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Robust Shifted Jacobi-Galerkin Method for Solving Linear Hyperbolic Telegraph Type Equation

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Abstract In this paper, we propose a numerical technique built on applying the shifted Jacobi Galerkin method (SJGM) for obtaining approximate solutions of the one-dimensional linear second-order hyperbolic telegraph differential equations (HTDEs). We convert the differential equation governed by its initial and boundary conditions into a modified one governed by boundary conditions only. The spectral Galerkin method aims to reduce the linear telegraph problem to a system of linear algebraic equations in the unknown expansion coefficients. The convergence and error analysis of the shifted Jacobi polynomials expansion are investigated. Several examples are carried out to illustrate the efficiency of the proposed method. Also, the results are compared with some other existing schemes in the literature.

1 Introduction

Hyperbolic partial differential equations (HPDEs) are important in the scope of differential equations. The telegraph equation is a second-order HPDE. It explains various phenomena in scientific fields such as energetic particle transport in the interplanetary medium (see, [13, 24]). This equation arises in wave propagation along transmission lines in the theory of electric circuits [30].

The standard Jacobi polynomials have an important place in mathematical analysis [38]. They include six well-known particular polynomials, the four kinds of Chebyshev polynomials, Legendre and ultraspherical polynomials [25]. The Jacobi polynomials are considered to be the only polynomials to arise as eigenfunctions of a singular Sturm-Liouville problem [14].

Various numerical methods and in particular, spectral methods are successfully applied in different fields such as chemistry, fluid dynamics, and astronomy. For some applications in this direction, see [33, 36, 26, 34, 10, 9, 8, 1, 39, 40]. These methods assume that the approximate solutions of differential equations are given in terms of certain combinations of special functions, and after that the coefficients of the proposed expansion are determined by minimizing the error between the numerical and the exact solutions as much as possible. Three techniques of spectral methods are popular, namely, collocation, Galerkin, and tau methods [14, 18]. The collocation method is based on selecting suitable collocation points and enforcing the differential equation to be satisfied precisely at these points and the initial/ boundary conditions are set as constraints. In the Galerkin method, the core concept for choosing the basis functions is to verify the underlying boundary/initial conditions of the given differential equation, then making the residual of the differential equation orthogonal to the basis functions. The tau method has been originally developed by Lanczos [31, 32]. Unlike the Galerkin approach, the tau method does not require that the basis functions satisfy the boundary constraint [17, 3, 5, 4, 6].

In the current years, considerable interest has been devoted to proposing some numerical algorithms for treating second-order HTDEs equations. For example, in [27], the author presented a spectral collocation algorithm for handling a system of one- and two-dimensional linear

HTDEs. The authors in [35] proposed a numerical algorithm to solve the one-dimensional HTDEs using the Chebyshev tau method. The authors in [22] solved linear HTDEs using Galerkin method and they presented approximate solutions using the double shifted Jacobi polynomials. The authors in [21] introduced an efficient Bernoulli-Laguerre collocation method for solving the nonlinear HTDEs in one dimension. The authors in [2] developed two new effective spectral algorithms for solving the HTDEs governed by specified initial and boundary conditions. Moreover, in [12], a numerical solution is proposed based on utilizing a Jacobi collocation method for treating the non-linear coupled HTDEs with variable coefficients subject to non-local conditions.

In this paper, we apply the shifted Jacobi Galerkin method for solving the linear second-order HTDEs governed by initial and boundary conditions. We handle both homogeneous and nonhomogeneous boundary conditions of linear second-order HTDEs. Furthermore, we discuss the convergence analysis of the shifted Jacobi expansion.

The paper is formed as follows. Section 2 displays some properties of Jacobi and shifted Jacobi polynomials. Section 3 is devoted to applying the proposed shifted Jacobi Galerkin method for the numerical treatment of one-dimensional linear HTDEs with homogeneous and nonhomogeneous boundary conditions. Section 4 discusses the convergence and error analysis of the suggested shifted Jacobi expansion. In Section 5, the proposed method is tested via presenting some illustrative examples. We end the paper by presenting some concluding remarks in Section 6.

2 Some properties of standard Jacobi polynomials and their shifted ones

This section displays some properties of the standard Jacobi polynomials and their shifted polynomials.

2.1 Standard Jacobi polynomials

Here, we present some fundamental properties of the standard Jacobi polynomials $J_r^{(\mu,\nu)}(x)$. The following special values are valid

$$J_r^{(\nu,\mu)}(-x) = (-1)^r\, J_r^{(\mu,\nu)}(x),\ J_r^{(\mu,\nu)}(-1) = \frac{(-1)^r\,\Gamma(r+\nu+1)}{r!\,\Gamma(\nu+1)},\ J_r^{(\mu,\nu)}(1) = \frac{\Gamma(r+\mu+1)}{r!\,\Gamma(\mu+1)}, \tag{2.1}$$

where $\mu > -1, \nu > -1$ and $x \in [-1,1]$.

In addition, the qth derivative (q is a whole number) of $J_r^{(\mu,\nu)}(x)$, could be acquired from

$$D^q J_r^{(\mu,\nu)}(x) = \frac{\Gamma(q+r+\mu+\nu+1)}{2^q\Gamma(r+\mu+\nu+1)} J_{r-q}^{(\mu+q,\nu+q)}(x). \tag{2.2}$$

Now, let $w^{(\mu,\nu)}(x) = (1-x)^\mu(1+x)^\nu$, and define the following norm and inner product for the weighted space $L^2_{w^{(\mu,\nu)}}[-1,1]$

$$(f,g)_{w^{(\mu,\nu)}} = \int_{-1}^{1} f(x)g(x)w^{(\mu,\nu)}(x)dx, \quad \|f\|_{w^{(\mu,\nu)}} = (f,f)^{\frac{1}{2}}_{w^{(\mu,\nu)}}. \tag{2.3}$$

The set of Jacobi polynomials forms a complete $L^2_{w^{(\mu,\nu)}}$[-1,1]-orthogonal system. In addition, and because of (2.3), we have

$$\|J_r^{(\mu,\nu)}\|^2_{w^{(\mu,\nu)}} = h_r^{(\mu,\nu)},$$

where

$$h_r^{(\mu,\nu)} = \frac{2^{\mu+\nu+1}\Gamma(r+\mu+1)\,\Gamma(r+\nu+1)}{(2r+\mu+\nu+1)\,r!\,\Gamma(r+\mu+\nu+1)}. \tag{2.4}$$

