NEW FIXED POINT THEOREMS IN OPERATOR VALUED EXTENDED HEXAGONAL b-METRIC SPACES

Kalpana Gopalan, Sumaiya Tasneem Zubair and Thabet Abdeljawad

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Abstract In the current work, we broaden the class of C*-algebra-valued hexagonal b-metric spaces and C*-algebra-valued extended b-metric spaces by defining the class of C*-algebra-valued extended hexagonal b-metric spaces and demonstrate a fixed point theorem with distinct contractive condition. In addition, an application is presented in the later part to demonstrate the existence and uniqueness of a particular type of operator equation in order to elucidate our results.

1 Introduction

The concept of Banach contraction is a basic outcome of the metric fixed point theory. It is a quite important and efficient tool in theoretical and applied sciences for solving the problems of Existence and uniqueness. In 2017, the conception of extended b-metric spaces was initiated by Tayyab Kamran et al. [10] as an extension of b-metric spaces [4]. Thereafter, the authors in [8] proposed the idea of extended hexagonal b-metric spaces by replacing the triangle inequality with hexagonal inequality. Recently, Asim et al. [1] developed a concept of C*-algebra-valued extended b-metric spaces and Kalpana et al. [9] established a common fixed point theorem in the setting of C*-algebra-valued hexagonal b-metric spaces. For further investigations on the concept of C*-algebra, the readers can view [2, 3, 5, 6, 7, 11, 12, 13].

Deeply influenced by the above facts, we reveal the conception of C*-algebra-valued extended hexagonal b-metric spaces and illustrate a fixed point theorem with distinctive contractive condition. Eventually, an application is provided to guarantee the existence and uniqueness for the specific type of operator equation under the framework of C*-algebra-valued extended hexagonal b-metric spaces.

2 Preliminaries

The conceptualization of extended b-metric spaces was commenced by Kamran et al. [10] that described in the following:

Definition 2.1. Given a nonempty set X and E : X × X → [1, ∞), and ˜dE : X × X → [0, ∞).
If for all a, b, c ∈ X
(1) ˜dE(a, b) = 0 ⇔ a = b;
(2) ˜dE(a, b) = ˜dE(b, a);
(3) ˜dE(a, b) ≤ E(a, b)[ ˜dE(a, c) + ˜dE(c, b)]
then we say that the pair (X, ˜dE) is an extended b-metric space.

Very recently, Kalpana et al. [8] generalized the above definition to the case of extended hexagonal b-metric spaces.

Definition 2.2. Let X be a non-empty set and E : X × X → [1, ∞). A function ˜dH : X × X → [0, ∞) is called an extended hexagonal b-metric if it satisfies:
(1) ˜dH(a, b) = 0 ⇔ a = b for all a, b ∈ X;
(2) \( \bar{d}_H(a, b) = \bar{d}_H(b, a) \) for all \( a, b \in X \);
(3) \( \bar{d}_H(a, b) \leq E(a, b) [\bar{d}_H(a, c) + \bar{d}_H(c, a) + \bar{d}_H(c, \delta) + \bar{d}_H(\delta, c) + \bar{d}_H(\delta, e) + \bar{d}_H(e, f) + \bar{d}_H(f, b)] \) for all \( a, b, c, d, e, f \in X \) and \( a \neq c, c \neq \delta, \delta \neq e, e \neq f, f \neq b \).

The pair \((X, \bar{d}_H)\) is called an extended hexagonal \(b\)-metric space.

We now discuss certain essential concepts and results in \(C^*\)-algebra.

Let \( \mathbb{A} \) signifies the unital \(C^*\)-algebra and set \( \mathbb{A}_h = \{ f \in \mathbb{A} : f^* = f \} \). An element \( f \in \mathbb{A} \) is said to be positive, if \( f \in \mathbb{A}_h \) and \( \sigma(f) \subseteq [0, \infty) \), where \( \theta \) is a zero element in \( \mathbb{A} \) and \( \sigma(f) \) is the spectrum of \( f \), which is denoted by \( \theta \leq f \). The partial ordering on \( \mathbb{A}_h \) given by \( g \leq h \) if and only if \( \theta \leq g - f \). The sets \( \{ f \in \mathbb{A} : \theta \leq f \} \) and \( \{ f \in \mathbb{A} : f g = gf, \forall g \in \mathbb{A} \} \) is represented as \( \mathbb{A}_+ \) and \( \mathbb{A}_h \) as and \( |\mathbb{A}| = (n^*w)^2 \) respectively.

