Retracted paper

Recently, we received the below email from the referee

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This is to inform you that I revised again the article titled.

A quasistatic frictional contact problem for thermo-electro-viscoelastic materials

because I knew something was wrong and made a big mistake. It concerns the variational formulation of the problem, which can never be provided.

In reality, for the variational formulation, I followed the wrong paper and did not know how I could make such an error.

Unfortunately, even though all the proofs of the primary existence and uniqueness result are now correct, the problem considered in this paper is senseless since it is in such a state that it is nearly impossible to provide its variational formulation. I am so sorry; I know that the author is disappointed. I emailed the author my findings.

Please accept my apologies.

The referee

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A QUASISTATIC FRICTIONAL CONTACT PROBLEM FOR THERMO-ELECTRO-VISCOELASTIC MATERIALS

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Abstract We consider a mathematical model which describes the quasis-static contact process between a piezoelectric body and a thermally-electrically conductive foundation. The behavior of the material is modeled with a nonlinear thermo-electro-viscoelastic constitutive law. The contact is modeled with normal compliance, unilateral constraint, memory term, Coulomb's law of dry friction, and a regularized electrical condition with thermal conductivity. We present the classical formulation of the problem, list the assumptions on the data and derive a variational formulation of the model. Then we prove the unique weak solvability of the problem. The proof is based on arguments of evolutionary quasivariational inequalities, a classical result on first order evolution equations and fixed point.

1 Introduction

Thermo-piezoelectric materials have attracted considerable attention because of their widespread use in industrial applications in various fields including the electronics industry, nuclear industry, smart structures, microelectromechanical systems, biomedical devices and super conducting devices, due to the intrinsic coupling effects that take place among thermal, mechanical and electrical fields. The theory of thermo-piezoelectricity was first proposed by Mindlin [14]. He also developed the governing equations of a three-dimensional linear thermo-piezoelectric medium (see, e.g. [13]). The physical laws for the thermo-piezoelectric materials have been explored by Nowacki (see, e.g [17, 16]). Chandrasekharaiah [8] has generalized Mindlins theory of thermo-piezoelectricity to account for the finite speed of propagation of thermal disturbances.

When a piezoelectric material is subjected to a mechanical load, it generates an electric charge. This effect is usually called the "direct piezoelectric effect". Conversely, when a piezoelectric material is stressed electrically by a voltage, its dimensions change. This phenomenon is known as the "inverse piezoelectric effect". Thermo-piezoelectric materials, on the other hand, can produce electric and mechanical fields when they are heated. The coupling properties among thermal, electric and mechanical fields make piezoelectric materials suitable.

General models for thermo-electro-elastic materials can be found in [1, 17, 22]. Static frictional contact problems for thermo-piezoelectric materials were studied in [3, 4]. Recent results on frictional contact in thermo-electro-viscoelasticity and thermo-electro-viscoplasticity can be found in [10, 11].

With respect to the papers mentioned in the previous paragraph, the current paper has two novelties that we describe in what follows. First, we model the behavior of the material with a nonlinear thermo-electro-viscoelastic constitutive law. Second, the model we consider involves Coulomb's law of dry friction and a version of contact conditions with normal compliance, unilateral constraint and memory effect. This condition takes into account the deformability, the rigidity, the memory effects and the thermally-electrically conductivity of the foundation. The contact condition with normal compliance and unilateral constraint can be found in a number of recent papers, including [5, 6, 18, 21]. The model considered in [5] was frictional; there, the material's behavior was described with a linear elastic constitutive law and the friction was modeled with a slip-dependent version of Coulomb's law. The mathematical model considered in [6] was frictional; there, the elasticity operator was assumed to be nonlinear and the friction law was able to describe the relationship between the Coulomb and the Tresca conditions, and points out to a possible transition from the first to the second one. The model considered in [21] was viscoelastic and the normal compliance function in the contact condition was assumed to be multivalued. In contrast, the model considered in [18] was viscoplastic, with internal state variable; there, the contact was described with normal compliance, finite penetration and memory term.

The rest of the paper is structured as follows. In Section 2 we state the model of a thermo-electro-viscoelastic body in frictional contact with a conductive foundation. We introduce notation and assumptions on the problem's data, derive the variational formulation of the problem and give main results (existence and uniqueness). Section 3 is devoted to the proofs of main results. More precisely, we prove the existence of a weak solution of the model and its uniqueness under additional assumptions.

