that there exists $c > 0$ such that $\psi(x, 1) \leq c$ a.e. in $\Omega$.*

*If $F$ is bounded in $W_0^{1,x} L_\varphi(Q)$ and is relatively compact in $L^1(0, T, Y)$ then $F$ is relatively compact in $L^1(Q)$ (and also in $E_\gamma(Q)$ for all Musielak function $\gamma \ll \varphi$ ).*

**Remark 5.3.** If $F \subset L^1(0, T, B)$ is such that $\left\{ \frac{\partial f}{\partial t} : f \in F \right\}$ is bounded in $F \subset L^1(0, T, B)$ then $\|\tau_h f - f\|_{L^1(0,T,B)} \longrightarrow 0$ as $h \longrightarrow 0$ uniformly with respect to $f \in F$.

**Lemma 5.4.** *[26] Let $\varphi$ be a Musielak function. If $F$ is bounded in $W^{1,x} L_\varphi(Q)$ and $\left\{ \frac{\partial f}{\partial t} : f \in F \right\}$ is bounded in $W^{-1,x} L_\psi(Q)$, then $F$ is relatively compact in $L^1(Q)$.*

**Theorem 5.5.** *Let $\varphi$ be a Musielak function. Let $(u_n)_n$ be a sequence of $W^{1,x} L_\varphi(Q)$ such that $u_n \to u$ weakly in $W^{1,x} L_\varphi(Q)$ for $\sigma(\Pi L_\varphi, \Pi L_\psi)$ and*

$$\frac{\partial u_n}{\partial t} = h_n + k_n \text{ in } D'(Q)$$

*with $(h_n)_n$ bounded in $W^{-1,x} L_\psi(Q)$ and $(k_n)_n$ bounded in the space $M(Q)$ set of measures on $Q$. Then $u_n \longrightarrow u$ strongly in $L^1_{\text{loc}}(Q)$ If further $u_n \in W_0^{1,x} L_\varphi(Q)$ then $u_n \longrightarrow u$ strongly in $L^1(Q)$.*

**Proof.** It is easily adapted from that given in [9] by using Theorem 4.4 and Remark 4.3 instead of Lemma 8 of [31] .

## 6 Assumptions and Existence results

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N (N \geq 2)$ satisfying the segment property, $T > 0$ and set $Q = \Omega \times ]0, T[$. In the sequel, we denote by $Q_\tau = \Omega \times ]\,0, \tau[$ for every $\tau \in [0, T]$. Let $\varphi$ and $\gamma$ two Musielak Orlicz functions such that $\gamma \ll \varphi$, we denote by $\psi$ the Musielak complementary function of $\varphi$. We assume that $\varphi$ and $\psi$ satisfy the assumptions of Lemma 3.2 and that $\varphi(x, t)$ decreases with respect to one of coordinates of $x$.

Let

$$b : \Omega \times \mathbb{R} \longrightarrow \mathbb{R} \text{ is a Carathédory function such that} \tag{6.1}$$

for every $x \in \Omega : b(x, s)$ is a strictly increasing $C^1$ -function, with $b(x, 0) = 0$.

For any $k > 0$, there exists $\lambda_k > 0$, a function $A_k$ in $L^\infty(\Omega)$ and a function $B_k \in L_\varphi(\Omega)$ such that

$$\lambda_k \leq \frac{\partial b(x, s)}{\partial s} \leq A_k(x) \text{ and } \left| \nabla_x \left( \frac{\partial b(x, s)}{\partial s} \right) \right| \leq B_k(x), \tag{6.2}$$

for almost every $x \in \Omega$, for every $s$ such that $|s| \leq k$.

Let $A : D(A) \subset W_0^{1,x} L_\varphi(Q) \longrightarrow W^{-1,x} L_\psi(Q)$ be a mapping given by

$$A(u) = -\text{div}(a(x, t, u, \nabla u)),$$

where $a : \Omega \times (0,T) \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (that is, measurable with respect to $x$ in $\Omega$ for every $(t,s,\xi)$ in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$, and continuous with respect to $(s,\xi)$ in $\mathbb{R} \times \mathbb{R}^N$ for almost every $x$ in $\Omega$) such that for all $\xi$ and $\xi^*$ in $\mathbb{R}^N$, $\xi \neq \xi^*$,

$$a(x,t,s,\xi).\xi \geq \alpha\varphi(x,|\xi|) + \varphi(x,|s|), \tag{6.3}$$

$$[a(x,t,s,\xi) - a(x,t,s,\xi^*)][\xi - \xi^*] > 0, \tag{6.4}$$

$$|a(x,t,s,\xi)| \leq \beta\left(a_0(x,t) + \psi_x^{-1}\gamma\left(x,k_1|s|\right) + \psi_x^{-1}\varphi\left(x,k_1|\xi|\right)\right), \tag{6.5}$$

with $a_0(.) \in E_\psi(Q)$, $k_1 \in \mathbb{R}^+$ and $\alpha, \beta > 0$. We assume that $g : \Omega \times [0,t] \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a Caratheodory function satisfying for a.e. $(x,t) \in \Omega \times [0,t]$ and $\forall s \in \mathbb{R}, \xi \in \mathbb{R}^N$ :

$$|g(x,t,s,\xi)| \leq h(x,t) + d(s)\varphi(x,|\xi|), \tag{6.6}$$

where $d : \mathbb{R} \to \mathbb{R}^+$ is continuous positive function which belong $L^1(\mathbb{R})$ and $P(.,.)$ belong $L^1(Q)$.

Furthermore $\Phi : Q \times \mathbb{R} \to \mathbb{R}^N$ is a Carathéodory function such that

$$|\Phi(x,t,s)| \leq c(x,t)\psi_x^{-1}\varphi\left(x,\alpha_0|s|\right) \tag{6.7}$$

where $\|c(.,.)\|_{L^\infty(Q)} \leq \alpha$ and $0 < \alpha_0 < \min\left(1, \frac{1}{\alpha}\right)$.

$$f \in L^1(Q), \quad \text{and} \quad F \in \left(E_\psi(Q)\right)^N, \tag{6.8}$$

$$u_0 \in L^1(\Omega) \text{ such that } b\left(.,u_0\right) \in L^1(\Omega). \tag{6.9}$$

We consider the following parabolic problem

$$\begin{cases} \frac{\partial b(x,u)}{\partial t} + A(u) + \operatorname{div}(\Phi(x,t,u)) + g(x,t,u,\nabla u) = f - \operatorname{div}(F) & \text{in } Q \\ u(x,t) = 0 & \text{on } \partial\Omega \times [0,T] \\ b(x,u)|_{t=0} = b\left(x,u_0\right) & \text{on } \Omega \end{cases} \tag{6.10}$$

We will show that the problem (6.10) has at least one entropy solution in the following sense.

**Definition 6.1.** A measurable function $u : \Omega \times [0,T] \longmapsto R$ is called entropy solution of (6.10) if, $T_k(u)$ belongs to $D(A) \cap W_0^{1,x}L_\varphi(\Omega)$ for every $k > 0$, $b(\cdot,u_0)$ belongs to $L^1(\Omega)$, and $u$ satisfies the following inequalities

$$b(x,u) \in L^\infty\left([0,T], L^1(\Omega)\right), \tag{6.11}$$

$$\lim_{m \to +\infty} \int_{\{m \leq |u| < m+1\}} a(x,t,u,\nabla u) \cdot \nabla u \, dx \, dt = 0, \tag{6.12}$$

and,

$$
\begin{aligned}
&\int_Q \int_0^u \frac{\partial b(x,r)}{\partial r} S'(r-v) T_k(r)\,dr\,dx\,dt \\
&+ \int_Q^T \int_0^t \left\langle \frac{\partial v}{\partial \sigma}, \int_0^u \frac{\partial b(x,r)}{\partial r} S''(r-v) T_k(r)\,dr \right\rangle d\sigma\,dt \\
&+ \int_Q \int_0^t a(x,\sigma,u,\nabla u)\cdot \nabla T_k(u) S'(u-v)\,d\sigma\,dx\,dt \\
&+ \int_Q \int_0^t S''(u-v) a(x,\sigma,u,\nabla u)\cdot(\nabla u - \nabla v) T_k(u)\,d\sigma\,dx\,dt \\
&+ \int_Q \int_0^t \Phi(x,\sigma,u)\cdot \nabla T_k(u) S'(u-v)\,d\sigma\,dx\,dt \\
&+ \int_Q \int_0^t \Phi(x,\sigma,u)\cdot \nabla(u-v) S''(u-v) T_k(u)\,d\sigma\,dx\,dt \\
&+ \int_Q \int_0^t g(x,\sigma,u,\nabla u) S'(u-v) T_k(u)\,d\sigma\,dx\,dt \\
&\le \int_Q \int_0^t f S'(u-v) T_k(u)\,dx\,dt \\
&+ \int_Q \int_0^t F\cdot \nabla(u-v) S''(u-v) T_k(u)\,d\sigma\,dx\,dt \\
&+ \int_Q \int_0^t F\cdot \nabla T_k(u) S'(u-v)\,d\sigma\,dx\,dt \\
&+ T\int_\Omega \int_0^{u_0} \frac{\partial b(x,r)}{\partial r} S'(r-v(0)) T_k(r)\,dr\,dx
\end{aligned}
\tag{6.13}
$$

for every $k>0$, and for all $v \in W_0^{1,x}L_\varphi(Q) \cap L^\infty(Q)$ such that $\frac{v}{\partial t}$ belongs to $W^{-1,x}L_\psi(Q) + L^1(Q)$ (recall that $T_k$ is the usual truncation at height $k$ defined on $R$ by $T_k(s) = \min(k, \max(s,-k))$ and for all increasing function $S \in W^{2,\infty}(\mathbb{R})$ with $S'$ has a compact support in $\mathbb{R}$ ).

Inequality (6.13) is formally obtained through pointwise multiplication of equation (6.10) by $\overline{S'}(u-v)T_k(u)$, and integration by parts. However, all the terms in (6.13) have a meaning in $D'(Q)$ Indeed, if $M>0$ is such that supp $S' \subset [-M,M]$, the following identifications are made in (6.13).

$*$ $S(u)$ belongs to $L^\infty(Q)$ since $S$ is a bounded function.

$*$ $\displaystyle\int_0^u \frac{\partial b(x,r)}{\partial r} S'(r-v)T_k(r)dr = \int_0^{T_{M+\|v\|_\infty}(u)} \frac{\partial b(x,r)}{\partial r} S'(r-v)T_k(r)dr \in L^\infty(Q)$

$*$ $\displaystyle\frac{\partial v}{\partial \sigma} \in W^{-1,x}L_\psi(Q), \int_0^{T_{M+\|v\|_\infty}(u)} \frac{\partial b(x,r)}{\partial r} S'(r-v)T_k(r)dr \in W_0^{1,x}L_\varphi(Q)$

$*$ $S'(u-v)a(x,\sigma,u,\nabla u)\cdot \nabla T_k(u)$ identifies with $S'(u-v)a\left(x,\sigma,T_k(u),\nabla T_k(u)\right)$.