Very recently, Asin et al. [1] set up the idea of extended \(b\)-metric spaces to the \(C^*\)-algebra.

**Definition 2.3.** Let \( X \neq \emptyset \) and \( E : X \times X \to \mathbb{A}_h \). The mapping \( \bar{d}_E : X \times X \to \mathbb{A}_h \) is called a \(C^*\)-algebra-valued extended \(b\)-metric on \( X \), if it satisfies the following (for all \( a, b, c \in X \)):

1. \( \theta \leq \bar{d}_E(a, b) \) for all \( a, b \in X \) and \( \bar{d}_E(a, b) = \theta \) iff \( a = b \);
2. \( \bar{d}_E(a, b) = \bar{d}_E(b, a) \) for all \( a, b \in X \);
3. \( \bar{d}_E(a, b) \leq E(a, b) [\bar{d}_E(a, c) + \bar{d}_E(c, a) + \bar{d}_E(c, \delta) + \bar{d}_E(\delta, c) + \bar{d}_E(\delta, e) + \bar{d}_E(e, f) + \bar{d}_E(f, b)] \) for all \( a, b, c, \delta, e, f \in X \) and \( \delta \neq e, e \neq f, f \neq b \).

Then \( d \) is called a \(C^*\)-algebra-valued \(b\)-metric on \( X \) and \((X, \mathbb{A}_h, \bar{d}_E)\) is called a \(C^*\)-algebra-valued extended \(b\)-metric space.

The definition of \(C^*\)-algebra-valued \(b\)-metric spaces was defined in the following way by Kalpana et al. [9].

**Definition 2.4.** Let \( X \) be a nonempty set, and \( A \in \mathbb{A}_h \) such that \( A \geq I \). Suppose the mapping \( \bar{d}_H : X \times X \to \mathbb{A}_h \) satisfies:

1. \( \theta \leq \bar{d}_H(a, b) \) for all \( a, b \in X \) and \( \bar{d}_H(a, b) = \theta \) iff \( a = b \);
2. \( \bar{d}_H(a, b) = \bar{d}_H(b, a) \) for all \( a, b \in X \);
3. \( \bar{d}_H(a, b) \leq A[\bar{d}_H(a, c) + \bar{d}_H(c, a) + \bar{d}_H(c, \delta) + \bar{d}_H(\delta, c) + \bar{d}_H(\delta, e) + \bar{d}_H(e, f) + \bar{d}_H(f, b)] \) for all \( a, b, c, \delta, e, f \in X \) and \( \delta \neq e, e \neq f, f \neq b \).

Then \( d \) is called a \(C^*\)-algebra-valued \(b\)-metric on \( X \) and \((X, \mathbb{A}_h, \bar{d}_H)\) is called a \(C^*\)-algebra-valued extended \(b\)-metric space.

3 Main Results

Through this main section, we implement the idea of \(C^*\)-algebra valued extended hexagonal \(b\)-metric spaces as follows.

Hereafter \( \mathbb{A}_h \) signify the set \( \{ a \in \mathbb{A} : ab = ba, \forall b \in \mathbb{A}, a \geq I \} \) respectively.

**Definition 3.1.** Let \( X \) be a nonempty set and \( E : X \times X \to \mathbb{A}_h \). Suppose the mapping \( \bar{d}_H : X \times X \to \mathbb{A}_h \) satisfies:

1. \( \theta \leq \bar{d}_H(a, b) \) and \( \bar{d}_H(a, b) = \theta \) iff \( a = b \) for all \( a, b \in X \);
2. \( \bar{d}_H(a, b) = \bar{d}_H(b, a) \) for all \( a, b \in X \);
3. \( \bar{d}_H(a, b) \leq E(a, b) [\bar{d}_H(a, c) + \bar{d}_H(c, a) + \bar{d}_H(c, \delta) + \bar{d}_H(\delta, c) + \bar{d}_H(\delta, e) + \bar{d}_H(e, f) + \bar{d}_H(f, b)] \) for all \( a, b, c, \delta, e, f \in X \) and \( \delta \neq e, e \neq f, f \neq b \).

The triplet \((X, \mathbb{A}_h, \bar{d}_H)\) is called an \(C^*\)-algebra-valued extended hexagonal \(b\)-metric space.