2 Problem's formulation and main result

2.1 The classical formulation

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a domain occupied by a viscoelastic-piezoelectric body with outer surface $\Gamma = \partial\Omega$, assumed to be sufficiently smooth and decomposed into three disjoint measurable parts Γ_1, Γ_2, and Γ_3, on the one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two disjoint measurable parts Γ_a and Γ_b on the other hand, such that $meas(\Gamma_1) > 0$ and $meas(\Gamma_a) > 0$. This body is supposed to be stress free and at a free temperature. Here the temperature variations, accompanying the deformations, produce changes in the material parameters which are considered as depending on temperature. Let us denote by $[0, T]$, $T > 0$ the time interval of interest. The body is clamped on Γ_1. A surface traction of density f_2 act on Γ_2. The body is submitted to the action of body forces of density f_0 and a volume electric charges of density q_0. We also assume that the electrical potential vanishes on $\Gamma_a \times (0, T)$, a surface electric charge of density q_2 is prescribed on Γ_b and the temperature is assumed to be zero on $\Gamma_1 \cup \Gamma_2$. Moreover, the body is subjected to a volume heat source q_{th} and it comes on Γ_3 in contact with an electrically and thermally conductive obstacle, the so-called foundation.

Let us recall now some classical notations, see e.g. [9, 15] for further details. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ (d = 2,3), while "." and $\|.\|$ represent the inner product and the Euclidean norm on $\mathbb{S}^d$and $\mathbb{R}^d$, respectively. We define the inner product and the Euclidean norm on $\mathbb{S}^d$ and $\mathbb{R}^d$ respectively, by

$$u \cdot v = u_i v_i, \quad \|u\| = (u \cdot u)^{1/2} \ \forall u, v \in \mathbb{R}^d,$$

$$\sigma \cdot \tau = \sigma_{ij}\tau_{ij}, \quad \|\sigma\| = (\sigma \cdot \sigma)^{1/2} \ \forall \sigma, \tau \in \mathbb{S}^d.$$

Since the boundary Γ is sufficiently regular, the unit outward normal field ν on Γ is defined. Then the normal and the tangential components of displacement vector and stress on the boundary are

$$v_\nu = v \cdot \nu, \quad v_\tau = v - v_\nu \nu,$$

$$\sigma_\nu = \sigma\nu \cdot \nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \nu.$$

Here and below, the indices i and j run from 1 to d, the summation convention over repeated indices is used and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable. We denote by $x \in \Omega \cup \Gamma$ and $t \in [0, T]$, the spatial and the time variable, respectively, and, to simplify the notation, we do not indicate in what follows the dependence of various functions on x and t. The dots above variable represent the time derivatives. Moreover, we denote by $\mathrm{Div}\,\sigma = (\sigma_{ij,j})$, $\mathrm{div}\,D = (D_{i,i})$ the divergence operator for tensor and vector valued functions, respectively.

The governing equations of thermo-piezoelectricity consist of the equilibrium equation, constitutive relations, strain-mechanical displacement and electric potential field relations.

The linearized strain tensor and potential field relations are given by

$$\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T), \ \ \varepsilon(u) = (\varepsilon_{ij}(u)) \ \text{ in } \ \Omega \times (0, T),$$

$$E(\varphi) = -\nabla\varphi = -(\varphi_{,i}), \ \ E(\varphi) = (E_i(\varphi)) \ \text{ in } \ \Omega \times (0, T),$$

where $u = (u_i)$ and φ are respectively, the displacement field and electric potential.
We suppose that the process is mechanically quasistatic and electrically static.
The equations of stress equilibrium and the equation of electric displacement field are, respectively, given by

$$\mathrm{Div}\,\sigma + f_0 = 0 \quad \text{in } \Omega \times (0,T), \tag{2.1}$$

$$\mathrm{div}\,D - q_0 = 0 \quad \text{in } \Omega \times (0,T), \tag{2.2}$$

where $\sigma = (\sigma_{ij})$ and $D = (D_i)$ represent the stress tensor and the electric displacement field, respectively.
The thermo-electro-viscoelastic constitutive law can be written as

$$\sigma = \mathcal{A}\varepsilon(\dot{u}) + \mathcal{B}\varepsilon(u) - \mathcal{E}^* E(\varphi) - \mathcal{M}\theta \quad \text{in } \Omega \times (0,T), \tag{2.3}$$

$$D = \mathcal{E}\varepsilon(u) + \mathcal{C}E(\varphi) + \mathcal{P}\theta \quad \text{in } \Omega \times (0,T), \tag{2.4}$$

$$\dot{\theta} - \mathrm{div}(\mathcal{K}\nabla\theta) = -\mathcal{M}.\nabla\dot{u} + q_{th} \quad \text{in } \Omega \times (0,T), \tag{2.5}$$