2.2 Shifted Jacobi Polynomials

The shifted Jacobi polynomials $J_{\ell,r}^{(\mu,\nu)}(x)$ of degree r (see,[20]) are defined on $[0,\ell]$ by

$$J_{\ell,r}^{(\mu,\nu)}(x) = J_r^{(\mu,\nu)}\left(\frac{2x}{\ell}-1\right), \quad r = 0,1,\dots, \tag{2.5}$$

where $J_r^{(\mu,\nu)}(x)$ are the standard Jacobi polynomials. Due to the identities in (2.1), it is easy to see that

$$J_{\ell,r}^{(\mu,\nu)}(0) = \frac{(-1)^r\,\Gamma(r+\nu+1)}{r!\,\Gamma(\nu+1)}, \; J_{\ell,r}^{(\mu,\nu)}(\ell) = \frac{\Gamma(r+\mu+1)}{r!\,\Gamma(\mu+1)}. \tag{2.6}$$

Let $w_\ell^{(\mu,\nu)}(x) = x^\nu\,(\ell-x)^\mu$. We can define the norm and inner product for the weighted space $L^2_{w_\ell^{(\mu,\nu)}}[0,\ell]$ as

$$(f,g)_{w_\ell^{(\mu,\nu)}} = \int_0^\ell f(x)g(x)w_\ell^{(\mu,\nu)}(x)dx, \quad \|f\|_{w_\ell^{(\mu,\nu)}} = (f(x),f(x))^{\frac{1}{2}}_{w_\ell^{(\mu,\nu)}}. \tag{2.7}$$

The set of polynomials $\{J_r^{(\mu,\nu)}(x)\}_{r\geq 0}$ forms a complete $L^2_{w^{(\mu,\nu)}}[0,\ell]$-orthogonal system. In addition, and due to (2.7), we have

$$\|J_{\ell,r}^{(\mu,\nu)}\|^2_{w_\ell^{(\mu,\nu)}} = h_{\ell,r}^{(\mu,\nu)} = \left(\frac{\ell}{2}\right)^{\mu+\nu+1} h_r^{(\mu,\nu)}, \tag{2.8}$$

where $h_r^{(\mu,\nu)}$ is defined as in (2.4).

It is worthy to mention here that for the polynomials in (2.5), the following polynomials can be obtained from the polynomials $J_r^{(\mu,\nu)}(x)$ as special cases.

- For $\mu=\nu$, the shifted ultraspherical polynomials are obtained.
- For $\mu=\nu=-\frac{1}{2}$, the shifted Chebyshev polynomials of the first kind are obtained.
- For $\mu=\nu=\frac{1}{2}$, the shifted Chebyshev polynomials of the second kind are obtained.
- For $\mu=\nu=0$, the shifted Legendre polynomials are obtained.
- For $\mu=-\frac{1}{2}$ and $\nu=\frac{1}{2}$, the shifted Chebyshev polynomials of the third kind are obtained.
- For $\mu=\frac{1}{2}$ and $\nu=-\frac{1}{2}$, the shifted Chebyshev polynomials of the fourth kind are obtained.

We denote $x_{N,j}^{(\mu,\nu)}$, $0\leq j\leq N$ and $\varpi_{N,j}^{(\mu,\nu)}$, $0\leq j\leq N$, by the nodes and Christoffel numbers of the standard Jacobi- Gauss interpolation in [-1,1]. The corresponding nodes and corresponding Christoffel numbers of the shifted Jacobi-Gauss interpolation in $[0,\ell]$ can be offered through

$$x_{\ell,N,j}^{(\mu,\nu)} = \frac{\ell}{2}\left(x_{N,j}^{(\mu,\nu)}+1\right),$$
$$\varpi_{\ell,N,j}^{(\mu,\nu)} = \left(\frac{\ell}{2}\right)^{\mu+\nu+1}\varpi_{N,j}^{(\mu,\nu)}.$$

From Jacobi-Gauss quadrature, we have for any positive integer N, $\phi\in S_{2N+1}[0,\ell]$,

$$\begin{aligned}\int_0^\ell (\ell-x)^\mu\,x^\nu\,\phi(x)dx &= \left(\frac{\ell}{2}\right)^{\mu+\nu+1}\int_{-1}^1 (1-x)^\mu(1+x)^\nu\phi\left(\frac{\ell}{2}(x+1)\right)dx \\ &= \left(\frac{\ell}{2}\right)^{\mu+\nu+1}\sum_{j=0}^N \varpi_{N,j}^{(\mu,\nu)}\phi\left(\frac{\ell}{2}(x_{N,j}^{(\mu,\nu)}+1)\right) \\ &= \sum_{j=0}^N \varpi_{\ell,N,j}^{(\mu,\nu)}\phi\left(x_{\ell,N,j}^{(\mu,\nu)}\right).\end{aligned} \tag{2.9}$$

Also, $J_{\ell,r}^{(\mu,\nu)}(x)$ may be constructed with the aid of the following recursive formula

$$\begin{aligned}
&2r(r+\mu+\nu)(2r+\mu+\nu-2)J_{\ell,r}^{(\mu,\nu)}(x) = \\
&(2r+\mu+\nu-1)\left[(2r+\mu+\nu)(2r+\mu+\nu-2)\left(\frac{2x}{\ell}-1\right)+\mu^2-\nu^2\right] J_{\ell,r-1}^{(\mu,\nu)}(x) \\
&-2(r+\mu-1)(r+\nu-1)(2r+\mu+\nu)\, J_{\ell,r-2}^{(\mu,\nu)}(x), \\
&\qquad J_{\ell,0}^{(\mu,\nu)}(x)=1,\; J_{\ell,1}^{(\mu,\nu)}(x)=\frac{1}{2}(\mu-\nu)+\frac{1}{2}(\mu+\nu+2)\left(\frac{2x}{\ell}-1\right),\; r\geq 2.
\end{aligned}$$

The upcoming two theorems which give explicit expressions for the derivatives and repeated integrals of the shifted Jacobi polynomials are useful in what follows.

Theorem 1. [20] The derivatives of $J_{\ell,r}^{(\mu,\nu)}(x)$ have the following link with their original polynomials

$$D^q J_{\ell,r}^{(\mu,\nu)}(x) = \sum_{m=0}^{r-q} C_q(r,m,\mu,\nu)\, J_{\ell,m}^{(\mu,\nu)}(x), \tag{2.10}$$

with

$$\begin{aligned}
C_q(r,m,\mu,\nu) =& \frac{(r+\eta)_q(r+\eta+q)_m(m+\mu+q+1)_{r-m-q}\Gamma(m+\eta)}{\ell^q(r-m-q)!\,\Gamma(2m+\eta)} \\
&\times {}_3F_2\left(\left.\begin{matrix} -r+m+q, & r+m+\eta+q, & m+\mu+1 \\ m+\mu+q+1, & 2m+\eta+1 & \end{matrix}\right| 1\right),
\end{aligned} \tag{2.11}$$

with $\eta=\mu+\nu+1$, $(z)_q=\dfrac{\Gamma(z+q)}{\Gamma(x)}$, and ${}_3F_2(z)$ denotes a specific hypergeometric function of the well-known standard generalized hypergeometric function (see [11]).