**Example 3.2.** Let \( X = \{1, 2, 3, 4, 5, 6\} \) and \( \mathbb{A} = \mathbb{R}^2 \). If \( a, b \in \mathbb{A} \) with \( a = (a_1, a_2), b = (b_1, b_2) \), then the addition, multiplication and scalar multiplication can be defined as follows

\[
a + b = (a_1 + b_1, a_2 + b_2), \quad ka = (ka_1, ka_2), \quad ab = (a_1b_1, a_2b_2).
\]

Now, we define the metric \( \bar{d}_H : X \times X \to \mathbb{A}_h \) such that \( \bar{d}_H \) is symmetric and the control function \( E : X \times X \to \mathbb{A}_h \) as

\[
\bar{d}_H(e, f) = (0, 0), \quad \forall e, f, \bar{d}_H(1, 2) = (700, 700),
\]

\[
\bar{d}_H(1, 3) = \bar{d}_H(1, 4) = \bar{d}_H(1, 5) = \bar{d}_H(2, 3) = \bar{d}_H(2, 4) = \bar{d}_H(2, 5) = \bar{d}_H(3, 4) = \bar{d}_H(3, 5) = \bar{d}_H(4, 5) = (50, 50), \quad \bar{d}_H(4, 6) = (150, 150), \quad \forall e = 2, 3, 4, 5 and the controlled function \( E(e, f) = e + f, \quad \forall e, f \in X \).
It is easy to verify that $\tilde{d}_H$ is a $C^*$-algebra-valued extended hexagonal $b$-metric type space. Indeed, we have

$$\tilde{d}_H(1, 2) = (700, 700) \succ E(1, 2)[\tilde{d}_H(1, 3) + \tilde{d}_H(3, 2)] = (300, 300).$$

Therefore, $\tilde{d}_H$ is not a $C^*$-algebra-valued extended $b$-metric space.

**Definition 3.3.** A sequence $\{\epsilon_n\}$ in a $C^*$-algebra-valued extended hexagonal $b$-metric space $(X, \mathcal{H}, \tilde{d}_H)$ is said to be:

(i) convergent sequence if $\exists \epsilon \in X$ such that $\tilde{d}_H(\epsilon_n, \epsilon) \to \theta \ (n \to \infty)$ and we denote it by $\lim_{n \to \infty} \epsilon_n = \epsilon$.

(ii) Cauchy sequence if $\tilde{d}_H(\epsilon_n, \epsilon_m) \to \theta \ (n, m \to \infty)$.

**Definition 3.4.** A $C^*$-algebra-valued extended hexagonal $b$-metric space $(X, \mathcal{H}, \tilde{d}_H)$ is said to be complete if every Cauchy sequence is convergent in $X$ with respect to $\mathcal{H}$.

**Theorem 3.5.** Let $(X, \mathcal{H}, \tilde{d}_H)$ be a complete $C^*$-algebra-valued extended hexagonal $b$-metric space and suppose $T : X \to X$ that meets the following criteria:

$$\tilde{d}_H(T\epsilon, T\tilde{f}) \preceq G^* E(\epsilon, \tilde{f}) \tilde{d}_H(\epsilon, \tilde{f}) G \quad \text{for all } \epsilon, \tilde{f} \in X$$

where $G \in \mathcal{H}$ with $\|G\| < 1$. For $\epsilon_0 \in X$, choose $\epsilon_n = T^n \epsilon_0$. Assume that

$$\sup_{m \geq 1} \lim_{i \to \infty} \|E(\epsilon_i, \epsilon_{i+1})\| E(\epsilon_{i+1}, \epsilon_m) \| < \frac{1}{\|G\|^8}\| (3.1)$$

and

$$\sup_{m \geq 1} \lim_{i \to \infty} \|E(\epsilon_{i+j}, \epsilon_{i+j+1})\| E(\epsilon_{i+1}, \epsilon_m) \| < \frac{1}{\|G\|^8}, \quad \text{for } j = 1, 2, 3. \quad (3.3)$$

Furthermore, presume that

$$\lim_{n, m \to \infty} \|E(\epsilon_n, \epsilon_m)\| < \frac{1}{\|G\|^2}, \quad \text{for each } \epsilon \in X. \quad (3.4)$$

Then, $T$ has a unique fixed point in $X$.