where $\mathcal{A}$ and $\mathcal{B}$ are the viscosity and elasticity operators, respectively, $\mathcal{E} = (e_{ijk})$ represents the third-order piezoelectric tensor, $\mathcal{E}^* = (e^*_{ijk}) = (e_{kij})$ is its transpose, $\mathcal{M} = (m_{ij})$, $\mathcal{C} = (c_{ij})$ and $\mathcal{P} = (p_i)$ denote the thermal expansion, the electric permittivity and the pyroelectric tensor, respectively. The differential equation (2.5) describes the evolution of the temperature field θ, where $\mathcal{K} = (k_{ij})$ represents the thermal conductivity tensor, q_{th} the density of volume heat sources.
Next, to complete the mechanical model according to the description of the physical setting, we have the following boundary conditions

$$u = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{2.6}$$

$$\sigma\nu = f_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{2.7}$$

$$\theta = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2 \times (0,T), \tag{2.8}$$

$$\varphi = 0 \quad \text{on } \Gamma_a \times (0,T), \tag{2.9}$$

$$D \cdot \nu = q_2 \quad \text{on } \Gamma_b \times (0,T). \tag{2.10}$$

We model the frictional contact on the contact surface Γ_3 with Coulomb's law of dry friction, a condition involving normal compliance, unilateral constraint and memory term and with regularized electrical and thermal conditions given by

$$\begin{aligned} &\|\sigma_\tau\| \le p_\tau(u_\nu), \\ &\sigma_\tau = -p_\tau(u_\nu)\tfrac{\dot{u}_\tau}{\|\dot{u}_\tau\|} \text{ if } \dot{u}_\tau \neq 0 \text{ on } \Gamma_3 \times (0,T), \end{aligned} \tag{2.11}$$

there exists $\xi : \Gamma_3 \times (0,T) \to \mathbb{R}$ which satisfies

$$\begin{cases} u_\nu(t) \le g\,, \quad \sigma_\nu(t) + \big(h_\nu(\varphi(t) - \varphi_F) + k_\nu(\theta(t) - \theta_F(t))\big)p_\nu(u_\nu(t)) + \xi(t) \le 0, \\ (u_\nu(t) - g)\Big(\sigma_\nu(t) + \big(h_\nu(\varphi(t) - \varphi_F) + k_\nu(\theta(t) - \theta_F(t))\big)p_\nu(u_\nu(t)) + \xi(t)\Big) = 0, \\ 0 \le \xi(t) \le \int_0^t b(t-s)u_\nu^+(s)ds, \\ \xi(t) = 0 \text{ if } u_\nu(t) < 0, \\ \xi(t) = \int_0^t b(t-s)u_\nu^+(s)ds \text{ if } u_\nu(t) > 0, \end{cases} \tag{2.12}$$

$$D \cdot \nu = h_e(u_\nu)\phi(\varphi - \varphi_F) \quad \text{on } \Gamma_3 \times (0,T), \tag{2.13}$$

$$-k_{ij}\tfrac{\partial\theta}{\partial x_i}\nu_j = k_{th}(\theta - \theta_F) - k_\tau(\|\dot{u}_\tau\|) \quad \text{on } \Gamma_3 \times (0,T). \tag{2.14}$$

The condition (2.11) represents the Coulomb's law of dry friction with friction bound $p_\tau(u_\nu - g)$.
We now describe the contact condition (2.12) in which our main interest lies, it incorporates a

version of the contact boundary conditions with unilateral constraint, in which the memory effects of the thermally-electrically conductive foundation are taken into account. Here u_ν is the normal displacement and, therefore, $u_\nu^+ = \max\{0, u\}$ being the penetration of the body's surface asperities and those of the foundation. Moreover, b is a positive function, h_ν and k_ν are prescribed functions which represent the stiffness coefficients and p_ν is a Lipschitz continuous increasing function. The physical meaning of this type of contact condition was derived in [21]. The condition shows that when there is a penetration the contact follows a normal compliance condition with memory term up to limit g and once the limit is reached the contact follows a Signorini type unilateral condition. The normal stress vanishes in cases where there is a separation between the body and the foundation.
The boundary conditions (2.13)-(2.14) describe respectively the electrical and the heat exchange conditions on the contact surface Γ_3 in which, as usual, φ_F and θ_F denote the electric potential and the temperature of the foundation respectively. First, the equation (2.13) represents the regularization of the electrical contact condition on Γ_3(for more details see [12]). The relation (2.14) represents the heat flux condition where k_{th} is the coefficient of heat exchange between the body and the obstacle. k_τ is a given function assumed to depend on the tangential pressure. The function ϕ, used in (2.13), is a real valued function.
The initial displacement and the initial temperature are given by

$$u(0) = u_0, \quad \theta(0) = \theta_0, \quad \text{in } \Omega. \tag{2.15}$$

Problem P: Find a displacement field $u : \Omega \times [0,T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0,T] \to \mathbb{S}^d$, an electric potential $\varphi : \Omega \times [0,T] \to \mathbb{R}$, an electric displacement field $D : \Omega \times [0,T] \to \mathbb{R}^d$ and a temperature $\theta : \Omega \times [0,T] \to \mathbb{R}_+$ such that (2.1)–(2.15) hold.