Theorem 2. The q times repeated integration of $J_{\ell,r}^{(\mu,\nu)}(x)$ ([19]) can be explicitly expressed as

$$\begin{aligned}
I_{\ell,r}^{(q,\mu,\nu)}(x) =& \frac{r!\,\Gamma(\mu+1)}{\Gamma(r+\mu+1)} \overbrace{\int\int\ldots\int}^{q\ times} J_{\ell,r}^{(\mu,\nu)}(x)\, \overbrace{dx\,dx\,...dx}^{q\ times} = \sum_{j=0}^{2q}\theta(j,q,r,\mu,\nu)\, J_{\ell,r+q-j}^{(\mu,\nu)}(x) \\
&+\pi_{q-1}(x),
\end{aligned} \tag{2.12}$$

where

$$\begin{aligned}
\theta(j,q,r,\mu,\nu) =& \frac{2^q\, r!\,(r+q-j+1)_\mu\,(r+\mu-j+1)_j\,(r-q+\eta)_{r+q-j}}{j!\,(\mu+1)_r\,(r+1)_\mu\,(r+q+\eta)_q\,(r+q+\eta-j)_{r+q-j}} \\
&\times {}_3F_2\left(\left.\begin{matrix} -j, & r+q+\mu-j+1, & 2r+\eta-j \\ r+\mu-j+1, & 2r+2q-2j+\eta+1 & \end{matrix}\right| 1\right),
\end{aligned} \tag{2.13}$$

and $\pi_{q-1}(x)$ is a polynomial whose degree does not exceed $(q-1)$.

Furthermore, and in virtue of Theorem 2, it can be shown that the following specific integral formulas hold

$$\int_0^t J_{\tau,i}^{(\mu,\nu)}(z)\,dz = \sum_{j=0}^{2}\theta(j,1,i,\mu,\nu)\, J_{\tau,i+1-j}^{(\mu,\nu)}(t) + A, \tag{2.14}$$

$$\int_0^t\int_0^z J_{\tau,i}^{(\mu,\nu)}(s)\,ds\,dz = \sum_{j=0}^{4}\theta(j,2,i,\mu,\nu)\, J_{\tau,i+2-j}^{(\mu,\nu)}(t) + B\,t + C, \tag{2.15}$$

where A, B, C are arbitrary constants.

3 Numerical treatment of the telegraph type equations

This section describes in detail how the shifted Jacobi Galerkin method (SJGM) can be employed for obtaining approximate solutions for the linear HTDEs.

3.1 The linear telegraph equations

Consider the following HTDE namely

$$\partial_{tt}v(x,t) + \alpha\,\partial_t v(x,t) + \beta\, v(x,t) = \partial_{xx}v(x,t) + f(x,t), \quad 0 < x < \ell,\, 0 < t \le \tau, \tag{3.1}$$

governed by the initial conditions

$$v(x,0) = p_0(x), \quad \partial_t v(x,0) = p_1(x), \quad 0 < x < \ell, \tag{3.2}$$

and the homogeneous boundary conditions

$$v(0,t) = v(\ell,t) = 0, \qquad 0 < t \le \tau. \tag{3.3}$$

In (3.1), α, β are constants and $f(x,t)$ is the source term which describes that the medium is heated ($f(x,t) > 0$) or cooled ($f(x,t) < 0$) at space x and time t.

In order to settle the initial conditions (3.2), Eq. (3.1) is integrated twice with respect to t. This means that instead of solving the HTDE (3.1) governed by (3.2) and (3.3), we solve alternatively its integrated formula

$$\begin{aligned} &v(x,t) + \alpha \int_0^t v(x,z)\,dz + \beta \int_0^t \int_0^z v(x,s)ds\,dz - \int_0^t \int_0^z \partial_{xx}v(x,s)ds\,dz = \tilde{f}(x,t), \\ &v(0,t) = v(\ell,t) = 0,\ 0 < x < \ell,\ 0 < t \le \tau, \end{aligned} \tag{3.4}$$

and

$$\tilde{f}(x,t) = \int_0^t \int_0^z f(x,s)ds\,dz + p_0(x)(\alpha\, t + 1) + p_1(x)\, t. \tag{3.5}$$

Shifted Jacobi Galerkin method

The core concept for the application of this method is to choose the basis functions satisfying the underlying boundary conditions in (3.4).

Now, consider the following spaces:

$$\begin{aligned} Y &= \{Y \in H^2_{w_{\ell,\tau}}(\Omega) : y(0,t) = y(\ell,t) = 0,\, t \in [0,\tau]\}, \\ Y_N &= \text{span}\{\psi^{(\mu,\nu)}_{\tau,i}(t)\phi^{(\mu,\nu)}_{\ell,j}(x),\ 0 \le i,j \le N\}, \end{aligned}$$

where $H^2_{w_{\ell,\tau}}(\Omega)$, $\Omega = (0,\tau) \times (0,\tau]$ is the Sobolev space defined in [16].

Let $v(x,t) \in Y$ and assume that it has the following expansion

$$v(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij}\, \psi^{(\mu,\nu)}_{\tau,i}(t)\, \phi^{(\mu,\nu)}_{\ell,j}(x), \tag{3.6}$$

and assume that it can be approximated as

$$v(x,t) \simeq v_N(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{N} c_{ij}\, \psi^{(\mu,\nu)}_{\tau,i}(t)\, \phi^{(\mu,\nu)}_{\ell,j}(x) = \mathbf{\Psi}^T_N(t)\, \mathbf{C}\, \mathbf{\Phi}_N(x), \tag{3.7}$$

where

$$c_{ij}=\frac{\ell^4}{\sigma^2_{j,\mu,\nu}h^{(\mu,\nu)}_{\tau,i}h^{(\mu+1,\nu+1)}_{\ell,j}}\int_0^{\tau}\int_0^{\ell}v(x,t)\psi^{(\mu,\nu)}_{\tau,i}(t)\phi^{(\mu,\nu)}_{\ell,j}(x)(\tau-t)^{\mu}\,t^{\nu}(\ell-x)^{\mu-1}x^{\nu-1}dx\,dt, \tag{3.8}$$

and

$$\mathbf{\Psi}^T_N(t)=[\psi^{(\mu,\nu)}_{\tau,0}(t),\psi^{(\mu,\nu)}_{\tau,1}(t),\dots,\psi^{(\mu,\nu)}_{\tau,N}(t)],\quad \mathbf{\Phi}_N(x)=[\phi^{(\mu,\nu)}_{\ell,0}(x),\phi^{(\mu,\nu)}_{\ell,1}(x),\dots,\phi^{(\mu,\nu)}_{\ell,N}(x)],$$