**Proof.** Let $\epsilon_0 \in X$ and set $\epsilon_{n+1} = T\epsilon_n = \ldots = T^{n+1}\epsilon_0, n = 1, 2, \ldots$. The element $\tilde{d}_H(\epsilon_1, \epsilon_0)$ in $\mathcal{H}$ is denoted by $G_0$. Then

$$\tilde{d}_H(\epsilon_n, \epsilon_{n+1}) = \tilde{d}_H(T\epsilon_{n-1}, T\epsilon_n) \preceq G^* E(\epsilon_{n-1}, \epsilon_n) \tilde{d}_H(\epsilon_{n-1}, \epsilon_n) G \preceq (G^*)^n \prod_{k=1}^{n} E(\epsilon_{k-1}, \epsilon_k) \tilde{d}_H(\epsilon_0, \epsilon_1) G^n. \quad (3.5)$$

Similarly, we get

$$\tilde{d}_H(\epsilon_n, \epsilon_{n+2}) \preceq (G^*)^n \prod_{k=1}^{n} E(\epsilon_{k-1}, \epsilon_{k+1}) \tilde{d}_H(\epsilon_0, \epsilon_2) G^n,$$

$$\tilde{d}_H(\epsilon_n, \epsilon_{n+3}) \preceq (G^*)^n \prod_{k=1}^{n} E(\epsilon_{k-1}, \epsilon_{k+2}) \tilde{d}_H(\epsilon_0, \epsilon_3) G^n \quad (3.6)$$

and

$$\tilde{d}_H(\epsilon_n, \epsilon_{n+4}) \preceq (G^*)^n \prod_{k=1}^{n} E(\epsilon_{k-1}, \epsilon_{k+3}) \tilde{d}_H(\epsilon_0, \epsilon_4) G^n.$$
Now, we demonstrate that \( \{ \epsilon_n \}_{n \in \mathbb{N}} \) is a Cauchy sequence i.e., \( \lim_{n \to \infty} \| \delta_H (\epsilon_n, \epsilon_{n+p}) \| = 0 \), for \( p \in \mathbb{N} \).

For \( p = 4m + 1 \), where \( m \geq 1 \), we consider

\[
\delta_H (\epsilon_n, \epsilon_{n+4m+1}) \leq E (\epsilon_n, \epsilon_{n+4m+1}) [\delta_H (\epsilon_{n+1}, \epsilon_{n+1}) + \delta_H (\epsilon_{n+2}, \epsilon_{n+2}) + \delta_H (\epsilon_{n+3}, \epsilon_{n+3})] \\
+ \delta_H (\epsilon_{n+3}, \epsilon_{n+4}) + \delta_H (\epsilon_{n+4}, \epsilon_{n+4+1}) + \delta_H (\epsilon_{n+4+1}, \epsilon_{n+4+2}) + \delta_H (\epsilon_{n+4+2}, \epsilon_{n+4+3}) \\
+ \delta_H (\epsilon_{n+4+3}, \epsilon_{n+4+4}) \]
\]

\[
\leq \sum_{i=\frac{p}{4}}^{\frac{n+4m+4}{4}} \prod_{j=\frac{p}{4}}^{i} E (\epsilon_{d_j}, \epsilon_{n+4m+1}) [(G^*)^{d_i}]^{\frac{4i+1}{4}} \prod_{k=1}^{4i+1} E (\epsilon_{k-1}, \epsilon_k) \delta_H (\epsilon_0, \epsilon_1) G^{4i+1} \\
+ (G^*)^{d_i+2} \prod_{k=1}^{4i+2} E (\epsilon_{k-1}, \epsilon_k) \delta_H (\epsilon_0, \epsilon_1) G^{4i+2} \\
+ \delta_H (\epsilon_{n+4m+1}, \epsilon_{n+4m+1}) [G^*]^{n+4m} \prod_{k=1}^{n+4m} E (\epsilon_{k-1}, \epsilon_k) \delta_H (\epsilon_0, \epsilon_1) G^{n+4m} \\
+ \delta_H (\epsilon_{n+4m+1}, \epsilon_{n+4m+1}) \]
where $I$ is the unit element in $\mathbb{A}$. Let