2.2 Preliminaries

We use standard notation for the L^p and the Sobolev spaces associated with Ω and Γ and, for a function $\zeta \in H^1(\Omega)$ we still write ζ to denote its trace on Γ. We recall that the summation convention applies to a repeated index.
For the electric displacement field we use the following space

$$\mathcal{W}_1 = \{\, D \in L^2(\Omega)^d : \operatorname{div} D \in L^2(\Omega)\},$$

endowed with the inner product

$$(D, E)_{\mathcal{W}_1} = (D, E)_{L^2(\Omega)^d} + (\operatorname{div} D, \operatorname{div} E)_{L^2(\Omega)},$$

and the associated norm $\|\cdot\|_{\mathcal{W}_1}$.
The electric potential field is to be found in

$$W = \{\zeta \in H^1(\Omega) : \zeta = 0 \text{ a.e. on } \Gamma_a\}.$$

Since $\operatorname{meas}\Gamma_a > 0$, the Friedrichs-Poincaré inequality holds, thus,

$$\|\nabla\zeta\|_{L^2(\Omega)^d} \geq C_F\,\|\zeta\|_W \quad \forall\,\zeta \in W, \tag{2.16}$$

where $C_F > 0$ is a constant which depends only on Ω and Γ_a. On W, we use the inner product

$$(\varphi, \zeta)_W = (\nabla\varphi, \nabla\zeta)_{L^2(\Omega)^d},$$

and the associated norm

$$\|\zeta\|_W = \|\nabla\zeta\|_{L^2(\Omega)^d} \quad \forall\,\zeta \in W. \tag{2.17}$$

It follows from (2.16) that $\|\cdot\|_{H^1(\Omega)}$ and $\|\cdot\|_W$ are equivalent norms on W and therefore $(W, \|\cdot\|_W)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant C_0, depending only on Ω, Γ_a and Γ_3, such that

$$\|\zeta\|_{L^2(\Gamma_3)} \leq C_0\|\zeta\|_W \quad \forall\,\zeta \in W. \tag{2.18}$$

We recall that when $D \in \mathcal{W}_1$ is a sufficiently regular function, the following Green formula holds:

$$(D, \nabla\zeta)_{L^2(\Omega)^d} + (\operatorname{div} D, \zeta)_{L^2(\Omega)} = \int_\Gamma D \cdot \nu\, \zeta\, da \quad \forall \zeta \in H^1(\Omega). \tag{2.19}$$

We introduce the real Hilbert space of the temperature denoted by

$$Q = \{\mu \in H^1(\Omega) : \mu = 0 \text{ on } \Gamma_1 \cup \Gamma_2\},$$

and we consider the inner product and the corresponding norm given by

$$(\theta, \mu)_Q = (\theta, \mu)_{H^1(\Omega)}, \;\; \|\mu\|_Q = \|\mu\|_{H^1(\Omega)} \;\; \forall \theta, \eta \in Q.$$

By Sobolev's trace theorem, there exists a constant $C_1 > 0$ which depends only on Ω and Γ such that

$$\|\mu\|_{L^2(\Gamma_3)} \leq C_1 \|\mu\|_Q \quad \forall \mu \in Q. \tag{2.20}$$

The following Friedrichs-Poincaré inequality holds on Q is

$$\|\nabla\mu\|_{L^2(\Omega)^d} \geq \tilde{C}_F \,\|\mu\|_Q \quad \forall \mu \in Q. \tag{2.21}$$

$L^2(\Omega)$ is identified with its dual and with a subspace of the dual Q' of Q, i.e., $Q \subset L^2(\Omega) \subset Q'$, and we say that the inclusions above define a Gelfand triple. The notation $\langle ., . \rangle_{Q',Q}$ represents the duality pairing between Q' and Q.
For the stress and strain variables, we use the real Hilbert spaces

$$H = L^2(\Omega)^d = \{u = (u_i) : u_i \in L^2(\Omega)\},$$
$$\mathcal{H} = \{\sigma = (\sigma_{ij}) : \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\},$$
$$H_1 = \{u = (u_i) : \varepsilon(u) \in \mathcal{H}\},$$
$$\mathcal{H}_1 = \{\sigma \in \mathcal{H} : \text{Div}(\sigma) \in H\}.$$