$\nabla T_k(v)$ a.e. in $Q$. since $S'(u-v) \in L^\infty(Q)$ and $\nabla T_k(u) \in \left(L_\varphi(Q)\right)^N$ , we obtain from (6.3) that $S'(u-v)a\left(x,\sigma,T_k(u),\nabla T_k(u)\right)\cdot \nabla T_k(v) \in L^1(Q)$

$*$ $S'(u-v)\Phi(x,\sigma,u)\cdot \nabla T_k(u)$ identifies with $S'(u-v)\Phi\left(x,\sigma,T_k(u)\right)$.

$\nabla T_k(v)$ a.e. in $Q$. since $S'(u-v) \in L^\infty(Q)$ and $\nabla T_k(u) \in \left(L_\varphi(Q)\right)^N$ , we obtain
$S'(u-v)\Phi\left(x,\sigma,T_k(u)\right)\cdot \nabla T_k(v) \in L^1(Q)$.

$*$ We have
$S''(u-v)a(x,\sigma,u,\nabla u)\cdot \nabla(u-v)T_k(u)$
$= S''(u-v)a\left(x,\sigma,T_{M+\|v\|_\infty}(u),\nabla T_{M+\|v\|_\infty}(u)\right)\cdot \nabla\left(T_{M+\|v\|_\infty}(u)-v\right)T_k(u)$ a.e. in $\Omega$,
and,
$S''(u-v)a\left(x,\sigma,T_{M+\|v\|_\infty}(u)\ \ \nabla T_{M+\|v\|_\infty}(u)\right)\cdot \nabla\left(T_{M+\|v\|_\infty}(u)-v\right)T_k(u) \in L^1(Q)$.

$*$ $S'(u-v)g(x,\sigma,u,\nabla u)T_k(u)$ identifies with $S'(u-v)g\left(x,\sigma,T_{M+\|v\|_\infty}(u)\ \ \nabla T_{M+\|v\|_\infty}(u)\right)T_k(u)$ a.e. in $Q$.

Since $S'(u-v)T_k(u) \in L^\infty(Q)$, we obtain from (6.3) and (6.6) that

$$S'(u-v)g\left(x,\sigma,T_{M+\|v\|_\infty}(u),\nabla T_{M+\|v\|_\infty}(u)\right)T_k(u) \in L^1(Q).$$

$*$ $S'(u-v)fT_k(u)$ belongs to $L^1(Q)$.

* Moreover Lemma 4.3 implies that $v \in C\left([0,T], L^1(\Omega)\right)$, then (6.2) gives

$$\int_\Omega \int_0^{T_{M+\|v\|_\infty}(u_0)} \frac{\partial b(x,r)}{\partial r} S'(r-v(0))T_k(r)drdx \mid \\ \leq k\left(M+\|v\|_\infty\right)\|S'\|_\infty \int_\Omega A_{M+\|v\|_\infty}(x)dx.$$

We shall prove the following existence theorem.

**Theorem 6.2.** *Assume that* (6.1)-(6.9) *hold true. Then the problem* (6.10) *admits at least one entropy solution solution (in the sense of Definition* (6.1)*)*

**Remark 6.3.** The results obtained in Theorem 6.2, remains true if we replace (6.7) by the growth condition

$$|\Phi(x,t,s)| \leq c(x)\bar{\gamma}_x^{-1}\gamma(x,|s|),$$

where $c(.) \in E_\gamma(\Omega)$ and $\gamma \prec\prec \varphi$.

**Remark 6.4.** We will use a Galerkin method due to Landes and Mustonen [24], we choose a sequence $\{w_1, w_2, \ldots\}$ in $D(\Omega)$ such that $\cup_{p=0}^\infty V_p$ with $V_p = \{w_1,\ldots,w_p\}$ is dense in $H_0^m(\Omega)$ with $m$ large enough such that $H_0^m(\Omega)$ is continuously embedded in $C^1(\bar{\Omega})$. For every $v \in H_0^m(\Omega)$ there exists a sequence $(v_j) \subset \cup_{p=0}V_p$ such that $v_n \longrightarrow v$ in $H_0^m(\Omega)$ and in $C^1(\bar{\Omega})$.

We denote further $V_p = C\left([0,T], V_p\right)$. It is easy to see that the closure of $\cup_{p=0}^\infty V_p$ with respect to the norm

$$\|v\|_{C^{1,0}(Q)} = \sup_{|\alpha|\leq 1}\left\{|D_x^\alpha v(x,t)| : (x,t) \in Q\right\},$$

contains $D(Q)$. This implies that, for any $f \in W^{-1,x}E_\psi(Q)$, there exists a sequence $(f_n) \subset \cup_{p=0}^\infty V_p$ such that $f_n \longrightarrow f$ strongly in $W^{-1,x}E_\psi(Q)$.

Indeed, let $\varepsilon > 0$ be given. Writing $f = \sum_{|\alpha|\leq 1} D_x^\alpha f^\alpha$ there exists $g^\alpha \in D(Q)$ such that $\|f^\alpha - g^\alpha\|_{\psi,Q} \leq \frac{\varepsilon}{2N+2}$. Moreover, by setting $g = \sum_{|\alpha|\leq 1} D_x^\alpha g^\alpha$, we see that $g \in D(Q)$, and so there exists $v \in \cup_{p=0}^\infty V_p$ such that $\|g-v\|_{\infty,Q} \leq \frac{\varepsilon}{2\,\text{meas}(Q)}$.

We deduce that

$$\|f-v\|_{W^{-1,x}L_\psi(Q)} \leq \sum_{|\alpha|\leq 1}\|f^\alpha - g^\alpha\|_{\psi,Q} + \|g-v\|_{\psi,Q} \leq \varepsilon.$$

We shall divide the theorem in several steps.

### Step 1: Approximate problems

For each $n > 0$, we define the approximation

$$b_n(x,s) = b\left(x, T_n(s)\right) + \frac{1}{n}s, \quad \text{a.e.} \quad x \in \Omega, \forall s \in \mathbb{R}, \tag{6.14}$$

$$a_n(x,t,s,\xi) = a\left(x,t,T_n(s),\xi\right) \quad \text{a.e. } (x,t) \in Q, \forall s \in \mathbb{R}, \forall \xi \in \mathbb{R}^N, \tag{6.15}$$

$$\Phi_n(x,t,s) = \Phi\left(x,t,T_n(s)\right) \quad \text{a.e. } (x,t) \in Q, \forall s \in \mathbb{R}, \tag{6.16}$$

$$g_n(x,t,s,\xi) = T_n(g(x,t,s,\xi)), \quad \text{a.e.} \quad x \in \Omega, \forall s \in R, \forall \xi \in \mathbb{R}^N, \tag{6.17}$$

$$u_{0n} \in \mathcal{C}_0^\infty(\Omega) \text{ such that } b_n\left(x,u_{0n}\right) \to b\left(x,u_0\right) \text{ strongly in } L^1(\Omega), \tag{6.18}$$

$f_n \in L^1(Q)$ such that $f_n \to f$ strongly in $L^1(Q)$, and $\|f_n\|_{L^1(Q)} \leq \|f\|_{L^1(Q)}$, and

$$\|b_n\left(x,u_{0n}\right)\|_{L^1} \leq \|b\left(x,u_0\right)\|_{L^1}. \tag{6.19}$$

Consider the nonlinear approximate problems

$$\begin{cases} u_n \in V_n, \frac{\partial u_n}{\partial t} \in L^1(0,T,V_n), u_n(x,0) = u_{0n}(x) \text{ a.e. in } \Omega \\ \frac{\partial b_n(x,u_n)}{\partial t} - \operatorname{div}(a(x,u_n,\nabla u_n)) + \operatorname{div}(\Phi_n(x,t,u_n)) + g_n(x,t,u_n,\nabla u_n) = f_n - \operatorname{div}(F) \quad \text{in} \quad D'(Q) \end{cases} \tag{6.20}$$

Since $g_n$ is bounded for any fixed $n > 0$, there exists at last one solution $u_n \in W_0^{1,x}L_\varphi(Q)$ of (6.20) ( see [24]).

### 6.1 Step 2: A priori estimates.

In this section we denote by $c_i, i = 1, 2, \ldots$ generic positive constants. Let $D(s) = \frac{2}{\alpha}\int_0^s d(\sigma)d\sigma$ where $d$ is the function in (6.6) For $k > 0$ taking $T_k(u_n)\exp(D(|u_n|))$ as a test function in (6.20), we get

$$\begin{aligned}
&\int_Q \frac{\partial b_n(x,u_n)}{\partial t} T_k(u_n)\exp(D(|u_n|))\,dxdt \\
&+ \int_Q a_n(x,t,u_n,\nabla u_n)\nabla(\exp(D(u_n))T_k(u_n))\,dxdt \\
&+ \int_Q \Phi_n(x,t,u_n)\nabla(\exp(D(u_n))T_k(u_n))\,dxdt \\
&+ \int_Q g_n(x,t,u_n,\nabla u_n)T_k(u_n)\exp(D(|u_n|))\,dxdt \\
&= \int_Q f_n T_k(u_n)\exp(D(|u_n|))\,dxdt + \int_Q F\cdot\nabla T_k(u_n)\exp(D(|u_n|))\,dxdt \\
&+ \frac{2}{\alpha}\int_Q F\cdot\nabla u_n\,|T_k(u_n)|\,d(u_n)\exp(D(|u_n|))\,dxdt.
\end{aligned} \tag{6.21}$$

For the first term of the left hand side of last equality, we have

$$\int_Q \frac{\partial b_n(x,u_n)}{\partial t} T_k(u_n)\exp(D(|u_n|))\,dxdt = \int_\Omega B_k^n(x,u_n(T))\,dx - \int_\Omega B_k^n(x,u_{0n})\,dx. \tag{6.22}$$

where $B_k^n(x,s) = \int_0^s T_k(t)\frac{\partial b_n(x,t)}{\partial t}\exp(D(|t|))dt$. Then, (6.21) becomes

$$\begin{aligned}
&\int_\Omega B_k^n(x,u_n(T))\,dx \\
&+ \int_Q a_n(x,t,u_n,\nabla u_n)\nabla(\exp(D(u_n))T_k(u_n))\,dxdt \\
&+ \int_Q \Phi_n(x,t,u_n)\nabla(\exp(D(u_n))T_k(u_n))\,dxdt \\
&+ \int_Q g_n(x,t,u_n,\nabla u_n)T_k(u_n)\exp(D(|u_n|))\,dxdt \\
&= \int_Q f_n T_k(u_n)\exp(D(|u_n|))\,dxdt + \int_Q F\cdot\nabla T_k(u_n)\exp(D(|u_n|))\,dxdt \\
&+ \frac{2}{\alpha}\int_Q F\cdot\nabla u_n\,|T_k(u_n)|\,d(u_n)\exp(D(|u_n|))\,dxdt + \int_\Omega B_k^n(x,u_{0n})\,dx.
\end{aligned} \tag{6.23}$$