$$a_i = \prod_{j=\frac{n}{2}}^{i} \|E(\mathbf{e}_j, \mathbf{e}_{n+4m+1})\| \prod_{k=1}^{4i} \|E(\mathbf{e}_{k-1}, \mathbf{e}_k)\| \|G\|^{2(4i)} \|G_0\|, \quad (3.7)$$

$$b_i = \prod_{j=\frac{n}{2}}^{i} \|E(\mathbf{e}_j, \mathbf{e}_{n+4m+1})\| \prod_{k=1}^{4i+1} \|E(\mathbf{e}_{k-1}, \mathbf{e}_k)\| \|G\|^{2(4i+1)} \|G_0\|, \quad (3.8)$$

$$c_i = \prod_{j=\frac{n}{2}}^{i} \|E(\mathbf{e}_j, \mathbf{e}_{n+4m+1})\| \prod_{k=1}^{4i+2} \|E(\mathbf{e}_{k-1}, \mathbf{e}_k)\| \|G\|^{2(4i+2)} \|G_0\|, \quad (3.9)$$

and

$$d_i = \prod_{j=\frac{n}{2}}^{i} \|E(\mathbf{e}_j, \mathbf{e}_{n+4m+1})\| \prod_{k=1}^{4i+3} \|E(\mathbf{e}_{k-1}, \mathbf{e}_k)\| \|G\|^{2(4i+3)} \|G_0\|. \quad (3.10)$$

It is clear that $\sup_{m \geq 1} \lim_{i \to \infty} \frac{a_{i+1}}{a_i} = \|E(\mathbf{e}_{4i+4}, \mathbf{e}_{n+4m+1})\| \|E(\mathbf{e}_{4i+3}, \mathbf{e}_{4i+4})\| \|G\|^8 < 1$ by the hypotheses of the theorem. In a similar manner, we can demonstrate that

$$\sup_{m \geq 1} \lim_{i \to \infty} \frac{b_{i+1}}{b_i} < 1, \quad \sup_{m \geq 1} \lim_{i \to \infty} \frac{c_{i+1}}{c_i} < 1 \quad \text{and} \quad \sup_{m \geq 1} \lim_{i \to \infty} \frac{d_{i+1}}{d_i} < 1.$$

Therefore,

$$\prod_{i=\frac{n}{2}}^{+\infty} \prod_{j=\frac{n}{2}}^{i} \|E(\mathbf{e}_j, \mathbf{e}_{n+4m+1})\| \prod_{k=1}^{4i} \|E(\mathbf{e}_{k-1}, \mathbf{e}_k)\| \|G\|^{2(4i)} \|G_0\| < +\infty,$$

$$\prod_{i=\frac{n}{2}}^{+\infty} \prod_{j=\frac{n}{2}}^{i} \|E(\mathbf{e}_j, \mathbf{e}_{n+4m+1})\| \prod_{k=1}^{4i+1} \|E(\mathbf{e}_{k-1}, \mathbf{e}_k)\| \|G\|^{2(4i+1)} \|G_0\| < +\infty,$$

$$\prod_{i=\frac{n}{2}}^{+\infty} \prod_{j=\frac{n}{2}}^{i} \|E(\mathbf{e}_j, \mathbf{e}_{n+4m+1})\| \prod_{k=1}^{4i+2} \|E(\mathbf{e}_{k-1}, \mathbf{e}_k)\| \|G\|^{2(4i+2)} \|G_0\| < +\infty$$

and

$$\prod_{i=\frac{n}{2}}^{+\infty} \prod_{j=\frac{n}{2}}^{i} \|E(\mathbf{e}_j, \mathbf{e}_{n+4m+1})\| \prod_{k=1}^{4i+3} \|E(\mathbf{e}_{k-1}, \mathbf{e}_k)\| \|G\|^{2(4i+3)} \|G_0\| < +\infty.$$