The spaces H, H_1, $\mathcal{H}$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products:

$$(u, v)_H = \int_\Omega u.v dx, \quad \forall u, v \in H,$$
$$(u, v)_{H_1} = \int_\Omega u.v dx + \int_\Omega \nabla u. \nabla v dx \quad \forall u, v \in H_1,$$

where

$$(\sigma, \tau)_{\mathcal{H}} = \int_\Omega \sigma.\tau dx \quad \forall \sigma, \tau \in \mathcal{H},$$
$$(\sigma, \tau)_{\mathcal{H}_1} = (\sigma, \tau)_{\mathcal{H}} + (\text{Div}(\sigma), \text{Div}(\tau))_H, \quad \forall \sigma, \tau \in \mathcal{H}_1.$$

The associated norms on the spaces H, H_1, $\mathcal{H}$ and $\mathcal{H}_1$ are denoted by $\|\cdot\|_H$, $\|\cdot\|_{H_1}$, $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}_1}$ respectively. Let $H_\Gamma = H^{1/2}(\Gamma)^d$ and $\gamma : H^1(\Omega)^d \to H_\Gamma$ be the trace map. For every element $v \in H_1$, we also use the notation v to denote the trace γv of v on Γ. For every $\sigma \in \mathcal{H}_1$ there exists an element $\sigma\nu \in H'_\Gamma$ satisfying the following Green formula

$$\langle \sigma\nu, \gamma v \rangle = (\sigma, \varepsilon(v))_{\mathcal{H}} + (Div(\sigma), v)_H \quad \forall v \in H^1. \tag{2.22}$$

Moreover, if Γ is continuously differentiable on Ω, then

$$\langle \sigma\nu, \gamma v \rangle = \int_\Gamma \sigma\nu \cdot v da \quad \forall v \in H_1, \tag{2.23}$$

where da is the surface element.
Let us now consider the closed subspace V defined by

$$V = \{v \in H_1 : v = 0 \text{ on } \Gamma_1\}. \tag{2.24}$$

Then, we consider the following closed convex subspace U of V given by

$$U = \{v \in V : \dot{v}_\nu - g \leq 0 \text{ on } \Gamma_3\}. \tag{2.25}$$

Since $\text{meas}(\Gamma_1) > 0$, Korn's inequality holds and thus, there exists a positive constant C_K depending only on Ω, Γ_1 such that

$$\|\varepsilon(v)\|_{\mathcal{H}} \geq C_K \|v\|_{H_1} \quad \forall v \in V.$$

On the space V we consider the inner product given by

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}},$$

and let $\|\cdot\|_V$ be the associated norm, defined by

$$\|v\|_V = \|\varepsilon(v)\|_{\mathcal{H}}. \tag{2.26}$$

It follows from Korn's inequality that $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent norms on V . Therefore $(V, \|\cdot\|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem there exists a positive constant $\tilde{C}_0$ which depends only on Ω, Γ_1 and Γ_3 such that

$$\|v\|_{L^2(\Gamma_3)^d} \leq \tilde{C}_0 \|v\|_V \; \forall v \in V. \tag{2.27}$$

Finally, for a real Banach space $(X, \|\cdot\|_X)$ we use the usual notation for the spaces $L^p(0, T; X)$ and $W^{k,p}(0, T; X)$ where $1 \leq p \leq \infty$, $k = 1, 2, \ldots$; we also denote by $C([0, T]; X)$ and $C^1([0, T]; X)$ the spaces of continuous and continuously differentiable functions on $[0, T]$ with values in X, with the respective norms

$$\|x\|_{C([0,T];X)} = \max_{t \in [0,T]} \|x(t)\|_X,$$

$$\|x\|_{C^1([0,T];X)} = \max_{t \in [0,T]} \|x(t)\|_X + \max_{t \in [0,T]} \|\dot{x}(t)\|_X.$$

Recall that the dot represents the time derivative.
We end this section by giving an existence, uniqueness and regularity result which was proved in [20, p.75-77]. Let X be a real Hilbert space with the inner product $(\cdot, \cdot)_X$ and the associated norm $\|\cdot\|_X$, and consider the problem of finding $u : [0, T] \to X$ such that

$$\begin{aligned} &(A\dot{u}(t), v - \dot{u}(t))_X + (Bu(t), v - \dot{u}(t))_X + j(u(t), v) - j(u(t), \dot{u}(t)) \\ &\geq (f(t), v - \dot{u}(t))_X \quad \forall\, v \in X,\; t \in [0, T], \end{aligned} \tag{2.28}$$

$$u(0) = u_0. \tag{2.29}$$

To study problem (2.28) and (2.29) we need the following assumptions:
The operator $A : X \to X$ is strongly monotone and Lipschitz continuous, i.e.