By (6.7) and Young inequality we have

$$
\begin{aligned}
&\int_Q \Phi_n\left(x,t,u_n\right)\nabla\left(\exp\left(D\left(u_n\right)\right)T_k\left(u_n\right)\right)dxdt\\
&\leq \frac{2\|c(.,.)\|_{L^\infty(Q)}}{\alpha}\left[\alpha_0\int_Q \varphi\left(x,u_n\right)d\left(u_n\right)\exp\left(D\left(u_n\right)\right)T_k\left(u_n\right)dxdt\right.\\
&\left.+\int_Q \varphi\left(x,\nabla u_n\right)d\left(u_n\right)\exp\left(D\left(u_n\right)\right)T_k\left(u_n\right)dxdt\right]\\
&+\|c(\ldots,)\|_{L^\infty(Q)}\alpha_0\int_Q \varphi\left(x,u_n\right)\exp\left(D\left(u_n\right)\right)dxdt\\
&+\|c(.,.)\|_{L^\infty(Q)}\int_Q \varphi\left(x,\left|\nabla T_k\left(u_n\right)\right|\right)\exp\left(D\left(u_n\right)\right)dxdt
\end{aligned}
\tag{6.24}
$$

then by using the above inequality (6.24) and thanks to (6.3) we have

$$
\begin{aligned}
&\int_\Omega B_k^n\left(x,u_n(T)\right)dx+\int_Q a\left(x,t,u_n,\nabla u_n\right)\cdot\nabla T_k\left(u_n\right)\exp\left(D\left(\left|u_n\right|\right)\right)dxdt\\
&\qquad \frac{2}{\alpha}\int_Q \varphi\left(x,u_n\right)d\left(u_n\right)\exp\left(D\left(u_n\right)\right)T_k\left(u_n\right)dxdt\\
&\qquad +2\int_Q \varphi\left(x,\left|\nabla u_n\right|\right)\left|T_k\left(u_n\right)\right|d\left(u_n\right)\exp\left(D\left(\left|u_n\right|\right)\right)dxdt\\
&\qquad +\int_Q g_n\left(x,t,u_n,\nabla u_n\right)T_k\left(u_n\right)\exp\left(D\left(\left|u_n\right|\right)\right)dxdt\\
&\leq \frac{2\|c(.,.)\|_{L^\infty(Q)}}{\alpha}\left[\alpha_0\int_Q \varphi\left(x,u_n\right)d\left(u_n\right)\exp\left(D\left(u_n\right)\right)T_k\left(u_n\right)dxdt\right.\\
&\qquad\left.+\int_Q \varphi\left(x,\nabla u_n\right)d\left(u_n\right)\exp\left(D\left(u_n\right)\right)T_k\left(u_n\right)dxdt\right]\\
&\qquad +\|c(\ldots,)\|_{L^\infty(Q)}\alpha_0\int_Q \varphi\left(x,u_n\right)\exp\left(D\left(u_n\right)\right)dxdt\\
&\qquad +\|c(.,.)\|_{L^\infty(Q)}\int_Q \varphi\left(x,\left|\nabla T_k\left(u_n\right)\right|\right)\exp\left(D\left(u_n\right)\right)dxdt\\
&\int_Q f_nT_k\left(u_n\right)\exp\left(D\left(\left|u_n\right|\right)\right)dxdt+\int_Q F\cdot\nabla T_k\left(u_n\right)\exp\left(D\left(\left|u_n\right|\right)\right)dxdt\\
&+\frac{2}{\alpha}\int_Q F\cdot\nabla u_n\left|T_k\left(u_n\right)\right|d\left(u_n\right)\exp\left(D\left(\left|u_n\right|\right)\right)dxdt+\int_\Omega B_k^n\left(x,u_{0n}\right)dx.
\end{aligned}
\tag{6.25}
$$

Using now the conditions (6.3) and (6.6), we get

$$
\begin{aligned}
&\int_\Omega B_k^n(x,u_n(T))\,dx+\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla T_k(u_n)\exp(D(|u_n|))\,dxdt\\
&\quad\frac{2}{\alpha}\int_Q \varphi(x,u_n)\,d(u_n)\exp(D(u_n))\,T_k(u_n)\,dxdt\\
&\quad+2\int_Q \varphi(x,|\nabla u_n|)\,|T_k(u_n)|\,d(u_n)\exp(D(|u_n|))\,dxdt\\
&\leq\frac{2\|c(.,.)\|_{L^\infty(Q)}}{\alpha}\left[\alpha_0\int_Q\varphi(x,u_n)\,d(u_n)\exp(D(u_n))\,T_k(u_n)\,dxdt\right.\\
&\quad\left.+\int_Q\varphi(x,\nabla u_n)\,\rho(u_n)\exp(D(u_n))\,T_k(u_n)\,dxdt\right]\\
&\quad+\|c(\dots,)\|_{L^\infty(Q)}\alpha_0\int_Q\varphi(x,u_n)\exp(D(u_n))\,dxdt\\
&\quad+\|c(.,.)\|_{L^\infty(Q)}\int_Q\varphi(x,|\nabla T_k(u_n)|)\exp(D(u_n))\,dxdt\\
&+\int_Q f_nT_k(u_n)\exp(D(|u_n|))\,dxdt+\int_Q F\cdot\nabla T_k(u_n)\exp(D(|u_n|))\,dxdt\\
&\quad+\int_Q[h_2(x,t)+d(u_n)\,\varphi(x,|\nabla u_n|)]\,|T_k(u_n)|\exp(D(|u_n|))\,dxdt\\
&+\frac{2}{\alpha}\int_Q F\cdot\nabla u_n\,|T_k(u_n)|\,d(u_n)\exp(D(|u_n|))\,dxdt+\int_\Omega B_k^n(x,u_{0n})\,dx
\end{aligned}
\tag{6.26}
$$

From (6.14) – (6.19), and since

$$
\begin{aligned}
\int_\Omega B_k^n(x,u_0)\,dx&\leq\exp\left(\frac{2\|d\|_{L^1(R)}}{\alpha}\right)k\,\|b_n(x,u_{0n})\|_{L^1(\Omega)}\\
&\leq\exp\left(\frac{2\|d\|_{L^1(R)}}{\alpha}\right)k\,\|b(x,u_0)\|_{L^1(\Omega)}
\end{aligned}
$$

we have

$$
\begin{aligned}
&\int_\Omega B_k^n(x,u_n(T))\,dx+\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla T_k(u_n)\exp(D(|u_n|))\,dxdt\\
&\quad+\int_Q\varphi(x,|\nabla u_n|)\,|T_k(u_n)|\,d(u_n)\exp(D(|u_n|))\,dxdt\\
&\leq\frac{2\|c(.,.)\|_{L^\infty(Q)}}{\alpha}\left[\alpha_0\int_Q\varphi(x,u_n)\,d(u_n)\exp(D(u_n))\,T_k(u_n)\,dxdt\right.\\
&\quad\left.+\int_Q\varphi(x,\nabla u_n)\,d(u_n)\exp(D(u_n))\,T_k(u_n)\,dxdt\right]\\
&\quad+\|c(\dots,)\|_{L^\infty(Q)}\alpha_0\int_Q\varphi(x,u_n)\exp(D(u_n))\,dxdt\\
&\quad+\|c(.,.)\|_{L^\infty(Q)}\int_Q\varphi(x,|\nabla T_k(u_n)|)\exp(D(u_n))\,dxdt\\
&\quad+\exp\left(\frac{2\|d\|_{L^1(R)}}{\alpha}\right)k\left(\|f\|_{L^1(Q)}+\|h_2\|_{L^1(Q)}+\|b(x,u_0)\|_{L^1(\Omega)}\right)\\
&\quad+\int_Q F\cdot\nabla T_k(u_n)\exp(D(|u_n|))\,dxdt\\
&\quad+\frac{2}{\alpha}\int_Q F\cdot\nabla u_n\,|T_k(u_n)|\,d(u_n)\exp(D(|u_n|))\,dxdt
\end{aligned}
$$

Then, by using Young's inequality on the second and third term of the last inequality, we obtain

$$\begin{aligned}
&\int_\Omega B_k^n(x,u_n(T))\,dx+\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla T_k(u_n)\exp(D(|u_n|))\,dxdt\\
&\frac{2}{\alpha}\int_Q \varphi(x,u_n)\,d(u_n)\exp(D(u_n))\,T_k(u_n)\,dxdt\\
&+2\int_Q \varphi(x,|\nabla u_n|)\,|T_k(u_n)|\,d(u_n)\exp(D(|u_n|))\,dxdt\\
&\le \frac{2\|c(.,.)\|_{L^\infty(Q)}}{\alpha}\Big[\alpha_0\int_Q \varphi(x,u_n)\,d(u_n)\exp(D(u_n))\,T_k(u_n)\,dxdt\\
&+\int_{Q_r}\varphi(x,\nabla u_n)\,d(u_n)\exp(D(u_n))\,T_k(u_n)\,dxdt\Big]\\
&+\|c(\ldots,)\|_{L^\infty(Q)}\alpha_0\int_Q \varphi(x,u_n)\exp(D(u_n))\,dxdt\\
&+\|c(.,.)\|_{L^\infty(Q)}\int_Q \varphi(x,|\nabla T_k(u_n)|)\exp(D(u_n))\,dxdt\\
&+\exp\left(\frac{2\|d\|_{L^1(R)}}{\alpha}\right)k\left(\|f\|_{L^1(\Omega)}+\|h_2\|_{L^1(\Omega)}+\|b(x,u_0)\|_{L^1(\Omega)}\right)\\
&+\exp\left(\frac{2\|d\|_{L^1(R)}}{\alpha}\right)\int_Q \psi\left(x,\frac{2(\alpha+1)}{\alpha}|F|\right)dxdt\\
&+\frac{\alpha}{2(\alpha+1)}\int_Q \varphi(x,|\nabla T_k(u_n)|)\exp(D(|u_n|))\,dxdt\\
&+\|d\|_\infty\exp\left(\frac{2\|d\|_{L^1(R)}}{\alpha}\right)k\int_Q \psi(x,c_\alpha|F|)\,dxdt.
\end{aligned}\tag{6.27}$$

where $c_\alpha$ is a positive constant depend only on $\alpha$ Then, by using the fact that $B_k^n(x,u_n(T))\ge 0$, we get

$$\begin{aligned}
&\frac{2\left[1-\alpha_0\|c(.,.)\|_{L^\infty(Q)}\right]}{\alpha}\int_Q \varphi(x,u_n)\,d(u_n)\exp(D(u_n))\,T_k(u_n)\,dxdt\\
&\quad+\frac{2(\alpha-\|c(...))\|_{L^\infty(Q)}}{\alpha}\int_Q \varphi(x,\nabla u_n)\,d(u_n)\exp(D(u_n))\,T_k(u_n)\,dxdt\\
&\frac{2\alpha+1}{2(\alpha+1)}\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla T_k(u_n)\exp(D(|u_n|))\,dxdt\\
&\le\frac{\|c(.,.)\|_{L^\infty(Q)}}{\alpha}\left[\alpha_0\alpha\int_Q \varphi(x,u_n)\exp(D(u_n))\,dxdt+\alpha\varphi(x,\nabla T_k(u_n))\exp(D(u_n))\,dxdt\right]\\
&\qquad\qquad+c_1k+c_2.
\end{aligned}\tag{6.28}$$