Consequently, we infer that

$$\left( \sum_{i=\frac{n}{2}}^{\frac{n+4m-4}{2}} \prod_{j=\frac{n}{2}}^{i} \|E(\mathbf{e}_j, \mathbf{e}_{n+4m+1})\| \prod_{k=1}^{4i} \|E(\mathbf{e}_{k-1}, \mathbf{e}_k)\| \|G\|^{2(4i)} \|G_0\| \right) I,$$

$$\left( \sum_{i=\frac{n}{2}}^{\frac{n+4m-4}{2}} \prod_{j=\frac{n}{2}}^{i} \|E(\mathbf{e}_j, \mathbf{e}_{n+4m+1})\| \prod_{k=1}^{4i+1} \|E(\mathbf{e}_{k-1}, \mathbf{e}_k)\| \|G\|^{2(4i+1)} \|G_0\| \right) I,$$

$$\left( \sum_{i=\frac{n}{2}}^{\frac{n+4m-4}{2}} \prod_{j=\frac{n}{2}}^{i} \|E(\mathbf{e}_j, \mathbf{e}_{n+4m+1})\| \prod_{k=1}^{4i+2} \|E(\mathbf{e}_{k-1}, \mathbf{e}_k)\| \|G\|^{2(4i+2)} \|G_0\| \right) I$$

and

$$\left( \sum_{i=\frac{n}{2}}^{\frac{n+4m-4}{2}} \prod_{j=\frac{n}{2}}^{i} \|E(\mathbf{e}_j, \mathbf{e}_{n+4m+1})\| \prod_{k=1}^{4i+3} \|E(\mathbf{e}_{k-1}, \mathbf{e}_k)\| \|G\|^{2(4i+3)} \|G_0\| \right) I.$$
are Cauchy sequences in $\mathbb{A}$. Thereby, we obtain that $\hat{d}_H(\varepsilon_n, \varepsilon_{n+4m+1}) \to \theta$ as $n \to \infty$. By following the above steps, we can easily deduce that

$$
\lim_{n \to \infty} \hat{d}_H(\varepsilon_n, \varepsilon_{n+4m+2}) = \lim_{n \to \infty} \hat{d}_H(\varepsilon_n, \varepsilon_{n+4m+3}) = \lim_{n \to \infty} \hat{d}_H(\varepsilon_n, \varepsilon_{n+4m+4}) = \theta. \quad (3.11)
$$

Therefore the sequence $\{\varepsilon_n\}$ is Cauchy. As $(X, \hat{d}_H)$ is complete, there exists $\varepsilon \in X$ such that $\lim_{n \to \infty} \varepsilon_n = \varepsilon$. We will reveal that $\varepsilon$ is a fixed point of $T$. Consider

$$
\hat{d}_H(T\varepsilon, \varepsilon) \leq E(T\varepsilon, \varepsilon)[\hat{d}_H(T\varepsilon, \varepsilon_{n+1}) + \hat{d}_H(\varepsilon_{n+1}, \varepsilon_{n+2}) + \hat{d}_H(\varepsilon_{n+2}, \varepsilon_{n+3})
\quad + \hat{d}_H(\varepsilon_{n+3}, \varepsilon_{n+4}) + \hat{d}_H(\varepsilon_{n+4}, \varepsilon)]
\leq E(T\varepsilon, \varepsilon)[G^*E(\varepsilon, \varepsilon)\hat{d}_H(\varepsilon, \varepsilon_n) + \hat{d}_H(\varepsilon_{n+1}, \varepsilon_{n+2}) + \hat{d}_H(\varepsilon_{n+2}, \varepsilon_{n+3})
\quad + \hat{d}_H(\varepsilon_{n+3}, \varepsilon_{n+4}) + \hat{d}_H(\varepsilon_{n+4}, \varepsilon)]
\iff ||\hat{d}_H(T\varepsilon, \varepsilon)|| \leq ||E(T\varepsilon, \varepsilon)||[||G^2||||E(\varepsilon, \varepsilon)||||\hat{d}_H(\varepsilon, \varepsilon_n)|| + ||\hat{d}_H(\varepsilon_{n+1}, \varepsilon_{n+2})||
\quad + ||\hat{d}_H(\varepsilon_{n+2}, \varepsilon_{n+3})|| + ||\hat{d}_H(\varepsilon_{n+3}, \varepsilon_{n+4})|| + ||\hat{d}_H(\varepsilon_{n+4}, \varepsilon)||]
$$

which yields $||\hat{d}_H(T\varepsilon, \varepsilon)|| \leq 0$ as $n \to \infty \iff \hat{d}_H(T\varepsilon, \varepsilon) \leq \theta$ as $n \to \infty$ i.e., $\varepsilon$ is a fixed point of $T$.