$$\begin{cases} \text{(a) There exists } M_A > 0 \text{ such that} \\ \qquad (Au_1 - Au_2, u_1 - u_2)_X \geq M_A \|u_1 - u_2\|_X^2 \quad \forall\, u_1, u_2 \in X. \\ \text{(b) There exists } L_A > 0 \text{ such that} \\ \qquad \|Au_1 - Au_2\|_X \leq L_A \|u_1 - u_2\|_X \quad \forall\, u_1, u_2 \in X. \end{cases} \tag{2.30}$$

The operator $B : X \to X$ is Lipschitz continuous, i.e., there exists $L_B > 0$ such that

$$\|Bu_1 - Bu_2\|_X \leq L_B \|u_1 - u_2\|_X \quad \forall\, u_1, u_2 \in X. \tag{2.31}$$

The functional $j : X \times X \to \mathbb{R}$ satisfies:

$$\begin{cases} \text{(a) } j(u, \cdot) \text{ is convex and l.s.c. on } X \text{ for all } u \in X. \\ \text{(b) There exists } m > 0 \text{ such that} \\ \qquad j(u_1, v_2) - j(u_1, v_1) + j(u_2, v_1) - j(u_2, v_2) \\ \qquad\quad \leq m \|u_1 - u_2\|_X \|v_1 - v_2\|_X \quad \forall\, u_1, u_2, v_1, v_2 \in X. \end{cases} \tag{2.32}$$

Finally, we assume that

$$f \in C([0, T]; X), \tag{2.33}$$

and

$$u_0 \in X. \tag{2.34}$$

Theorem 2.1. *Let* (2.30)-(2.34) *hold. Then, if* $M_A > m$, *there exists a unique solution* $u \in C^1([0,T];X)$ *of problem* (2.28) *and* (2.29).

2.3 Weak formulation and main result

In the study of the mechanical problem (2.1)–(2.15), we need to assume that the viscosity operator $\mathcal{A}$, the elasticity operator $\mathcal{B}$, the piezoelectric operator $\mathcal{E}$, the electric permittivity operator $\mathcal{C}$, the functions p_r, k_r (for $r = \nu, \tau$), h_ν and h_e satisfy the following conditions

$$\left\{\begin{array}{ll} \text{(a)} & \mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\ \text{(b)} & \text{There exists } L_{\mathcal{A}} > 0 \text{ such that} \\ & \|\mathcal{A}(x,\varepsilon_1) - \mathcal{A}(x,\varepsilon_2)\| \le L_{\mathcal{A}}\|\varepsilon_1 - \varepsilon_2\| \\ & \text{for all } \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \text{ a.e. } x \in \Omega. \\ \text{(c)} & \text{There exists } m_{\mathcal{A}} > 0 \text{ such that} \\ & (\mathcal{A}(x,\varepsilon_1) - \mathcal{A}(x,\varepsilon_2)).(\varepsilon_1 - \varepsilon_2) \ge m_{\mathcal{A}}\|\varepsilon_1 - \varepsilon_2\|^2, \\ & \text{for any } \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d \text{ a.e } x \in \Omega. \\ \text{(d)} & \text{The mapping } x \mapsto \mathcal{A}(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega \\ & \text{for any } \varepsilon \in \mathbb{S}^d. \\ \text{(e)} & \text{The mapping } x \mapsto \mathcal{A}(x,0) \in \mathcal{H}. \end{array}\right. \tag{2.35}$$

$$\left\{\begin{array}{ll} \text{(a)} & \mathcal{B} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\ \text{(b)} & \text{There exists } L_{\mathcal{B}} > 0 \text{ such that} \\ & \|\mathcal{B}(x,\varepsilon_1) - \mathcal{B}(x,\varepsilon_2)\| \le L_{\mathcal{B}}\|\varepsilon_1 - \varepsilon_2\| \\ & \text{for any } \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \text{ a.e. } x \in \Omega \\ \text{(c)} & \text{The mapping } x \mapsto \mathcal{B}(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega \\ & \text{for any } \varepsilon \in \mathbb{S}^d \\ \text{(d)} & \text{The mapping } x \mapsto \mathcal{B}(x,0) \in \mathcal{H}. \end{array}\right. \tag{2.36}$$