If we choose $\alpha$ such that $\alpha>\|c(.,.)\|_{L^\infty(Q)}$ and using again (6.3) we get

$$\left[\frac{2\alpha+1}{2(\alpha+1)}-\frac{\|c(.,.)\|_{L^\infty(Q)}}{\alpha}\right]\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla T_k(u_n)\exp(D(|u_n|))\,dxdt\le c_1k+c_2.\tag{6.29}$$

Thus

$$\left[1-\frac{\|c(.,.)\|_{L^\infty(Q)}}{2\alpha(\alpha+1)}\right]\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla T_k(u_n)\exp(D(|u_n|))\,dxdt\le c_1k+c_2.\tag{6.30}$$

Taking $\frac{1}{c_3}=\left[1-\frac{\|c(.,.)\|_{L^\infty(Q)}}{2\alpha(\alpha+1)}\right]$
It follow that

$$\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla T_k(u_n)\exp(D(|u_n|))\,dxdt\le c_3c_1k+c_3c_2.\tag{6.31}$$

Then by using (6.3), we have

$$\int_Q \varphi\left(x, \left|\nabla T_k\left(u_n\right)\right|\right) dxdt \leq c_4 k + c_5 \tag{6.32}$$

By using Lemma 3.8, we have $(T_k(u_n))$ is bounded in $W_0^{1,x}L_\varphi(Q)$, then there exists $v_k$ such that

$$\begin{cases} T_k\left(u_n\right) \rightharpoonup v_k & \text{in } W_0^{1,x}L_\varphi(Q) \text{ for } \sigma\left(\Pi L\varphi, \Pi E_\psi\right) \\ T_k\left(u_n\right) \longrightarrow v_k & \text{strongly in } E_\varphi(Q). \end{cases} \tag{6.33}$$

Therefore, we can assume that $\left(T_k\left(u_n\right)\right)_n$ is a Cauchy sequence in measure in $\Omega$, then for all $k>0$ and $\delta, \varepsilon > 0$ there exists $n_0 = n_0(k, \delta, \varepsilon)$ such that

$$\text{meas}\left\{\left|T_k\left(u_n\right) - T_k\left(u_m\right)\right| > \delta\right\} \leq \frac{\varepsilon}{3}, \quad \forall m, n \geq n_0. \tag{6.34}$$

We have by simple calculus

$$\begin{aligned} \inf_{x\in\Omega} \varphi\left(x, \frac{k}{c}\right) \text{meas}\left\{\left|u_n\right| > k\right\} &= \int_{\{|u_n|>k\}} \inf_{x\in\Omega} \varphi\left(x, \frac{k}{c}\right) dxdt \\ &\leq \int_\Omega \inf_{x\in\Omega} \varphi\left(x, \frac{\left|T_k\left(u_n\right)\right|}{c}\right) dxdt \\ &\leq \int_\Omega \varphi\left(x, \frac{\left|T_k\left(u_n\right)\right|}{c}\right) dxdt \\ &\leq \int_\Omega \varphi\left(x, \left|\nabla T_k\left(u_n\right)\right|\right) dxdt, \quad (\text{ using Lemma 3.8 }) \\ &\leq c_4 k + c_5, \quad (\text{ using (6.32)}), \end{aligned}$$

where $c$ is the constant of Lemma 3.8.

Then, by the definition of $\varphi$, we get

$$\text{meas}\left\{\left|u_n\right| > k\right\} \leq \frac{c_4 k + c_5}{\inf_{x\in\Omega} \varphi\left(x, \frac{k}{c}\right)} \longrightarrow 0, \quad as \quad k \longrightarrow +\infty. \tag{6.35}$$

since $\forall \delta > 0$

$$\text{meas}\left\{\left|u_n - u_m\right| > \delta\right\} \leq \text{meas}\left\{\left|u_n\right| > k\right\} + \text{meas}\left\{\left|u_m\right| > k\right\} + \text{meas}\left\{\left|T_k\left(u_n\right) - T_k\left(u_m\right)\right| > \delta\right\} \tag{6.36}$$

Then, we have $\forall \varepsilon > 0$, there exists $k_0 > 0$ such that

$$\text{meas}\left\{\left|u_n\right| > k\right\} \leq \frac{\varepsilon}{3}, \quad \text{meas}\left\{\left|u_m\right| > k\right\} \leq \frac{\varepsilon}{3}, \quad \forall k \geq k_0(\varepsilon). \tag{6.37}$$

Combining (6.34), (6.36) and (6.37), we obtain that for all $\delta, \varepsilon > 0$, there exists $n_0 = n_0(\delta, \varepsilon)$ such that

$$\text{meas}\left\{\left|u_m - u_m\right| > \delta\right\} \leq \varepsilon, \quad \forall n, m \geq n_0.$$

It follows that $\left(u_n\right)_n$ is a Cauchy sequence in measure, then converges in measure. Now, we turn to prove the almost every convergence of $u_n$.

Consider now a $C^2(\mathbb{R})$, and nondecreasing function $r_k$ such that $r_k(s) = s$ for $|s| \leq \frac{k}{2}$ and $r_k(s) = k\,\text{sign}(s)$ if $|s| > k$. Multiplying the approximate equation (6.20) by $r_k'\left(u_n\right)$, one has

$$\begin{aligned} &\frac{\partial B_k^n\left(x, u_n\right)}{\partial t} - \text{div}\left(a\left(x, t, u_n, \nabla u_n\right) r_k'\left(u_n\right)\right) + a\left(x, t, u_n, \nabla u_n\right) \cdot \nabla u_n r_k''\left(u_n\right) \\ &\quad + \text{div}\left(r_k'\left(u_n\right) \Phi_n\left(x, t, u_n\right)\right) - r_k''\left(u_n\right) \Phi_n\left(x, t, u_n\right) \nabla u_n \\ &\quad + g_n\left(x, t, u_n, \nabla u_n\right) r'\left(u_n\right) = f_n r'\left(u_n\right) + F \cdot \nabla u_n r_k''\left(u_n\right) \quad in \quad D'(Q), \end{aligned}$$

with $B_k^n(x,s) = \int_0^s \frac{\partial b_n(x,\sigma)}{\partial t} r'(\sigma) d\sigma$.

Which yields easily that $\frac{\partial B_k^n(x,u_n)}{\partial t}$ is bounded in $W^{-1,x}L_\varphi(Q) + L^1(Q)$.

Due to the properties of $r_k$ and Lemma (5.5), we conclude that $\frac{\partial r_k(u_n)}{\partial t}$ is bounded in $W^{-1,x}L_\varphi(Q)+ L^1(Q)$ Thanks to Lemma (5.4), we deduce that $r_k(u_n)$ is compact in $L^1(Q)$.

Due to the choice of $r_k$, we conclude that for each $k$, the sequence $T_k(u_n)$ converges almost everywhere in $Q$, which implies that the sequence $u_n$ converges almost everywhere to some measurable function $u$ in Q. Consequently, we get

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u) & \text{in } W_0^1 L_\varphi(\Omega) \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi) \\ T_k(u_n) \longrightarrow T_k(u) & \text{strongly in } E_\varphi(\Omega). \end{cases} \tag{6.38}$$

### Step 3: Boundness of $(a(x,t,T_k(u_n), \nabla T_k(u_n)))_n$ in $(L_\psi(\Omega))^N$

Let $w \in (E_\varphi(Q))^N$ be arbitrary such that $\|w\|_{\varphi,Q} = 1$, by (6.4) we have

$$\left(a(x,t,T_k(u_n), \nabla T_k(u_n)) - a\left(x,t,T_k(u_n), \frac{w}{\nu}\right)\right)\left(\nabla T_k(u_n) - \frac{w}{\nu}\right) \exp(D(|u_n|)) > 0.$$

Hence

$$\begin{aligned} &\int_Q a(x,t,T_k(u_n), \nabla T_k(u_n)) \frac{w}{\nu} \exp(D(|u_n|))\, dxdt \\ &\leq \int_Q a(x,t,T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \exp(D(|u_n|))\, dxdt \\ &- \int_Q a\left(x,t,T_k(u_n), \frac{w}{\nu}\right)\left(\nabla T_k(u_n) - \frac{w}{\nu}\right) \exp(D(|u_n|))\, dxdt, \end{aligned} \tag{6.39}$$

hence, by using (6.31)

$$\int_Q a(x,t,T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \exp(D(|u_n|))\, dxdt \leq c_6 k + c_7. \tag{6.40}$$

For $\mu$ large enough ($\mu > \beta$), we have by using (6.5)

$$\begin{aligned} &\int_Q \psi_x\left(\frac{a\left(x,t,T_k(u_n), \frac{w}{k_1}\right)}{3\mu}\right) dxdt \\ &\leq \int_Q \psi_x\left(\frac{\beta\left(a_0(x,t) + \psi_x^{-1}(\gamma(x,k_1|T_k(u_n)|)) + \psi_x^{-1}(\varphi(x,|w|))\right)}{3\mu}\right) dxdt \\ &\leq \int_Q \psi_x\left(\frac{\beta\left(a_0(x,t) + \psi_x^{-1}(\gamma(x,k_1|T_k(u_n)|)) + \psi_x^{-1}(\varphi(x,|w|))\right)}{3\mu}\right) dxdt \\ &\leq \frac{\beta}{\mu}\int_Q \psi_x\left(\frac{a_0(x,t) + \psi_x^{-1}(\gamma(x,k_1|T_k(u_n)|)) + \psi_x^{-1}(\varphi(x,|w|))}{3}\right) dxdt \\ &\leq \frac{\beta}{3\mu}\left(\int_Q \psi_x(a_0(x,t))\, dxdt + \int_Q \gamma(x,k_1|T_k(u_n)|)\, dxdt + \int_Q \varphi(x,|w|) dxdt\right) \\ &\leq c_6(k). \end{aligned} \tag{6.41}$$

Now, since $\gamma$ grows essentially less rapidly than $\varphi$ near infinity and by using the Remark2.1, there exists $r'(k) > 0$ such that $\gamma(x,k_1k) \leq r'(k)\varphi(x,1)$ and so we have

$$\begin{aligned} &\int_Q \psi_x\left(\frac{a\left(x,t,T_k(u_n), \frac{w}{k_1}\right)}{3\mu}\right) dxdt \\ &\leq \frac{\beta}{3\mu}\left(\int_Q \psi_x(a_0(x,t))\, dxdt + r'(k)\int_Q \varphi(x,1) dxdt + \int_Q \varphi(x,|w|) dxdt\right) \end{aligned} \tag{6.42}$$

hence $a\left(x, t, T_k\left(u_n\right), \frac{w}{k_1}\right)$ is bounded in $\left(L_\psi(Q)\right)^N$ Which implies that second term of the right hand side of is bounded, consequently, we obtain

$$\int_Q a\left(x, t, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) w d x d t \leq c_2(k). \tag{6.43}$$

for all $w \in\left(L_{\varphi}(Q)\right)^N$ with $\|w\|_{\varphi, Q} \leq 1$. Hence by the theorem of Banach-Steinhous, the sequence $\left(a\left(x, t, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)\right)_n$ remains bounded in $\left(L_\psi(Q)\right)^N$ Which implies that, for all $k>0$ there exists a function $l_k \in\left(L_\psi(Q)\right)^N$ such that

$$a\left(x, t, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \rightarrow l_k \text{ weak star in } \left(L_\psi(Q.)\right)^N \text{ for } \sigma\left(\Pi L_\psi, \Pi E_{\varphi}\right). \tag{6.44}$$