**Unicity:**
Let $\hat{f}(\neq \varepsilon)$ be an another fixed point of $T$. As $\theta \leq \hat{d}_H(\varepsilon, \hat{f}) = \hat{d}_H(T\varepsilon, T\hat{f}) \leq G^*E(\varepsilon, \hat{f})\hat{d}_H(\varepsilon, \hat{f})G$, we have

$$
0 \leq ||\hat{d}_H(\varepsilon, \hat{f})|| = ||\hat{d}_H(T\varepsilon, T\hat{f})||
\leq ||G^*E(\varepsilon, \hat{f})\hat{d}_H(\varepsilon, \hat{f})G||
\leq ||G^*G||||E(\varepsilon, \hat{f})||||\hat{d}_H(\varepsilon, \hat{f})||
= ||G||^2||E(T^n\varepsilon, T^n\hat{f})||||\hat{d}_H(\varepsilon, \hat{f})||.
$$

Taking limit $n \to \infty$ in the equation mentioned above and employing (3.4), we get $||\hat{d}_H(\varepsilon, \hat{f})|| < ||\hat{d}_H(\varepsilon, \hat{f})||$, which is impossible. Henceforth the fixed point $\varepsilon$ is unique.

**Example 3.6.** Let $X = [0, 8]$ and $\mathbb{A} = M_2(\mathbb{R})$. Define partial ordering on $\mathbb{A}$ as

$$
\begin{pmatrix}
\varepsilon_1 & \varepsilon_2 \\
\varepsilon_3 & \varepsilon_4
\end{pmatrix} \succeq
\begin{pmatrix}
\hat{f}_1 & \hat{f}_2 \\
\hat{f}_3 & \hat{f}_4
\end{pmatrix}
\iff \varepsilon_i \geq \hat{f}_i \text{ for } i = 1, 2, 3, 4.
$$

For any $G \in \mathbb{A}$, its norm can be defined as, $||G|| = \max_{1 \leq i \leq 4} |a_i|$. Define $\hat{d}_H : X \times X \to \mathbb{A}$ for all $\varepsilon, \hat{f} \in X$

$$
\hat{d}_H(\varepsilon, \hat{f}) = \begin{pmatrix}
(\varepsilon - \hat{f})^6 & 0 \\
0 & (\varepsilon - \hat{f})^6
\end{pmatrix}
$$

with the controlled function

$$
E(\varepsilon, \hat{f}) = \begin{cases}
\begin{pmatrix}
2 + |\varepsilon - \hat{f}|^5 & 0 \\
0 & 2 + |\varepsilon - \hat{f}|^5
\end{pmatrix}, & \text{if } \varepsilon \neq \hat{f} \\
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}, & \text{if } \varepsilon = \hat{f}
\end{cases}
$$
It is easy to verify that \((X, \mathcal{A}, \bar{d}_H)\) is a complete \(C^*\)-algebra-valued extended hexagonal \(b\)-metric space. Define \(T : X \to X\) by \(T\varepsilon = \frac{\varepsilon}{4}\). We have

\[
\bar{d}_H(T\varepsilon, T\f) = \begin{pmatrix}
\left(\frac{\varepsilon}{4} - \frac{1}{2}\right)^6 & 0 \\
0 & \left(\frac{\varepsilon}{4} - \frac{1}{4}\right)^6
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\frac{1}{\sqrt{6}}(\varepsilon - f)^6 & 0 \\
0 & \frac{1}{\sqrt{6}}(\varepsilon - f)^6
\end{pmatrix}
\]

\[
\leq \begin{pmatrix}
\frac{1}{4} & 0 \\
0 & \frac{1}{4}
\end{pmatrix}
\begin{pmatrix}
2 + |\varepsilon - f|^5 (\varepsilon - f)^6 & 0 \\
0 & [2 + |\varepsilon - f|^5](\varepsilon - f)^6
\end{pmatrix}
\begin{pmatrix}
\frac{1}{4} & 0 \\
0 & \frac{1}{4}
\end{pmatrix}
\]

\[
= G^* E(\varepsilon, f) \bar{d}_H(\varepsilon, f) G
\]

where \(\|G\| = \frac{1}{4} < 1\). Notice that for each \(\varepsilon \in X\), \(T^n\varepsilon = \frac{\varepsilon}{2^n}\). Thus

\[
\sup_{m \geq 1} \lim\limits_{i \to \infty} \|E(\varepsilon_i, \varepsilon_{i+1})\| \|E(\varepsilon_{i+1}, \varepsilon_m)\| = \sup_{m \geq 1} \left[4 + 2\left(\frac{\varepsilon}{4^m}\right)^5\right] < 4^8 = \frac{1}{\|G\|^8}
\]

and

\[
\lim_{n \to \infty} \|E(\varepsilon_n, \varepsilon_m)\| = 2 < \infty.
\]

As a result, all of the conditions of Theorem 3.5 are fulfilled. Accordingly \(T\) has a unique fixed point \((\varepsilon = 0)\).