$$\boldsymbol{C}=(c_{ij})_{0\le i,j\le N}. \tag{3.9}$$

Then the shifted Jacobi Galerkin approximation to (3.4) is to find $v_N\in Y_N$ such that

$$\left.\begin{aligned}&(v_N,u)_{w(x,t)_{\ell,\tau}}+\alpha\left(\int_0^t v_N\,dt,u\right)_{w(x,t)_{\ell,\tau}}+\beta\left(\int_0^t\int_0^t v_N dt\,dt,u\right)_{w(x,t)_{\ell,\tau}}\\&-\left(\int_0^t\int_0^t\partial_{xx}v_N dt\,dt,u\right)_{w(x,t)_{\ell,\tau}}=(\tilde f,u)_{w(x,t)_{\ell,\tau}},\quad u\in Y_N,\end{aligned}\right\} \tag{3.10}$$

where $w(x,t)_{\ell,\tau}=(\ell-x)^{\mu-1}x^{\nu-1}(\tau-t)^{\mu}\,t^{\nu}$ and $(v,u)_{w_{\ell,\tau}}=\displaystyle\int_0^{\tau}\int_0^{\ell}vuw_{\ell,\tau}\,dx\,dt$ is the inner products in the weighted space $L^2_w(0,\ell)\times(0,\tau]$.

For the sake of minimizing the bandwidth of the coefficient matrix corresponding to (3.4), we choose the following basis functions:

$$\psi^{(\mu,\nu)}_{\tau,i}(t)=J^{(\mu,\nu)}_{\tau,i}(t), \tag{3.11}$$

$$\phi^{(\mu,\nu)}_{\ell,j}(x)=J^{(\mu,\nu)}_{\ell,j}(x)+\eta_j J^{(\mu,\nu)}_{\ell,j+1}(x)+\bar\eta_j J^{(\mu,\nu)}_{\ell,j+2}(x). \tag{3.12}$$

From the boundary conditions: $\phi^{(\mu,\nu)}_{\ell,j}(0)=\phi^{(\mu,\nu)}_{\ell,j}(\ell)=0$, and the two identities in (2.6), we obtain immediately

$$\eta_j=\frac{(j+1)(\mu-\nu)(2j+\mu+\nu+3)}{(j+\nu+1)(j+\mu+1)(2j+\mu+\nu+4)}, \tag{3.13}$$

$$\bar\eta_j=\frac{-(j+1)(j+2)(2j+\mu+\nu+2)}{(j+\nu+1)(j+\mu+1)(2j+\mu+\nu+4)}. \tag{3.14}$$

It is now clear that (3.10) is equivalent to:

$$\begin{aligned}&\left(v_N,\psi^{(\mu,\nu)}_{\tau,i}(t)\phi^{(\mu,\nu)}_{\ell,j}(x)\right)_{w(x,t)_{\ell,\tau}}+\alpha\left(\int_0^t v_N\,dt,\psi^{(\mu,\nu)}_{\tau,i}(t)\phi^{(\mu,\nu)}_{\ell,j}(x)\right)_{w(x,t)_{\ell,\tau}}\\&+\beta\left(\int_0^t\int_0^t v_N dt\,dt,\psi^{(\mu,\nu)}_{\tau,i}(t)\phi^{(\mu,\nu)}_{\ell,j}(x)\right)_{w(x,t)_{\ell,\tau}}\\&-\left(\int_0^t\int_0^t\partial_{xx}v_N dt\,dt,\psi^{(\mu,\nu)}_{\tau,i}(t)\phi^{(\mu,\nu)}_{\ell,j}(x)\right)_{w(x,t)_{\ell,\tau}}=\left(\tilde f(x,t),\psi^{(\mu,\nu)}_{\tau,i}(t)\phi^{(\mu,\nu)}_{\ell,j}(x)\right)_{w(x,t)_{\ell,\tau}}.\end{aligned} \tag{3.15}$$

Let us denote

$$\begin{aligned}\tilde f_{ij}&=\left(\tilde f(x,t),\psi^{(\mu,\nu)}_{\tau,i}(t)\phi^{(\mu,\nu)}_{\ell,j}(x)\right)_{w(x,t)_{\ell,\tau}}\\&=\int_0^t\int_0^{\ell}\tilde f(x,t)\psi^{(\mu,\nu)}_{\tau,i}(t)(\tau-t)^{\mu}\,t^{\nu}\phi^{(\mu,\nu)}_{\ell,j}(x)(\ell-x)^{\mu-1}x^{\nu-1}dx\,dt,\end{aligned}$$

then making use of (2.9) allows one to write

$$\tilde{f}_{ij}=\sum_{n=0}^{N}\sum_{m=0}^{N}\tilde{f}(x^{(\mu,\nu)}_{\ell,N,m},t^{(\mu,\nu)}_{\tau,N,n})\varpi^{(\mu,\nu)}_{\tau,N,n}J^{(\mu,\nu)}_{\tau,i}(t^{(\mu,\nu)}_{\tau,N,n})\varpi^{(\mu,\nu)}_{\ell,N,m}\phi^{(\mu,\nu)}_{\ell,j}(x^{(\mu,\nu)}_{\ell,N,m}),$$

and

$$F=(\tilde{f}_{ij}),i=0,1,\dots,N,j=0,1,\dots,N.$$

Now, if $v(x,t)$ is approximated as in (3.7), then relations (2.14) and (2.15) lead to the following approximations:

$$\int_0^t v(x,t)\,dt=\boldsymbol{C}\,\Phi(x)\sum_{j=0}^{2}\theta(j,1,i,\mu,\nu)\,J^{(\mu,\nu)}_{\tau,i+1-j}(t), \tag{3.16}$$

$$\int_0^t\int_0^t v(x,t)dt\,dt=\boldsymbol{C}\,\Phi(x)\sum_{j=0}^{4}\theta(j,2,i,\mu,\nu)\,J^{(\mu,\nu)}_{\tau,i+2-j}(t), \tag{3.17}$$

$$\int_0^t\int_0^t \partial_{xx}v(x,t)dt\,dt=\boldsymbol{C}\,\partial_{xx}\Phi(x)\sum_{j=0}^{4}\theta(j,2,i,\mu,\nu)\,J^{(\mu,\nu)}_{\tau,i+2-j}(t). \tag{3.18}$$