$$\left\{\begin{array}{ll} \text{(a)} & \mathcal{E} : \Omega \times S^d \to \mathbb{R}^d. \\ \text{(b)} & \mathcal{E}(x,\tau) = (e_{ijk}(x)\tau_{jk}), \ \forall \tau = (\tau_{ij}) \in \mathbb{S}^d, \ \text{ a.e. } x \in \Omega, \\ \text{(c)} & (e_{ijk}) = (e_{ikj}) \in L^\infty(\Omega), \ 1 \le i,j,k \le d, \\ \text{(d)} & \mathcal{E}\sigma.v = \sigma.\mathcal{E}^* v, \ \ \forall \sigma \in \mathbb{S}^d, \ v \in \mathbb{R}^d. \end{array}\right. \tag{2.37}$$

$$\left\{\begin{array}{ll} \text{(a)} & \mathcal{C} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d, \\ \text{(b)} & \mathcal{C}(x,E) = (c_{ij}(x)E_j) \ \ \forall E = (E_i) \in \mathbb{R}^d, \text{ a.e. } x \in \Omega. \\ \text{(c)} & c_{ij} = c_{ji} \in L^\infty(\Omega), \ 1 \le i,j \le d. \\ \text{(d)} & \text{There exists } m_{\mathcal{C}} > 0 \text{ such that} \\ & c_{ij}(x)E_i.E_j \ge m_{\mathcal{C}}\|E\|^2 \ \forall E = (E_i) \in \mathbb{R}^d \ \text{ a.e. } x \in \Omega. \end{array}\right. \tag{2.38}$$

$$\left\{\begin{array}{ll} \text{(a)} & \pi : \Gamma_3 \times \mathbb{R} \to \mathbb{R}, \text{ for } \pi = h_\nu \text{ or } \pi = k_\nu. \\ \text{(b)} & \text{There exists } L_\pi > 0 \text{ such that} \\ & |\pi(x,u_1) - \pi(x,u_2)| \le L_\pi|u_1 - u_2| \\ & \text{for any } u_1, u_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3 \\ \text{(c)} & \text{There exists } M_\pi > 0 \text{ such that } |\pi(x,u)| \le M_\pi \\ & \text{for any } u \in \mathbb{R}, \text{ a.e } x \in \Gamma_3. \\ \text{(d)} & \text{The mapping } x \mapsto \pi(x,u) \text{ is Lebesgue measurable on } \Gamma_3, \\ & \text{for any } u \in \mathbb{R} \\ \text{(e)} & \text{The mapping } x \mapsto \pi(x,u) = 0 \text{ for all } u \le 0, \text{ a.e } x \in \Gamma_3. \end{array}\right. \tag{2.39}$$

$$\begin{cases} \text{(a)} & h_e : \Gamma_3 \times \mathbb{R} \to \mathbb{R}. \\ \text{(b)} & \text{There exists } L_{h_e} > 0 \text{ such that} \\ & |h_e(x,u_1) - h_e(x,u_2)| \le L_{h_e}|u_1 - u_2| \\ & \forall\, u_1, u_2 \ \in \mathbb{R},\ \text{a.e. } x \in \Gamma_3. \\ \text{(c)} & \text{There exists } M_{h_e} > 0 \text{ such that } 0 \le h_e(x,u) \le M_{h_e} \\ & \text{ for any } u \in \mathbb{R},\ \text{a.e } x \in \Gamma_3. \\ \text{(d)} & x \mapsto h_e(x,u) \text{ is measurable on } \Gamma_3,\ \text{for all } u \in \mathbb{R}. \\ \text{(e)} & x \mapsto h_e(x,u) = 0,\ \text{for all } u \le 0,\ \text{a.e } x \in \Gamma_3. \end{cases} \tag{2.40}$$

$$\begin{cases} \text{(a)} & p_r(r = \nu, \tau) : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+. \\ \text{(b)} & \text{There exists } L_r > 0 \text{ such that} \\ & |p_r(x,u_1) - p_r(x,u_2)| \le L_r|u_1 - u_2| \\ & \forall\, u_1, u_2 \ \in \mathbb{R},\ \text{a.e. } x \in \Gamma_3. \\ \text{(c)} & \text{There exists } M_{p_r} > 0 \text{ such that } 0 \le p_r(x,u) \le M_{p_r} \\ & \text{ for any } u \in \mathbb{R},\ \text{a.e } x \in \Gamma_3. \\ \text{(d)} & x \mapsto p_r(x,u) \text{ is measurable on } \Gamma_3,\ \text{for all } u \in \mathbb{R}. \\ \text{(e)} & x \mapsto p_r(x,u) = 0,\ \text{for all } u \le 0,\ \text{a.e } x \in \Gamma_3. \end{cases} \tag{2.41}$$