**Step 4 : Modular convergence of truncations.**

Let $\left(v_j\right)_j$ be a sequence in $D(Q)$ such that

$$v_j \longrightarrow u \text{ with respect to the modular convergence in } W_0^{1, x} L_{\varphi}(Q), \tag{6.45}$$

and let $w_i \in D(\Omega)$ be a sequence which converges strongly to $u_0$ in $L^2(\Omega)$. Set $w_{\mu, j}^i= T_k\left(v_j\right)_\mu+\exp(-\mu t) T_k\left(w_i\right)$ where $T_k\left(v_j\right)_\mu$ is the mollification with respect to time of $T_k\left(v_j\right)$ Note that $w_{\mu, j}^i$ a smooth function having the following proprieties

$$\begin{cases}\frac{\partial}{\partial t}\left(w_{\mu, j}^i\right)=\mu\left(T_k\left(v_j\right)-w_{\mu, j}^i\right), \quad w_{\mu, j}^i(0)=T_k\left(v_j\right), \quad\left|w_{\mu, j}^i\right| \leq k; \\ w_{\mu, j}^i \longrightarrow T_k(u)_\mu+\exp(-\mu t) T_k\left(w_i\right) \text{ in } W_0^{1, x} L_{\varphi}(Q) \\ \text{for the modular convergence as } j \longrightarrow+\infty; \\ T_k(u)_\mu+\exp(-\mu t) T_k\left(w_i\right) \longrightarrow T_k(u) \text{ in } W_0^{1, x} L_{\varphi}(Q) \\ \text{for the modular convergence as } j \longrightarrow+\infty.\end{cases}$$

For $m>k$ we define the function $\rho_m$ on $R$ by

$$\rho_m(s)= \begin{cases}1 & \text{if } |s| \leq m; \\ m+1-|s| & \text{if } m \leq|s| \leq m+1; \\ 0 & \text{if } |s|>m+1.\end{cases}$$

For the sake of simplicity, we denote by $\varepsilon(n, j, \mu, s)$ any quantity (possible different) such that

$$\lim_{s \longrightarrow \infty} \lim_{\mu \longrightarrow \infty} \lim_{j \longrightarrow \infty} \lim_{n \longrightarrow \infty} \varepsilon(n, j, \mu, s)=0.$$

If the quantity we consider does not depend on one of parameters $n, j, \mu$ and $s$, we will omit the dependence on the corresponding parameter as an example, $\varepsilon(n)$ is any quantity such that

$$\lim_{j \rightarrow \infty} \lim_{n \rightarrow \infty} \varepsilon(n, j)=0$$

We denote also $\chi_s$ the characteristic functions of the set

$$Q_s=\left\{(x, t,) \in Q:\left|\nabla T_k(u)\right| \leq s\right\}$$

Let $D(s)=\frac{1}{\alpha} \int_0^s d(t) d t$, taking $\left(T_k\left(u_n\right)-w_{\mu, j}^i\right) \rho_m\left(u_n\right) \exp \left(D\left(\left|u_n\right|\right)\right)$ as a test function in

(6.20), one has

$$
\begin{aligned}
&\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\left(T_k(u_n)-w^i_{\mu,j}\right)\rho_m(u_n)\exp\left(D\left(|u_n|\right)\right)dxdt\\
&\quad+\int_Q a(x,t,u_n,\nabla u_n)\cdot\left(\nabla T_k(u_n)-\nabla w^i_{\mu,j}\right)\rho_m(u_n)\exp\left(D\left(|u_n|\right)\right)dxdt\\
&\quad+\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\rho'_m(u_n)\left(T_k(u_n)-w^i_{\mu,j}\right)\exp\left(D\left(|u_n|\right)\right)dxdt\\
&\quad+\frac{1}{\alpha}\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla u_n \,\mathrm{sign}(u_n)\left(T_k(u_n)-w^i_{\mu,j}\right)d(u_n)\rho_m(u_n)\\
&\quad\times\exp\left(D\left(|u_n|\right)\right)dxdt\\
&\quad+\int_Q \Phi(x,t,u_n)\cdot\left(\nabla T_k(u_n)-\nabla w^i_{\mu,j}\right)\rho_m(u_n)\exp\left(D\left(|u_n|\right)\right)dxdt\\
&\quad+\int_Q \Phi(x,t,u_n)\cdot\nabla u_n\rho'_m(u_n)\left(T_k(u_n)-w^i_{\mu,j}\right)\exp\left(D\left(|u_n|\right)\right)dxdt\\
&\quad+\frac{1}{\alpha}\int_Q \Phi(x,t,u_n)\cdot\nabla u_n\,\mathrm{sign}(u_n)\left(T_k(u_n)-w^i_{\mu,j}\right)d(u_n)\rho_m(u_n)\\
&\quad\times\exp\left(D\left(|u_n|\right)\right)dxdt\\
&\quad+\int_Q g_n(x,t,u_n,\nabla u_n)\left(T_k(u_n)-w^i_{\mu,j}\right)\rho_m(u_n)\exp\left(D\left(|u_n|\right)\right)dxdt\\
&=\int_Q f_n\left(T_k(u_n)-w^i_{\mu,j}\right)\rho_m(u_n)\exp\left(D\left(|u_n|\right)\right)dxdt\\
&+\int_Q F\cdot\left(\nabla T_k(u_n)-\nabla w^i_{\mu,j}\right)\rho_m(u_n)\exp\left(D\left(|u_n|\right)\right)dxdt\\
&+\int_Q F\cdot\nabla u_n\left(T_k(u_n)-w^i_{\mu,j}\right)\rho'_m(u_n)\exp\left(D\left(|u_n|\right)\right)dxdt\\
&+\frac{1}{\alpha}\int_Q F\cdot\nabla u_n\,\mathrm{sign}(u_n)\left(T_k(u_n)-w^i_{\mu,j}\right)d(u_n)\rho_m(u_n)\exp\left(D\left(|u_n|\right)\right)dxdt.
\end{aligned}
\tag{6.46}
$$

**For the first term of the left hand side of** (6.46)**,** by the definition of $w^i_{\mu,j}, j$ and Lemma 5.6 of [28], we get

$$
\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\left(T_k(u_n)-w^i_{\mu,j}\right)\rho_m(u_n)\exp\left(D\left(|u_n|\right)\right)dxdt \geq \varepsilon(n,\mu,j,i). \tag{6.47}
$$

**For the third term of the left hand side of** (6.46)**,** we get

$$
\begin{aligned}
&\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\rho'_m(u_n)\left(T_k(u_n)-w^i_{\mu,j}\right)\exp\left(D\left(|u_n|\right)\right)dxdt\\
&\qquad\leq 2k\exp\left(\frac{\|d\|_{L^1(R)}}{\alpha}\right)\int_{m\leq|u_n|\leq m+1} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n dxdt
\end{aligned}
$$

Hence by Lemma 5.1 of [3], we get

$$
\begin{aligned}
&\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\rho'_m(u_n)\left(T_k(u_n)-w^i_{\mu,j}\right)\exp\left(D\left(|u_n|\right)\right)dxdt\\
&\quad=\varepsilon(n,j,\mu,i,m)
\end{aligned}
\tag{6.48}
$$

**For the fifth term of the left hand side of** (6.46)**:**

If we take $n > m+1$, we get

$$
\begin{aligned}
\Phi_n(x,t,u_n)\exp\left(D(u_n)\right)\rho_m(u_n) =&\Phi\left(x,t,T_{m+1}(u_n)\right)\exp\left(D\left(T_{m+1}(u_n)\right)\right)\\
&\times\rho_m\left(T_{m+1}(u_n)\right)
\end{aligned}
$$

then $\Phi_n(x,t,u_n)\exp\left(G(u_n)\right)\rho_m(u_n)$ is bounded in $L_\psi(Q)$, thus, by using the pointwise convergence of $u_n$ and Lebesgue's theorem we obtain

$$
\Phi_n(x,t,u_n)\exp\left(D(u_n)\right)\rho_m(u_n)\to\Phi(x,t,u)\exp(D(u))\rho_m(u),
$$

with the modular convergence as $n \to +\infty$ then

$$\Phi_n(x,t,u_n)\exp(D(u_n))\rho_m(u_n) \to \Phi(x,t,u)\exp(D(u))\rho_m(u)$$

for $\sigma\left(\prod L_\psi, \prod L_\varphi\right)$.

In the other hand $\nabla T_k(u_n) - \nabla(T_k(v_j))_\mu$ converge to $\nabla T_k(u) - \nabla(T_k(v_j))_\mu$ weakly in $(L_\varphi(Q))^N$, then

$$\begin{aligned}&\int_Q \Phi_n(x,t,u_n)\exp(D(u_n))\rho_m(u_n)\nabla T_k(u_n) - \nabla(T_k(v_j))_\mu\,dxdt\\ &\to \int_Q \Phi(x,t,u)\rho_m(u)\exp(D(u))\nabla T_k(u) - \nabla(T_k(v_j))_\mu\,dxdt, \text{ as } n \to +\infty\end{aligned}$$

By using the modular convergence of $\nabla T_k(u) - \nabla(T_k(v_j))_\mu$ as $j \to +\infty$ and letting $\mu$ tends to infinity, we get

$$\int_Q \Phi_n(x,t,u_n)\rho_m(u_n)\exp(D(u_n))\left(\nabla T_k(u_n) - \nabla(T_k(v_j))_\mu\right)dxdt = \epsilon(n,j,\mu) \quad \text{for any } m \geq 1. \tag{6.49}$$

**For the sixth term of the left hand side of** (6.46)**,** if we take $n > m+1 > k$, we have

$$\nabla u_n \rho_m'(u_n) = \nabla T_{m+1}(u_n) \text{ a.e. in } Q.$$

By the almost every where convergence of $u_n$ we have

$$\exp(D(u_n))\left(T_k(u_n) - (T_k(v_j))_\mu\right) \to \exp(D(u))\left(T_k(u) - (T_k(v_j))_\mu\right) \text{ in } L^\infty(Q) \text{ weak-* },$$

and since the sequence $(\Phi_n(x,t,T_{m+1}(u_n)))_n'$ converge strongly in $E_\psi(Q)$, then

$$\Phi_n(x,t,T_{m+1}(u_n))\exp(D(u_n))\left(T_k(u_n) - (T_k(v_j))_\mu\right) \to$$

$$\Phi(x,t,T_{m+1}(u))\exp(D(u))\left(T_k(u) - (T_k(v_j))_\mu\right)$$

converge strongly in $E_\psi(Q)$ as $n \to +\infty$.