Then, in matrix form, Eq. (3.15) can be written as:

$$(A\,D+\alpha\,A\,E+\beta\,A\,G-K\,G)\,\boldsymbol{C}-\boldsymbol{F}=0, \tag{3.19}$$

where $\boldsymbol{F}$ is written in a column vector as

$$\boldsymbol{F}=[\tilde{f}_{00},\tilde{f}_{10},\cdots,\tilde{f}_{N0},\tilde{f}_{01},\tilde{f}_{11},\cdots,\tilde{f}_{N1},\tilde{f}_{0N},\tilde{f}_{1N},\cdots,\tilde{f}_{NN}]^T,$$

and A,D,E,G and K are $(N+1)\times(N+1)$ matrices, and

$$\begin{aligned}
a_{ij}&=\left(\phi^{(\mu,\nu)}_{\ell,j}(x),\phi^{(\mu,\nu)}_{\ell,j}(x)\right)_{w_\ell}, & A&=(a_{ij})_{0\le i,j\le N},\\
d_{ij}&=\left(\psi^{(\mu,\nu)}_{\tau,i}(t),\psi^{(\mu,\nu)}_{\tau,i}(t)\right)_{w_\tau}, & D&=(d_{ij})_{0\le i,j\le N},\\
e_{ij}&=\left(\psi^{(\mu,\nu)}_{\tau,i}(t),\int_0^\tau \psi^{(\mu,\nu)}_{\tau,i}(t)dt\right)_{w_\tau}, & E&=(e_{ij}))_{0\le i,j\le N},\\
g_{ij}&=\left(\psi^{(\mu,\nu)}_{\tau,i}(t),\int_0^t\int_0^t \psi^{(\mu,\nu)}_{\tau,i}(t)dt\,dt\right)_{w_\tau}, & G&=(g_{ij})_{0\le i,j\le N},\\
k_{ij}&=\left(\partial_{xx}\phi^{(\mu,\nu)}_{\ell,j}(x),\phi^{(\mu,\nu)}_{\ell,j}(x)\right)_{w_\ell}, & K&=(k_{ij})_{0\le i,j\le N},
\end{aligned}$$

with $w_\ell=x^\nu\,(\ell-x)^\mu,\ {}_{w_\tau}=x^\nu\,(\tau-x)^\mu$.
The nonzero elements of the vectors A,D,E,G and K are given explicitly in the following theorem.

Theorem 3. If we take the basis functions $\psi_i(t)$ and $\phi_j(x)$ as in (3.11) and (3.12), then the nonzero elements of the matrices A, D, E, G and K are given explicitly by

$$a_{jj} = h_{\ell,j}^{(\mu,\nu)} + \epsilon_j^2 h_{\ell,j+1}^{(\mu,\nu)} + \vartheta_j^2 h_{\ell,j+2}^{(\mu,\nu)}, \tag{3.20}$$

$$d_{ii} = h_{\tau,i}^{(\mu,\nu)}, \tag{3.21}$$

$$e_{ii} = \theta(1,1,i,\mu,\nu) h_{\tau,i}^{(\mu,\nu)}, \tag{3.22}$$

$$g_{ii} = \theta(2,2,i,\mu,\nu) h_{\tau,i}^{(\mu,\nu)}, \tag{3.23}$$

$$k_{jj} = \bar{\eta}_j\, C_2(j+2,j,\mu,\nu) h_{\ell,j}^{(\mu,\nu)}, \tag{3.24}$$

where $0 \le i, j \le N$.

Proof. To show (3.20) and (3.24), we make use of relations (2.10) and (3.12), to get

$$\begin{aligned}
a_{jj} &= \left(\phi_{\ell,j}^{(\mu,\nu)}(x), \phi_{\ell,j}^{(\mu,\nu)}(x)\right)_{w_\ell} \\
&= \left(J_{\ell,j}^{(\mu,\nu)}(x) + \epsilon_j J_{\ell,j+1}^{(\mu,\nu)}(x) + \vartheta_j J_{\ell,j+2}^{(\mu,\nu)}(x), J_{\ell,j}^{(\mu,\nu)}(x) + \epsilon_j J_{\ell,j+1}^{(\mu,\nu)}(x) + \vartheta_j J_{\ell,j+2}^{(\mu,\nu)}(x)\right)_{w_\ell}, \\
k_{jj} &= \left(\partial_{xx}\phi_{\ell,j}^{(\mu,\nu)}(x), \phi_{\ell,j}^{(\mu,\nu)}(x)\right)_{w_\ell} \\
&= \Bigg(\sum_{i=0}^{j-2} C_2(j,i,\mu,\nu)\, J_{\ell,i}^{(\mu,\nu)}(x) + \epsilon_j \sum_{i=0}^{j-1} C_2(j+1,i,\mu,\nu)\, J_{\ell,i}^{(\mu,\nu)}(x) + \vartheta_j \sum_{i=0}^{j} C_2(j+2,i,\mu,\nu)\, J_{\ell,i}^{(\mu,\nu)}(x), \\
&\qquad J_{\ell,j}^{(\mu,\nu)}(x) + \epsilon_j J_{\ell,j+1}^{(\mu,\nu)}(x) + \vartheta_j J_{\ell,j+2}^{(\mu,\nu)}(x)\Bigg)_{w_\ell}.
\end{aligned}$$

Hence, the application of the orthogonality relation (2.8) yields the required results in (3.20) and (3.24).

Relation (3.21) can be acquired with the aid of Eqs. (3.11) and (2.8).

To prove relations (3.22) and (3.23). In virtue of relations (3.11), (2.14) and (2.15) , we have

$$\begin{aligned}
e_{ii} &= \left(\psi_{\tau,i}^{(\mu,\nu)}(t), \int_0^t \psi_{\tau,i}^{(\mu,\nu)}(t)dt\right)_{w_\tau} \\
&= \left(J_{\tau,i}^{(\mu,\nu)}(t), \int_0^t J_{\tau,i}^{(\mu,\nu)}(t)dt\right)_{w_\tau} \\
&= \left(J_{\tau,i}^{(\mu,\nu)}(t), \theta(0,1,i,\mu,\nu)J_{\tau,i+1}^{(\mu,\nu)}(t) + \theta(1,1,i,\mu,\nu)J_{\tau,i}^{(\mu,\nu)}(t) + \theta(2,1,i,\mu,\nu)J_{\tau,i-1}^{(\mu,\nu)}(t)\right)_{w_\tau}, \\
g_{ii} &= \left(\psi_{\tau,i}^{(\mu,\nu)}(t), \int_0^t\int_0^t \psi_{\tau,i}^{(\mu,\nu)}(t)dt\right)_{w_\tau} \\
&= \left(J_{\tau,i}^{(\mu,\nu)}(t), \int_0^t\int_0^t J_{\tau,i}^{(\mu,\nu)}(t)dt\right)_{w_\tau} \\
&= \Big(J_{\tau,i}^{(\mu,\nu)}(t), \theta(0,2,i,\mu,\nu)J_{\tau,i+2}^{(\mu,\nu)}(t) + \theta(1,2,i,\mu,\nu)J_{\tau,i+1}^{(\mu,\nu)}(t) + \theta(2,2,i,\mu,\nu)J_{\tau,i}^{(\mu,\nu)}(t) \\
&\quad + \theta(3,2,i,\mu,\nu)J_{\tau,i-1}^{(\mu,\nu)}(t) + \theta(4,2,i,\mu,\nu)J_{\tau,i-2}^{(\mu,\nu)}(t)\Big)_{w_\tau}.
\end{aligned}$$