**Corollary 3.7.** Let \((X, \mathcal{A}, \bar{d}_H)\) be a complete \(C^*\)-algebra-valued hexagonal \(b\)-metric space and suppose \(T : X \to X\) is a mapping satisfying the following condition:

\[
\bar{d}_H(T\varepsilon, T\f) \leq G^* F\bar{d}_H(\varepsilon, f) G \text{ for all } \varepsilon, f \in X
\]

(3.12)

where \(G \in \mathcal{A}, F \in \mathcal{A}'_b\) with \(\|G\| < 1\) and \(\|F\| > 1\). Then, \(T\) has a unique fixed point in \(X\).

**Proof.** The proof follows from Theorem 3.5 by defining \(E : X \times X \to \mathcal{A}_b\) via \(E(\varepsilon, f) = F\).

### 4 Application

In this section, we show that a type of operator equation exists and is unique in the context of complete \(C^*\)-algebra-valued extended hexagonal \(b\)-metric spaces.

**Example 4.1.** Assume \(H\) is a Hilbert space, \(L(H)\) is the set of linear bounded operators on \(H\). Let \(F_1, F_2, \ldots, F_n, \ldots \in L(H)\) that satisfy \(\sum_{n=1}^\infty \|F_n\|^6 < 1\) and \(R \in L(H)_+\). Then the operator equation

\[
C - \sum_{n=1}^\infty F_n^* C F_n = R
\]

has a unique solution in \(L(H)\).

**Proof.** Set \(G = \left(\sum_{n=1}^\infty \|F_n\|^6\right)^{\frac{1}{6}}\), therefore it is obvious that \(\|G\| < 1\) and \(G > 0\). Now, select an operator \(M \in L(H)\) that is positive. For \(C, D \in L(H)\), set

\[
\bar{d}_H(C, D) = \|C - D\|^6 M.
\]

Thereby \(\bar{d}_H\) is a \(C^*\)-algebra-valued extended hexagonal \(b\)-metric with a controlled function

\[
E(C, D) = \begin{cases} 
I + \|C - D\|^5 M, & \text{if } C \neq D \\
I, & \text{if } C = D
\end{cases}
\]
As $L(H)$ is a Banach space, $(L(H), \tilde{d}_H)$ is a complete $C^*$-algebra-valued extended hexagonal $b$-metric space. Consider the map $T : L(H) \to L(H)$ defined by

$$TC = \sum_{n=1}^{\infty} F_n^* CF_n + R.$$ 

Then

$$\tilde{d}_H(T(C), T(D)) = \|T(C) - T(D)\|_b^6 M$$

$$= \left\| \sum_{n=1}^{\infty} F_n^*(C - D)F_n \right\|_b^6 M$$

$$\leq \sum_{n=1}^{\infty} \|F_n\|^6 \|C - D\|_b^6 M$$

$$\leq \sum_{n=1}^{\infty} \|F_n\|^6 \left[ I + \|C - D\|_b^5 M \right] \|C - D\|_b^6 M$$

$$= C^2 E(C, D) \tilde{d}_H(C, D)$$

$$= (GI)^* E(C, D) \tilde{d}_H(C, D)(GI).$$

Using Theorem 3.5, there exists a unique fixed point $C$ in $L(H)$.

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Author information

Kalpana Gopalan, Department of Mathematics, Sri Sivasubramaniya Nadar College of Engineering, Kalavakkam, Chennai-603 110, India.
E-mail: kalpanag@ssn.edu.in

Sumaiya Tasneem Zubair, Department of Mathematics, Sri Sivasubramaniya Nadar College of Engineering, Kalavakkam, Chennai-603 110, India.
E-mail: sumaiyatasneemz@ssn.edu.in

Thabet Abdeljawad, Department of Mathematics and General Sciences, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia.
E-mail: tabdeljawad@psu.edu.sa

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