By virtue of $\nabla T_{m+1}(u_n) \to \nabla T_{m+1}(u)$ weakly in $(L_\varphi(Q))^N$ as $n \to +\infty$ we have

$$\begin{aligned}&\int_{m\leq|u_n|\leq m+1} \Phi_n(x,t,T_{m+1}(u_n))\nabla u_n\rho_m'(u_n)\exp(D(u_n))\left(T_k(u_n) - (T_k(v_j))_\mu\right)dxdt\\ &\qquad\to \int_{m\leq|u|\leq m+1} \Phi(x,t,u))\nabla u\exp(D(u))\left(T_k(u) - (T_k(v_j))_\mu\right)dxdt\end{aligned}$$

as $n \to +\infty$ with the modular convergence of $\left(T_k(u) - (T_k(v_j))_\mu\right)$ as $j \to +\infty$ and letting $\mu \to +\infty$ we get

$$\int_Q \Phi_n(x,t,u_n)\nabla u_n\rho_m'(u_n)\exp(D(u_n))\left(T_k(u_n) - (T_k(v_j))_\mu\right)dxdt = \epsilon(n,j,\mu) \quad \text{for any } m \geq 1. \tag{6.50}$$

By a similar calculus, we get

$$\frac{1}{\alpha}\int_Q \Phi(x,t,u_n)\cdot\nabla u_n \,\text{sign}(u_n)\left(T_k(u_n) - w_{\mu,j}^i\right)d(u_n)\rho_m(u_n) = \epsilon(n,j,\mu) \quad \text{for any } m \geq 1. \tag{6.51}$$

**for the fourth term of the right hand side of** (6.46)**,**we get

$$\begin{aligned}&\left|\frac{1}{\alpha}\int_Q F\cdot\nabla u_n \,\text{sign}(u_n)\left(T_k(u_n) - w_{\mu,j}^i\right)d(u_n)\rho_m(u_n)\exp(D(|u_n|))\,dxdt\right|\\ &\leq \frac{\|d\|_\infty}{\alpha}\exp\left(\frac{\|d\|_{L^1(R)}}{\alpha}\right)\int_Q |F|\cdot|\nabla T_{m+1}(u_n)|\left|T_k(u_n) - w_{\mu,j}^i\right|dxdt\end{aligned}$$

Then, by using the fact that $T_k(u_n) - w^i_{\mu,j}$ converges to $T_k(u) - w^i_{\mu,j}$ strongly in $E_\varphi(Q)$ and $\nabla T_{m+1}(u_n)$ converges weakly to $\nabla T_{m+1}(u)$ in $(L_\varphi(Q))^N$ as $n \longrightarrow +\infty$ then by using the modular convergence on $\mu$ and $j$, we get

$$\frac{1}{\alpha}\int_Q F\cdot\nabla u_n \,\text{sign}\,(u_n)\left(T_k(u_n) - w^i_{\mu,j}\right) d(u_n)\,\rho_m(u_n)\exp\left(D\left(|u_n|\right)\right)dxdt = \varepsilon(n,j,\mu,i). \tag{6.52}$$

By a similar calculus, we get

$$\int_Q f_n\left(T_k(u_n) - w^i_{\mu,j}\right)\rho_m(u_n)\exp\left(D\left(|u_n|\right)\right)dxdt = \varepsilon(n,j,\mu,i) \tag{6.53}$$

$$\int_Q F\cdot\left(\nabla T_k(u_n) - \nabla w^i_{\mu,j}\right)\rho_m(u_n)\exp\left(D\left(|u_n|\right)\right)dxdt = \varepsilon(n,j,\mu,i) \tag{6.54}$$

and

$$\int_Q F\cdot\nabla u_n\left(T_k(u_n) - w^i_{\mu,j}\right)\rho'_m(u_n)\exp\left(D\left(|u_n|\right)\right)dxdt = \varepsilon(n,j,\mu,i) \tag{6.55}$$

Now, combining (6.46) – (6.55) and using (6.6),we obtain

$$\begin{aligned}
&\int_Q a(x,t,u_n,\nabla u_n)\cdot\left(\nabla T_k(u_n) - \nabla w^i_{\mu,j}\right)\rho_m(u_n)\exp\left(D\left(|u_n|\right)\right)dxdt\\
&\quad+\frac{1}{\alpha}\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\,\text{sign}\,(u_n)\left(T_k(u_n) - w^i_{\mu,j}\right)\\
&\quad\times d(u_n)\,\rho_m(u_n)\exp\left(D\left(|u_n|\right)\right)dxdt\\
&\quad\le \varepsilon(n,j,\mu,i,m) + \int_Q h_2(x,t)\left|T_k(u_n) - w^i_{\mu,j}\right|\rho_m(u_n)\exp\left(D\left(|u_n|\right)\right)dxdt\\
&+\int_Q d(u_n)\,\varphi\left(x,|\nabla u_n|\right)\left|T_k(u_n) - w^i_{\mu,j}\right|\rho_m(u_n)\exp\left(D\left(|u_n|\right)\right)dxdt
\end{aligned} \tag{6.56}$$

Splitting the second term of the left hand side and the third term of the right hand side of (6.56) on $\{|u_n| \le k\}$ and $\{|u_n| > k\}$ , and using (6.3) and the fact that

$$\left(T_k(u_n) - w^i_{\mu,j}\right)u_n \ge 0 \text{ on } \{|u_n| > k\}\,,\ \text{one has}$$

$$\begin{aligned}
&\int a(x,t,u_n,\nabla u_n)\cdot\left(\nabla T_k(u_n) - \nabla w^i_{\mu,j}\right)\rho_m(u_n)\Big)\exp\left(D\left(|u_n|\right)\right)dxdt\\
&-\frac{1}{\alpha}\int_Q a\left(x,t,T_k(u_n),\nabla T_k(u_n)\right)\cdot\nabla T_k(u_n)\left|T_k(u_n) - w^i_{\mu,j}\right|\\
&\times d(u_n)\,\rho_m(u_n))\exp\left(D\left(|u_n|\right)\right)dxdt + \frac{1}{\alpha}\int_{\{|u_n|>k\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\left|T_k(u_n) - w^i_{\mu,j}\right| d(u_n)\\
&\times\rho_m(u_n))\exp\left(D\left(|u_n|\right)\right)dxdt\\
&\le \varepsilon(n,j,\mu,i,m) + \int_Q h_2(x,t)\left|T_k(u_n) - w^i_{\mu,j}\right|\rho_m(u_n)\Big)\exp\left(D\left(|u_n|\right)\right)dxdt\\
&+\int_Q d(u_n)\,a\left(x,t,T_k(u_n),\nabla T_k(u_n)\right)\cdot\nabla T_k(u_n)\left|T_k(u_n) - w^i_{\mu,j}\right|\\
&\times\rho_m(u_n))\exp\left(D\left(|u_n|\right)\right)dxdt\\
&+\frac{1}{\alpha}\int_{\{|u_n|>k\}} d(u_n)\,a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\left|T_k(u_n) - w^i_{\mu,j}\right|\\
&\times\rho_m(u_n))\exp\left(D\left(|u_n|\right)\right)dxdt
\end{aligned} \tag{6.57}$$

Then, by simplification, we have

$$\begin{aligned}&\int_Q a(x,t,u_n,\nabla u_n)\cdot\left(\nabla T_k(u_n)-\nabla w^i_{\mu,j}\right)\rho_m(u_n)\exp\left(D(|u_n|)\right)dxdt\\&\leq\varepsilon(n,j,\mu,i,m)+\int_Q h(x,t)\left|T_k(u_n)-w^i_{\mu,j}\right|\rho_m(u_n)\exp\left(D(|u_n|)\right)dxdt\\&+\frac{2}{\alpha}\int_Q d(u_n)\,a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)\left|T_k(u_n)-w^i_{\mu,j}\right|\\&\times\rho_m(u_n)\exp\left(D(|u_n|)\right)dxdt\end{aligned}\tag{6.58}$$

Similarly, like in (6.53) and (6.48), we get

$$\begin{aligned}&\int_Q h(x,t)\left|T_k(u_n)-w^i_{\mu,j}\right|\rho_m(u_n)\exp\left(D(|u_n|)\right)dxdt\\&=\varepsilon(n,j,\mu,i)\end{aligned}\tag{6.59}$$

and

$$\begin{aligned}&\left|\frac{2}{\alpha}\int_Q d(u_n)\,a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)\right|T_k(u_n)-w^i_{\mu,j}\,|\\&\times\rho_m(u_n)\exp\left(D(|u_n|)\right)dxdt\,|\leq\frac{4\|d\|_\infty}{\alpha}\exp\left(\frac{\|d\|_{L^1(R)}}{\alpha}\right)\int_{m\leq|u_n|\leq m+1}a(x,t,T_k(u_n),\nabla T_k(u_n))\\&\cdot\nabla T_k(u_n)\,dxdt\\&=\varepsilon(n,\mu,j,i,m)\end{aligned}\tag{6.60}$$

Thus, by combining (6.58),(6.59) and (6.60), one has

$$\int_Q a(x,t,u_n,\nabla u_n)\cdot\left(\nabla T_k(u_n)-\nabla w^i_{\mu,j}\right)\rho_m(u_n)\exp\left(D(|u_n|)\right)dxdt\quad\leq\varepsilon(n,j,\mu,i,m)\tag{6.61}$$

since $\rho_m(u_n)=0$ if $|u_n|>m+1$, one has

$$\begin{aligned}&\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot\left(\nabla T_k(u_n)-\nabla w^i_{\mu,j}\right)\rho_m(u_n)\exp\left(D(|u_n|)\right)dxdt\\&-\int_{\{|u_n|>k\}}a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\cdot\nabla w^i_{\mu,j}\rho_m(u_n)\exp\left(D(|u_n|)\right)dxdt\\&\leq\varepsilon(n,j,\mu,i,m)\end{aligned}\tag{6.62}$$

since $a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))$ converges weak star to $l_{m+1}$ in $(L_\psi(Q))^N$ and $\rho_m$ is continuous, we get

$$\begin{aligned}&\int_{\{|u_n|>k\}}a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\cdot\nabla w^i_{\mu,j}\rho_m(u_n)\exp\left(D(|u_n|)\right)dxdt\\&=\int_{\{|u|>k\}}l_{m+1}\cdot\nabla w^i_{\mu,j}\rho_m(T_{m+1}(u))\exp\left(D(|T_{m+1}(u)|)\right)dxdt+\varepsilon(n)\end{aligned}\tag{6.63}$$

Then, by passing to the limit on $j,\mu$ and $i$, we get

$$\int_{\{|u_n|>k\}}a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\cdot\nabla w^i_{\mu,j}\rho_m(u_n)\exp\left(D(|u_n|)\right)dxdt\quad=\varepsilon(n,j,\mu,i).\tag{6.64}$$

Thus, we deduce that,

$$\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot\left(\nabla T_k(u_n)-\nabla w^i_{\mu,j}\right)\rho_m(u_n)\exp\left(D(|u_n|)\right)dxdt\quad\leq\varepsilon(n,j,\mu,i,m).\tag{6.65}$$