The orthogonality relation (2.8) again yields (3.22) and (3.23). □

3.2 Treatment of the non-homogeneous Boundary Conditions

Eq. (3.1) subject to the initial conditions (3.2) and the following non-homogeneous boundary conditions

$$v(0,t) = q_0(t) \qquad v(\ell,t) = q_1(t), \qquad\qquad 0 < t \leq \tau, \tag{3.25}$$

can be converted with the aid of a suitable transformation ([2]) into a modified one similar to (3.1) subject to (3.2) and (3.3). More precisely, the following modified equation can be obtained:

$$V(x,t) + \alpha \int_0^t V(x,t)\, dt + \beta \int_0^t \int_0^t V(x,t) dt\, dt - \int_0^t \int_0^t \partial_{xx} V(x,t) dt\, dt = \tilde{\tilde{f}}(x,t), \tag{3.26}$$

governed by the homogeneous boundary conditions

$$V(0,t) = V(\ell,t) = 0, \qquad 0 < t \leq \tau, \tag{3.27}$$

where

$$\begin{aligned}\tilde{\tilde{f}}(x,t) =& \tilde{f}(x,t) - q_0(t) - \frac{q_1(t) - q_0(t)}{\ell} x \\ & - \alpha \int_0^t \left[q_0(t) + \frac{q_1(t) - q_0(t)}{\ell} x \right] dt - \beta \int_0^t \int_0^t \left[q_0(t) + \frac{q_1(t) - q_0(t)}{\ell} x \right] dt\, dt,\end{aligned}$$

and $\tilde{f}(x,t)$ is as defined in (3.5). Now, we can apply SJGM to (3.26) subject to the homogeneous boundary conditions (3.27).

4 Convergence and error analysis of the suggested expansion

In this section, we give a comprehensive study for the convergence and error analysis of the suggested shifted Jacobi expansion. Some lemmas in this respect are needed.

Lemma 1. The basis functions in (3.12) can be written alternatively as

$$\phi_{\ell,j}^{(\mu,\nu)}(x) = \frac{\sigma_{j,\mu,\nu}}{\ell^2} x(\ell - x) J_{\ell,j}^{(\mu+1,\nu+1)}(x), \tag{4.1}$$

where

$$\sigma_{j,\mu,\nu} = \frac{(2j+\eta+1)(2j+\eta+2)}{(j+\mu+1)(j+\nu+1)}\ ,\ \eta = \mu + \nu + 1.$$

Lemma 2. [15] (Bernstien-type inequality of Jacobi polynomial) If $\max\{|\mu|, |\nu|\} \leq \frac{1}{2}$, then the following inequality is valid:

$$\left| J_{\ell,k}^{(\mu,\nu)}(x) \right| \leq \left(\frac{\ell}{2}\right)^{\frac{\eta}{2}} \frac{\Gamma(q+1)\binom{n+q}{n}}{\Gamma(\frac{1}{2})(n+\frac{\eta}{2})^{q+\frac{1}{2}}} w^{(-\frac{2\mu+1}{4}, -\frac{2\nu+1}{4})}(x),$$

with $q = \max\{\mu, \nu\}$.

Lemma 3. The following integral formula holds:

$$\int J_{\ell,k}^{(\mu,\nu)}(x)\, dx = \frac{\ell}{k+\eta-1} J_{\ell,k+1}^{(\mu-1,\nu-1)}(x).$$

Lemma 4. The following orthogonality relations hold:

$$\int_0^\ell \phi_{\ell,i}^{(\mu,\nu)}(x) \phi_{\ell,j}^{(\mu,\nu)}(x)\ (\ell - x)^{\mu-1}\ x^{\nu-1}\, dx = \frac{\sigma_{j,\mu,\nu}^2}{\ell^4} h_{\ell,j}^{(\mu+1,\nu+1)} \delta_{ij},$$

$$\int_0^\tau \psi_{\tau,i}^{(\mu,\nu)}(t)\ \psi_{\tau,j}^{(\mu,\nu)}(t)\ (\tau - t)^\mu\ t^\nu\, dt = h_{\tau,i}^{(\mu,\nu)} \delta_{ij},$$

where δ_{ij} is the well-known Kronecker delta function.

Lemma 5. [28] (Stirling approximation of Gamma function) For all $n > 4$ and $|a| < 2$, one has:

$$\frac{1}{2}n^{a-1}n! < \Gamma(n+a) < 2n^{a-1}n!.$$

Theorem 4. The following estimates are valid:

(i) $|h_{\ell,j}^{(\mu+1,\nu+1)}| = \mathcal{O}(\frac{1}{j}), \quad |h_{\tau,i}^{(\mu,\nu)}| = \mathcal{O}(\frac{1}{i})$.

(ii) Under the assumptions of Lemma 2, there exists a generic positive constant ς such that $|J_{\ell,k}^{(\mu,\nu)}(x)| \leq (\frac{\varsigma}{\sqrt{k}})w^{(-\frac{2\mu+1}{4},-\frac{2\nu+1}{4})}(x)$.

Theorem 5. (Convergence) Any function $v(x,t) = x(\ell - x)\, g_1(t)g_2(x) \in Z$ with $|g_1^{(i)}(t)| \leqslant \gamma_i$ and $|g_2^{(i)}(x)| \leqslant \varrho_i$; $i = 0,1,2$, $|\mu| \leqslant \frac{1}{2}$, $|\nu| \leqslant \frac{1}{2}$ can be expanded as an infinite double sum in terms of the basis functions given in (3.6) that converges uniformly to $v(x,t)$. Moreover, the coefficients c_{ij} in (3.8) satisfy the inequality

$$|c_{ij}| = \mathcal{O}\left(\frac{1}{i^{\frac{3}{2}}j^{\frac{3}{2}}}\right).$$

Proof. If we start with the coefficients c_{ij} in (3.8), then we have

$$c_{ij} = \frac{\ell^4}{\sigma_{j,\mu,\nu}^2 h_{\tau,i}^{(\mu,\nu)} h_{\ell,j}^{(\mu+1,\nu+1)}} \int_0^\tau \int_0^\ell v(x,t)\psi_{\tau,i}^{(\mu,\nu)}(t)\phi_{\ell,j}^{(\mu,\nu)}(x)t^\nu(\tau-t)^\mu x^{\nu-1}(\ell-x)^{\mu-1}dx\,dt, \tag{4.2}$$

then by the hypotheses of the theorem and (3.11), (4.1) enables one to write

$$c_{ij} = \frac{\ell^2}{\sigma_{j,\mu,\nu} h_{\tau,i}^{(\mu,\nu)} h_{\ell,j}^{(\mu+1,\nu+1)}} \int_0^\tau g_1(t)J_{\tau,i}^{(\mu,\nu)}(t)w_\tau^{(\mu,\nu)}(t)dt \int_0^\ell g_2(x)J_{\ell,j}^{(\mu+1,\nu+1)}(x)w_\ell^{(\mu+1,\nu+1)}(x)dx.$$

Let

$$c_{ij} = I_{g_1}(i)I_{g_2}(j),$$

where

$$I_{g_1}(i) = \frac{1}{h_{\tau,i}^{(\mu,\nu)}} \int_0^\tau g_1(t)J_{\tau,i}^{(\mu,\nu)}(t)w_\tau^{(\mu,\nu)}(t)dt, \tag{4.3}$$

and

$$I_{g_2}(j) = \frac{\ell^2}{\sigma_{j,\mu,\nu} h_{\ell,j}^{(\mu+1,\nu+1)}} \int_0^\ell g_2(x)J_{\ell,j}^{(\mu+1,\nu+1)}(x)w_\ell^{(\mu+1,\nu+1)}(x)dx. \tag{4.4}$$

If the right hand side of (4.3) and (4.4) are integrated twice by parts, then we get

$$\begin{aligned} I_{g_1}(i) = &\frac{\tau^2}{(i+\eta)(i+\eta-1)h_{\tau,i}^{(\mu,\nu)}} \int_0^\tau \Big(g_1(t)\Big[\nu(\nu-1)w_\tau^{(\mu,\nu-2)} - 2\mu\nu w_\tau^{(\mu-1,\nu-1)} \\ &+ \mu(\mu-1)w_\tau^{(\mu-2,\nu)}\Big] + 2\nu g_1'(t)w_\tau^{(\mu,\nu-1)} - 2\mu g_1'(t)w_\tau^{(\mu-1,\nu)} + g_1''(t)w_\tau^{(\mu,\nu)}\Big)J_{\tau,i+2}^{(\mu-2,\nu-2)}(t)dt, \end{aligned}$$

$$\begin{aligned} I_{g_2}(j) = &\frac{\ell^4}{\sigma_{j,\mu,\nu}(j+\eta-1)(j+\eta+1)h_{\ell,j}^{(\mu+1,\nu+1)}} \int_0^\ell \Big(g_2(x)\Big[\nu(\nu+1)w_\ell^{(\mu+1,\nu-1)} \\ &- 2(\mu+1)(\nu+1)w_\ell^{(\mu,\nu)} + \mu(\mu+1)w_\ell^{(\mu-1,\nu+1)}\Big] + g_2'(x)\Big[2(\nu+1)w_\ell^{(\mu+1,\nu)} - 2(\mu+1)w_\ell^{(\mu,\nu+1)}\Big] \\ &+ g_2''(x)w_\ell^{(\mu+1,\nu+1)}\Big)J_{\ell,i+2}^{(\mu-1,\nu-1)}(x)dx. \end{aligned}$$

Now, and due to the condition $|\mu| \leqslant \frac{1}{2}$ and $|\nu| \leqslant \frac{1}{2}$, an the boundedness of g_1^i, g_2^i, $i = 0,1,2$ along with Theorem 4, the desired result can be achieved which completes the proof of the theorem. □

Theorem 6. If $v(x,t) \in Y$ satisfies the hypotheses of Theorem 5 and if we consider the approximate solution $v_N(x,t)$ as in (3.7), we have the following truncation error estimate

$$\|v - v_N\|_2 = \mathcal{O}(N^{-4}).$$

Proof. From relations (3.11), (3.12), (3.6) and (3.7), we get

$$\|v - v_N\|^2 = \left\| \sum_{i=0}^{N} \sum_{j=N+1}^{\infty} c_{ij}\, \psi_{\tau,i}^{(\mu,\nu)}\, \phi_{\ell,j}^{(\mu,\nu)} + \sum_{i=N+1}^{\infty} \sum_{j=0}^{\infty} c_{ij}\, \psi_{\tau,i}^{(\mu,\nu)}\, \phi_{\ell,j}^{(\mu,\nu)} \right\|^2,$$

and therefore

$$\|v - v_N\|^2 \leqslant \sum_{i=0}^{N} \sum_{j=N+1}^{\infty} |c_{ij}|^2 \|\psi_{\tau,i}^{(\mu,\nu)}\|^2 \|\phi_{\ell,j}^{(\mu,\nu)}\|^2 + \sum_{i=N+1}^{\infty} \sum_{j=0}^{\infty} |c_{ij}|^2 \|\psi_{\tau,i}^{(\mu,\nu)}\|^2 \|\phi_{\ell,j}^{(\mu,\nu)}\|^2.$$

By Lemma 4 and Theorem 4 part (1), we get

$$\|v - v_N\|^2 = \mathcal{O}\left(\sum_{i=N+1}^{\infty} \sum_{j=N+1}^{\infty} \frac{|c_{ij}|^2}{ij} \right),$$

and Theorem 5 leads to

$$\|v - v_N\|^2 = \mathcal{O}\left(\sum_{i=N+1}^{\infty} \sum_{j=N+1}^{\infty} \frac{1}{i^3 j^3} \right).$$

From the integral test (see, [37]), we obtain the result

$$\|v - v_N\|^2 = \mathcal{O}\left(\int_N^{\infty} \int_N^{\infty} \frac{1}{x^3 y^3} dx dt \right),$$

and accordingly, we have

$$\|v - v_N\|^2 = \mathcal{O}(N^{-4}),$$

which ends the proof. □

5 Illustrative Examples

This section is confined to presenting some numerical experiments to demonstrate the use of the proposed algorithm for solving the telegraph type equations. The absolute errors (in L_2 norm) can be computed by

$$L_2 = \left(\int_0^{\tau} \int_0^{\ell} (v(x,t) - v_N(x,t))^2 dx dt \right)^{\frac{1}{2}},$$

and the maximum absolute errors are given by

$$L_\infty = Max\{|v(x,t) - v_N(x,t)|,\ 0 \leq x \leq \ell,\ 0 \leq t \leq \tau\}.$$

Example 1. [35] Consider the linear telegraph equation (3.1) governed by the conditions (3.2) and (3.25), respectively, with

$f(x,t) = x^2 + t - 1$, $p_0(x) = x^2$, $p_1(x) = 1$, $q_0(t) = t$, $q_1(t) = t+1$, $\alpha = \beta = 1$ and $\ell = \tau = 1$.

We apply the proposed algorithm in Section 3.2, when $N = 1$, we get a system of four linear algebraic equations in $\{c_{0,0}, c_{0,1}, c_{1,0}, c_{1,1}\}$, then solving this system we endup with

$$c_{00} = -\frac{(\nu+1)(\mu+1)}{(\nu+\mu+2)(\nu+\mu+3)}, \quad c_{01} = 0, \quad c_{10} = 0, \quad c_{11} = 0.$$

Hence, $v_N(x,t) = x^2 + t$, which coincides with the exact solution of Example 1.