Remark that,

$$\begin{aligned}
&\int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)] \\
&\quad \times (\nabla T_k(u_n) - \nabla T_k(u)\chi_s)\rho_m(u_n)\exp(D(|u_n|))\,dxdt \\
&\quad \le -\int_Q a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)(\nabla T_k(u_n) - \nabla T_k(u)\chi_s) \\
&\times \rho_m(u_n)\exp(D(|u_n|))\,dxdt \\
&\quad -\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\left(\nabla T_k(u)\chi_s - \nabla w^i_{\mu,j}\right) \\
&\times \rho_m(u_n)\exp(D(|u_n|))\,dxdt \\
&+\varepsilon(n,j,\mu,i,m) \\
&= J_1 + J_2 + \varepsilon(n,j,\mu,i,m).
\end{aligned} \tag{6.66}$$

We shall go to the limit as $n,\mu,j,i$ and $s$ to infinity in the integrals of the right-hand side. Starting by $J_1$, we have

$$\begin{aligned}
J_1 = &\int_Q a(x,t,T_k(u),\nabla T_k(u)\chi_s)(\nabla T_k(u) - \nabla T_k(u)\chi_s) \\
&\times \rho_m(u)\exp(D(|u|))dxdt + \varepsilon(n) \\
=&\varepsilon(n,j,\mu,i,m,s).
\end{aligned} \tag{6.67}$$

Concerning $J_2$, one has

$$J_2 = \int_Q l_k(\nabla T_k(u)\chi_s - \nabla T_k(u))\rho_m(u)\exp(D(|u|))dxdt + \varepsilon(n,j,\mu,i) \quad = \varepsilon(n,j,\mu,i,m,s). \tag{6.68}$$

Combining (6.66),(6.67) and (6.68), follows

$$\begin{aligned}
&\int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)] \\
&\quad \times (\nabla T_k(u_n) - \nabla T_k(u)\chi_s)\rho_m(u_n)\exp(D(|u_n|))\,dxdt \le \varepsilon(n,j,\mu,i,m,s),
\end{aligned} \tag{6.69}$$

since $\rho_m(u_n) = 1$ in $\{|u_n| \le m\}$ and $\{|u_n| \le k\} \subset \{|u_n| \le m\}$, for $m$ large enough, we get

$$\begin{aligned}
&\int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)] \\
&\quad \times (\nabla T_k(u_n) - \nabla T_k(u)\chi_s)\exp(D(|u_n|))\,dxdt \\
&\quad = \int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)] \\
&\times (\nabla T_k(u_n) - \nabla T_k(u)\chi_s)\rho_m(u_n)\exp(D(|u_n|))\,dxdt \\
&+ \int_{\{|u_n|>k\}} [a(x,t,T_k(u_n),0) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)] \\
&\times (\nabla T_k(u_n) - \nabla T_k(u)\chi_s)(1-\rho_m(u_n))\exp(D(|u_n|))\,dxdt
\end{aligned} \tag{6.70}$$

It is easy to see that the last terms of the last equality tend to zero as $n$ tends to infinity. Which yields

$$\begin{aligned}
\exp(D(-\infty)) &\int_Q [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)] \\
&\times (\nabla T_k(u_n) - \nabla T_k(u)\chi_s)\,dxdt \\
&\le \varepsilon(n,j,\mu,i,m,s).
\end{aligned} \tag{6.71}$$

Passing to the limit in (6.71)as $n$ and $s$ tends infinity, we get

$$\lim_{n,s\to+\infty}\int_Q \left[a\left(x,t,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)-a\left(x,t,T_k\left(u_n\right),\nabla T_k(u)\chi_s\right)\right] \times\left(\nabla T_k\left(u_n\right)-\nabla T_k(u)\chi_s\right)dxdt=0. \tag{6.72}$$

Using Lemma 3.11 , we have

$$T_k\left(u_n\right)\longrightarrow T_k(u) \text{ for the modular convergence in } W_0^{1,x}L_\varphi(Q). \tag{6.73}$$

**Step 6: Equi-integrability of g**

We shall prove that $g_n\left(x,t,u_n,\nabla u_n\right)\to g(x,t,u,\nabla u)$ strongly in $L^1(\Omega)$.

Consider $\vartheta_0\left(u_n\right)=\displaystyle\int_0^{u_n} d(s)\chi_{\{s>h\}}ds$ and multiply (6.20) by $\exp\left(D\left(u_n\right)\right)\vartheta_0\left(u_n\right)$, we get

$$\begin{aligned}
&\int_\Omega B_n^h\left(x,u_n(T)\right)dx+\int_Q a\left(x,u_n,\nabla u_n\right)\nabla\left(\exp\left(D\left(u_n\right)\right)\vartheta_0\left(u_n\right)\right)dxdt\\
&\quad+\int_Q \Phi_n\left(x,u_n,\nabla u_n\right)\nabla\left(\exp\left(D\left(u_n\right)\right)\vartheta_0\left(u_n\right)\right)dxdt\\
&\quad+\int_Q g_n\left(x,t,u_n,\nabla u_n\right)\exp\left(D\left(u_n\right)\right)\vartheta_0\left(u_n\right)\Big)\,dxdt\\
&\le\left(\int_h^{+\infty} d(u_n)dx\right)\exp\left(\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha}\right)\left[\|f\|_{L^1(Q)}+\|b(x,u_0)\|_{L^1(\Omega)}+\|h(.,.)\|_{L^1(Q)}\right]\\
&\quad+\int_Q\psi\left(x,\frac{2(\alpha+1)|F|}{\alpha}\right)d\left(u_n\right)\chi_{\{u_n>h\}}\exp\left(D\left(u_n\right)\right)dxdt\\
&\quad+\frac{\alpha}{2(\alpha+1)}\int_Q\varphi\left(x,|\nabla u_n|\right)d\left(u_n\right)\chi_{\{u_n>h\}}\exp\left(D\left(u_n\right)\right)dxdt\\
&\quad+\int_Q\psi\left(x,\frac{2|F|}{\alpha}\right)d\left(u_n\right)\left(\int_0^{u_n} d(s)\chi_{\{s>h\}}ds\right)\exp\left(D\left(u_n\right)\right)dxdt\\
&\quad;+\textstyle\int_Q\varphi\left(x,\nabla u_n\right)d\left(u_n\right)\left(\int_0^{u_n} d(s)\chi_{\{s>h\}}ds\right)\exp\left(D\left(u_n\right)\right)dxdt
\end{aligned}$$

where $B_n^h(x,r)=\displaystyle\int_0^r\frac{\partial b_n(x,\tau)}{\partial t}\left(\int_0^\tau d(\sigma)\chi_{\{\sigma>\tau\}}d\sigma\right)\exp(D(\tau))d\tau$, and $B_n^h\left(x,u_n(T)\right)\ge 0$, then using same technique in a priori estimates step we can have

$$\int_{\{u_n>h\}}\rho\left(u_n\right)\varphi\left(x,\nabla u_n\right)dxdt\le C\left(\int_h^{+\infty} d(s)dx\right).$$

since $d\in L^1(\mathbb{R})$, we get

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n>h\}} d\left(u_n\right)\varphi\left(x,\nabla u_n\right)dxdt=0$$

Similarly, let $\vartheta_0\left(u_n\right)=\displaystyle\int_{u_n}^0 d(s)\chi_{\{s<-h\}}dx$ in (6.20) we have also

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n<-h\}} d\left(u_n\right)\varphi\left(x,\nabla u_n\right)dxdt=0$$

We conclude that

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{|u_n|>h\}} d\left(u_n\right)\varphi\left(x,\nabla u_n\right)dxdt=0. \tag{6.74}$$

Let $D\subset\Omega$ then

$$\begin{aligned}
\int_D\rho\left(u_n\right)\varphi\left(x,\nabla u_n\right)dxdt\le&\max_{\{|u_n|\le h\}}(\rho(x))\int_{D\cap\{|u_n|\le h\}}\varphi\left(x,\nabla u_n\right)dxdt\\
&+\int_{D\cap\{|u_n|>h\}}\rho\left(u_n\right)\varphi\left(x,\nabla u_n\right)dxdt.
\end{aligned}$$

Consequently $d(u_n)\varphi(x,\nabla u_n)$ is equi-integrable. Then

$$d(u_n)\varphi(x,\nabla u_n) \text{ converge to } d(u)\varphi(x,\nabla u) \text{ strongly in } L^1(\mathbb{R}).$$

By (6.6) we get

$$g_n(x,t,u_n,\nabla u_n) \to g(x,t,u,\nabla u) \text{ strongly in } L^1(Q). \tag{6.75}$$

**Step 7: Passing to the limit.**

In this step, we shall prove that $u$ is an entropy solution to the problem (6.10) in the sense of Definition 6.1. Firstly, we prove that $u$ satisfies (6.11). For $\tau \in ]0,T]$, considering $T_k(u_n)\exp(D(|u_n|))\chi_{[0,\tau]}$ as a test function in (6.20) then like Step 2, we get

$$\int_{Q_\tau} \frac{\partial b_n(x,u_n)}{\partial t} T_k(u_n)\exp(D(u_n))\,dxdt \le c_1 k + c_2$$

Then, for $k \ge c_2$, we get

$$\int_{Q_\tau} \frac{\partial b_n(x,u_n)}{\partial t} T_k(u_n)\exp(D(u_n))\,dxdt \le (c_1+1)k$$

By passing to the limit inf with respect to $n$, we obtain

$$\frac{1}{k}\int_{Q_\tau} \frac{\partial b(x,u)}{\partial t} T_k(u)\exp(D(u_n))\,dxdt \le c_1+1$$
$$\int_0^{u(\tau)} \operatorname{sgn}(r)\frac{\partial b(x,r)}{\partial r}\exp(D(|r|))dr \le c_1+1$$

Observe that,

$$|b(x,u(\tau))| \le \int_0^{u(\tau)} \operatorname{sgn}(r)\frac{\partial b(x,r)}{\partial r}\exp(D(|r|))dr.$$

which shows that $b(x,u) \in L^\infty([0,T],L^1(\Omega))$ Secondly, we shall show that $u$ fulfills the condition (6.12). Indeed, since

$$a(x,t,u_n,\nabla u_n)\cdot\nabla u_n = a(x,t,T_{M+1}(u_n),\nabla T_{M+1}(u_n))\cdot\nabla T_{M+1}(u_n) \text{ a.e. in } Q,$$

by a simple calculus, we get

$$\begin{aligned}
&\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n dxdt\\
&= \int_{\{m\le|u|\le m+1\}} (a(x,t,T_{M+1}(u_n),\nabla T_{M+1}(u_n))\\
&-a(x,t,T_{M+1}(u_n),\nabla T_{M+1}(u)\chi_s))(\nabla T_{M+1}(u_n)-\nabla T_{M+1}(u)\chi_s)\,dxdt\\
&+\int_{\{m\le|u|\le m+1\}} (a(x,t,T_{M+1}(u_n),\nabla T_{M+1}(u_n))\\
&-a(x,t,T_{M+1}(u_n),\nabla T_{M+1}(u)\chi_s))\dot{\nabla} T_{M+1}(u)\chi_s dxdt\\
&+\int_{\{m\le|u|\le m+1\}} a(x,t,T_{M+1}(u_n),\nabla T_{M+1}(u)\chi_s)\cdot\nabla T_{M+1}(u_n)\,dxdt.
\end{aligned}$$

Then, by (6.72),(6.73) and the fact that $a(x,t,T_{M+1}(u_n),\nabla T_{M+1}(u_n))$ converges weak star to $a(x,t,T_{M+1}(u),\nabla T_{M+1}(u))$ and the strong convergence of $a(x,t,T_{M+1}(u_n),\nabla T_{M+1}(u)\chi_s)$ to $a(x,t,T_{M+1}(u),\nabla T_{M+1}(u)\chi_s)$ , we get

$$\begin{aligned}
\limsup_{n\to+\infty}\int_{\{m\le|u_n|\le m+1\}} & a(x,t,u_n,\nabla u_n)\cdot\nabla u_n dxdt\\
&= \int_{\{m\le|u|\le m+1\}} a(x,t,u,\nabla u)\cdot\nabla u dxdt
\end{aligned} \tag{6.76}$$

and then by Lemma 5.1 of [30] the condition (6.12) is fulfill.

Finally, we show that $u$ fulfills the condition (6.13). Let $S$ be an increasing function in $W^{2,\infty}(\mathbb{R})$ such that $S'$ has a compact support and $M>0$ such that. $\operatorname{supp}(S')\subset[-M,M]$ Let $v\in W_0^{1,x}L_\varphi(Q)\cap L^\infty(Q)$ such that , $\frac{\partial v}{\partial t}\in W^{-1,x}L_\psi(Q)$. Using $S'(u_n-v)T_k(u_n)$ as test function in (6.20) by using the integration by parts, we get

$$\begin{aligned}
&\int_Q\int_0^{u_n}\frac{\partial b_n(x,r)}{\partial r}S'(r-v)T_k(r)drdxdt\\
&+\int_0^T\int_0^t\left\langle\frac{\partial v}{\partial\sigma},\int_0^{u_n}\frac{\partial b_n(x,r)}{\partial r}S''(r-v)T_k(r)dr\right\rangle d\sigma dt\\
&+\int_Q\int_0^t a(x,\sigma,u_n,\nabla u_n)\cdot\nabla T_k(u_n)S'(u_n-v)\,d\sigma dxdt\\
&+\int_Q\int_0^t S''(u_n-v)\,a(x,\sigma,u_n,\nabla u_n)\cdot\nabla(u_n-v)\,T_k(u_n)\,d\sigma dxdt\\
&+\int_Q\int_0^t\Phi(x,\sigma,u)\cdot\nabla T_k(u)S'(u-v)d\sigma dxdt\\
&+\int_Q\int_0^t\Phi(x,\sigma,u)\cdot\nabla(u_n-v)\,S''(u_n-v)\,T_k(u_n)\,d\sigma dxdt\\
&+\int_Q\int_0^t g_n(x,\sigma,u_n,\nabla u_n)\,S'(u_n-v)\,T_k(u_n)\,d\sigma dxdt\\
&=\int_Q\int_0^t f_nS'(u_n-v)\,T_k(u_n)\,d\sigma dxdt+\int_Q\int_0^t F\cdot\nabla T_k(u_n)\,S'(u_n-v)\,d\sigma dxdt\\
&+\int_Q\int_0^t F\cdot\nabla(u_n-v)\,S''(u_n-v)\,T_k(u_n)\,d\sigma dxdt\\
&+T\int_\Omega\int_0^{u_{0n}}\frac{\partial b_n(x,r)}{\partial r}S'(r-v(0))T_k(r)drdx.
\end{aligned}\tag{6.77}$$

Now, we pass to the limit in each term of (6.77) as $n$ tends to infinity. since $S$ is bounded and continuous, one has

$$\int_Q\int_0^{u_n}\frac{\partial b_n(x,r)}{\partial r}S'(r-v)T_k(r)drdxdt=\int_Q\int_0^{u}\frac{\partial b(x,r)}{\partial r}S'(r-v)T_k(r)drdxdt+\varepsilon(n)$$
$$\text{and, }\int_0^{u_n}\frac{\partial b_n(x,r)}{\partial r}S''(r-v)T_k(r)dr$$

tends to $\int_0^{u}\frac{\partial b(x,r)}{\partial r}S''(r-v)T_k(r)dr$ a.e. in $Q$ and weakly in $W_0^{1,x}L_\varphi(Q)$ and $L^\infty$ weak $*$, and , $\int_0^{u_{0n}}\frac{\partial b_n(x,r)}{\partial r}S'(r-v)T_k(r)dr$ tends to $\int_0^{u_0}\frac{\partial b(x,r)}{\partial r}S'(r-v)T_k(r)dr+\varepsilon(n)$ a.e. in $\Omega$ and $L^\infty$ weak $*$, then

$$\begin{aligned}
&\int_0^T\int_0^t\left\langle\frac{\partial v}{\partial\sigma},\int_0^{u_n}\frac{\partial b_n(x,r)}{\partial r}S''(r-v)T_k(r)dr\right\rangle d\sigma dt\\
&=\int_0^T\int_0^t\left\langle\frac{\partial v}{\partial\sigma},\int_0^{u}\frac{\partial b(x,r)}{\partial r}S''(r-v)T_k(r)dr\right\rangle d\sigma dt+\varepsilon(n)
\end{aligned}\tag{6.78}$$

and,

$$\begin{aligned}
&\int_\Omega\int_0^{u_{0n}}\frac{\partial b_n(x,r)}{\partial r}S'(r-v(0))T_k(r)drdx\\
&=\int_\Omega\int_0^{u_0}\frac{\partial b(x,r)}{\partial r}S'(r-v(0))T_k(r)drdx+\varepsilon(n)
\end{aligned}\tag{6.79}$$

Since $\operatorname{supp}(S')\subset[-M,M]$ and since for $n\geq k$, one has $S'(u_n-v)\,a(x,\sigma,u_n,\nabla u_n)\,\nabla T_k(u_n)=S'(u_n-v)\,a(x,\sigma,T_k(u_n),\nabla T_k(u_n))\,\nabla T_k(u_n)$ a.e. in $Q$. Thus, the almost everywhere convergence of $\nabla u_n$ to $\nabla u$ and the bounded character of $S'$ permit us to conclude that $S'(u_n-v)\,a(x,\sigma,u_n,\nabla u_n)\,\nabla T$ tends to $S'(u-v)a(x,\sigma,T_k(u),\nabla T_k(u))\,\nabla T_k(u)$ weak star in $(L_\psi(Q))^N$ , for the topology

$\sigma\left(\Pi L_{\psi}, \Pi E_{\varphi}\right)$ , as $n$ tends to infinity, which yields, by using the modular convergence of $T_k\left(u_n\right)$ in $W_0^{1,x}L_{\varphi}(Q)$

$$\begin{aligned}&\int_Q \int_0^t a\left(x, \sigma, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \cdot \nabla T_k\left(u_n\right) S'\left(u_n-v\right) d\sigma dx dt \\ &\quad = \int_Q \int_0^t a\left(x, \sigma, T_k(u), \nabla T_k(u)\right) \cdot \nabla T_k(u) S'(u-v) d\sigma dx dt + \varepsilon(n)\end{aligned}$$

and,

$$\begin{aligned}&\int_Q \int_0^t S''\left(u_n-v\right) a\left(x, \sigma, T_{M+\|v\|_\infty}\left(u_n\right), \nabla T_{M+\|v\|_\infty}\left(u_n\right)\right) \\ &\qquad \cdot \nabla\left(T_{M+\|v\|_\infty}\left(u_n\right)-v\right) T_k\left(u_n\right) d\sigma dx dt \\ &\quad = \int_Q \int_0^t S''(u-v) a\left(x, \sigma, T_{M+\|v\|_\infty}(u), \nabla T_{M+\|v\|_\infty}(u)\right) \\ &\qquad \cdot \nabla\left(T_{M+\|v\|_\infty}(u)-v\right) T_k(u) d\sigma dx dt + \varepsilon(n)\end{aligned}$$

$g_n\left(x, \sigma, u_n, \nabla u_n\right) S'\left(u_n-v\right) \longrightarrow g(x, \sigma, u, \nabla u)$ strongly in $L^1(Q)$, as $n \longrightarrow +\infty$ and since $T_k\left(u_n\right)$ converges to $T_k(u)$ weak star in $L^{\infty}(Q)$, then

$$\begin{aligned}&\int_Q \int_0^t g_n\left(x, \sigma, u_n, \nabla u_n\right) S'\left(u_n-v\right) T_k\left(u_n\right) d\sigma dx dt \\ &\quad = \int_Q \int_0^t g(x, \sigma, u, \nabla u) S'(u-v) T_k(u) d\sigma dx dt + \varepsilon(n)\end{aligned} \tag{6.80}$$

Due to the strong convergence of $\left(f_n\right)_n$ to $f$ in $L^1(Q)$ and weak star convergence of $T_k\left(u_n\right)$ to $T_k(u)$ in $L^{\infty}(Q)$ and since $S'$ is bounded and $\left(u_n\right)_n$ converges to $u$ almost everywhere in $Q$, we get

$$\int_Q \int_0^t f_n S'\left(u_n-v\right) T_k\left(u_n\right) d\sigma dx dt = \int_Q \int_0^t f S'(u-v) T_k(u) d\sigma dx dt \tag{6.81}$$

Similarly as above, we get

$$\begin{aligned}&\int_Q \int_0^t \Phi(x, \sigma, u) \cdot \nabla T_k\left(u_n\right) S'\left(u_n-v\right) d\sigma dx dt \\ &\quad = \int_Q \int_0^t \Phi(x, \sigma, u) \cdot \nabla T_k(u) S'(u-v) d\sigma dx dt + \varepsilon(n)\end{aligned} \tag{6.82}$$

$$\begin{aligned}&\int_Q \int_0^t \Phi(x, \sigma, u) \cdot \nabla\left(u_n-v\right) S''\left(u_n-v\right) T_k\left(u_n\right) d\sigma dx dt \\ &\quad = \int_Q \int_0^t \Phi(x, \sigma, u) \cdot \nabla(u-v) S''(u-v) T_k(u) d\sigma dx dt + \varepsilon(n)\end{aligned} \tag{6.83}$$

$$\begin{aligned}&\int_Q \int_0^t F \cdot \nabla T_k\left(u_n\right) S'\left(u_n-v\right) d\sigma dx dt \\ &\quad = \int_Q \int_0^t F \cdot \nabla T_k(u) S'(u-v) d\sigma dx dt + \varepsilon(n)\end{aligned} \tag{6.84}$$

and,

$$\begin{aligned}&\int_Q \int_0^t F \cdot \nabla\left(u_n-v\right) S''\left(u_n-v\right) T_k\left(u_n\right) d\sigma dx dt \\ &\quad = \int_Q \int_0^t F \cdot \nabla(u-v) S''(u-v) T_k(u) d\sigma dx dt + \varepsilon(n)\end{aligned} \tag{6.85}$$

Consequently, combining (6.77) – (6.85) we conclude that (6.13) is fulfill. Which means that $u$ is an entropy solution of (6.10) in the sense of Definition 6.1 . This completes the proof of the Theorem 6.2.

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