Example 2. [23] Consider the linear equation (3.1) governed by the conditions in (3.2), (3.3)

$$p_0(x) = p_1(x) = 0,\ \alpha = \beta = 1 \text{ and } \ell = \tau = 1.$$

The exact solution is $v(x,t) = x^2(1-x)\sin^2 t$.

In Table 1, we list the maximum absolute error (MAE) for some choices of μ, ν and N In Figure 1, we depict the LogError plot $t = 0.5$. The table and the figure shows that our method is in good agreement with the analytical solution.

Table 1. MAE of Example 2

N	μ	ν	E	μ	ν	E
6	$-\frac{1}{2}$	$-\frac{1}{2}$	$5.268\ .\ 10^{-10}$	$\frac{1}{2}$	$\frac{1}{2}$	$3.259\ .\ 10^{-10}$
10			$2.361\ .\ 10^{-14}$			$6.982\ .\ 10^{-14}$
6	$-\frac{1}{2}$	$\frac{1}{2}$	$2.361\ .\ 10^{-10}$	$\frac{1}{2}$	$-\frac{1}{2}$	$6.837\ .\ 10^{-11}$
10			$2.473\ .\ 10^{-14}$			$9.927\ .\ 10^{-15}$
6	0	0	$4.425\ .\ 10^{-10}$	1	1	$3.392\ .\ 10^{-9}$
10			$3.251\ .\ 10^{-14}$			$2.634\ .\ 10^{-13}$

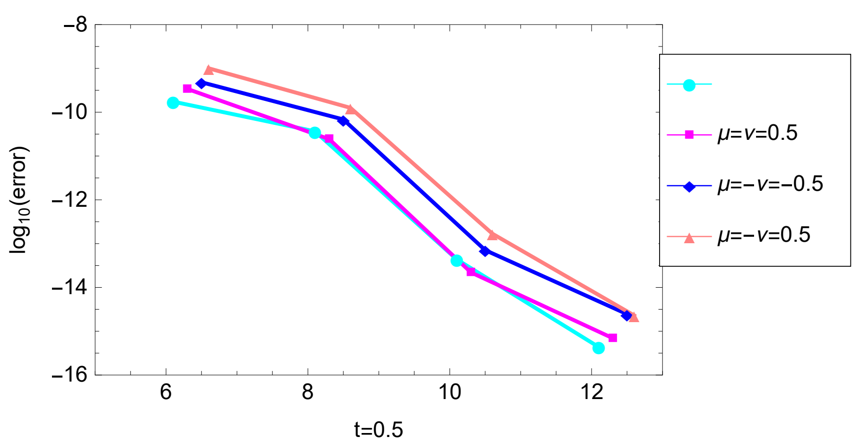

Figure 1. LogError Plot of Example 2 at $t = 0.5$

Example 3. [27] Consider the following telegraph equation

$$\partial_{tt}v(x,t) + 2\alpha\,\partial_t v(x,t) + \beta^2\,v(x,t) = \partial_{xx}v(x,t) + f(x,t), \quad (x,t) \in [0,1]\times[0,1],\ \alpha > \beta \geq 0, \tag{5.1}$$

with the following initial and boundary conditions:

$$\begin{aligned} &v(x,0) = \sin x, && \partial_t v(x,0) = 0, && x \in [0,1],\\ &v(0,t) = 0, && v(1,t) = \sin 1\,\cos t, && t \in [0,1], \end{aligned}$$

and $f(x,t)$ is selected to be compatible with the exact solution $v(x,t) = \cos t\,\sin x$.

Tables 2 and 3 display, respectively, a comparison between the maximum absolute errors (L_∞) for $N = 8$ and $N = 16$, obtained by our method (SJGM) and the shifted Jacobi collocation method presented in [27] for various choices of the parameters (μ, ν). Furthermore, Table 4 displays a comparison between the maximum absolute errors (L_∞) for $N = 16$, obtained by our method (SJGM) and the Modified Cubic B-Spline Based Differential Quadrature Method (MCB-DQM) presented in [29]. These tables show that our results are more accurate than those obtained by using the methods in [27] and [29].

Table 2. Comparison of MAE for Example 3, $N = 8$, $\alpha = 20$, $\beta = 10$

	Method in [27]			Our method				
	$(-\frac{1}{2},-\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$(0,0)$	$(-\frac{1}{2},-\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$(0,0)$	$(-\frac{1}{2},\frac{1}{2})$	$(-\frac{1}{2},-\frac{1}{2})$
L_∞	1.8×10^{-11}	4.5×10^{-10}	4.1×10^{-10}	2.2×10^{-12}	8.1×10^{-12}	6.4×10^{-12}	3.5×10^{-12}	4.8×10^{-12}

Table 3. Comparison of MAE for Example 3, $N = 16$, $\alpha = 10$, $\beta = 5$

	Method in [27]			Our method				
	$(-\frac{1}{2},-\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$(0,0)$	$(-\frac{1}{2},-\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$(0,0)$	$(-\frac{1}{2},\frac{1}{2})$	$(-\frac{1}{2},-\frac{1}{2})$
L_∞	1.6×10^{-16}	5.0×10^{-15}	1.7×10^{-15}	3.7×10^{-16}	4.5×10^{-16}	2.8×10^{-16}	1.4×10^{-16}	5.7×10^{-16}

Table 4. Comparison of MAE for Example 3, $N = 16$, $\alpha = 10$, $\beta = 5$ and $t = 0.5$

		Our method				
	$MCB - DQM$ in [29]	$\mu = \nu = -\frac{1}{2}$	$\mu = \nu = \frac{1}{2}$	$\mu = \nu = 0$	$\mu = -\frac{1}{2}, \nu = \frac{1}{2}$	$\mu = \frac{1}{2}, \nu = -\frac{1}{2}$
L_∞	4.0346×10^{-5}	2.3×10^{-16}	8.6×10^{-16}	4.6×10^{-16}	8.4×10^{-16}	7.2×10^{-16}

6 Conclusions

In this paper, we presented an algorithm for treating the hyperbolic telegraph equation based on the application of the spectral Galerkin algorithm. This method is built on the construction of the shifted Jacobi basis functions, which satisfy the boundary conditions of the differential equation. The suggested algorithm reduces the telegraph equation governed by its initial and boundary conditions into a linear algebraic system which can be solved efficiently through a suitable numerical algorithm. Of the advantages of the presented algorithm is that we can obtain various solutions due to the presence of two parameters. In addition, it generalizes some of the previously presented algorithms. The convergence and error analysis of the suggested expansion were carefully investigated. Some numerical results are included to demonstrate the applicability and efficiency of the proposed algorithm.

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