$$\begin{cases} \text{(a)} & k_\tau : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+ \\ \text{(b)} & \text{There exists } M_\tau > 0 \text{ such that} \\ & |k_\tau(x,r_1) - k_\tau(x,r_2)| \le M_\tau\,|r_1 - r_2| \\ & \text{for all } r_1,\ r_2 \in \mathbb{R}_+,\ \text{a.e. } x \in \Gamma_3. \\ \text{(c)} & \text{The mapping } x \mapsto k_\tau(x,r) \in L^2(\Gamma_3) \text{ is measurable on } \ \Gamma_3\ \forall\, r \in \mathbb{R}_+. \end{cases} \tag{2.42}$$

The thermal expansion tensor $\mathcal{M} = (m_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, the pyroelectric tensor $\mathcal{P} = (p_i) : \Omega \to \mathbb{R}^d$ and the thermal tensors $\mathcal{K} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfy

$$\mathcal{M} = (m_{ij}),\ m_{ij} = m_{ji} \in L^\infty(\Omega),\ \ \mathcal{P} = (p_i) \in L^\infty(\Omega). \tag{2.43}$$

$$\mathcal{K} = (k_{ij}),\ k_{ij} = k_{ji} \in L^\infty(\Omega). \tag{2.44}$$

$$\exists c_k > 0 \text{ such that } k_{ij}\xi_i\xi_j \ge c_k\xi_i\xi_j,\ \forall \xi = (\xi_i) \in \mathbb{R}^d. \tag{2.45}$$

The forces, tractions, volume and surface free charge densities and the heat sources density have the regularity

$$f_0 \in C([0,T]; L^2(\Omega)^d), \quad f_2 \in C([0,T]; L^2(\Gamma_2)^d), \tag{2.46}$$

$$q_0 \in C([0,T]; L^2(\Omega)), \quad q_2 \in C([0,T]; L^2(\Gamma_b)), \tag{2.47}$$

$$q_{th} \in C(0,T; L^2(\Omega)). \tag{2.48}$$

The real valued function ϕ satisfies

$$\begin{cases} \text{(a)} & \phi : \Gamma_3 \times \mathbb{R} \to \mathbb{R}. \\ \text{(b)} & \text{There exists } l_e > 0 \text{ such that} \\ & |\phi(x,\varphi_1) - \phi(x,\varphi_2)| \le l_e|\varphi_1 - \varphi_2| \\ & \forall\, \varphi_1, \varphi_2 \ \in \mathbb{R},\ \text{a.e. } x \in \Gamma_3. \\ \text{(c)} & \text{There exists } \bar{l}_e > 0 \text{ such that } |\phi(x,\varphi)| \le \bar{l}_e \\ & \text{ for any } \varphi \in \mathbb{R},\ \text{a.e } x \in \Gamma_3. \\ \text{(d)} & \text{The mapping } x \mapsto \phi(x,\varphi) \text{ is measurable on } \Gamma_3,\ \text{for all } \varphi \in \mathbb{R}. \end{cases} \tag{2.49}$$

The boundary potential and thermic data satisfy

$$\varphi_F \in L^2(\Gamma_3),\ \theta_F \in L^2([0,T]; L^2(\Gamma_3)),\ k_{th} \in L^\infty(\Omega; \mathbb{R}_+), \tag{2.50}$$

The surface memory function verify

$$b \in C([0,T]; L^\infty(\Gamma_3)), b(t,x) \geq 0 \text{ for all } t \in [0,T], \text{ a.e. } x \in \Gamma_3. \tag{2.51}$$

We assume that the initial conditions satisfy

$$u_0 \in U, \ \ \theta_0 \in Q. \tag{2.52}$$

Using Riesz's representation theorem, we define the functions $f : [0,T] \to V$, and $q : [0,T] \to W$ by

$$(f(t),v)_V = \int_\Omega f_0(t).v dx + \int_{\Gamma_2} f_2(t).v da, \tag{2.53}$$

$$(q(t),\psi)_W = \int_\Omega q_0(t)\psi dx - \int_{\Gamma_b} q_2(t)\psi da. \tag{2.54}$$

Next, we define the mappings $j : V \times V \to \mathbb{R}$, $J_e : V \times W \times W \to \mathbb{R}$, $J_{el} : W \times V \times V \to \mathbb{R}$, $J_{te} : Q \times V \times V \to \mathbb{R}$, $S : [0,T] \to \mathbb{R}$ and the functions $\mathcal{Z} : Q \to Q'$ and $\mathcal{R} : V \to Q'$, respectively, by

$$j(u,v) = \int_{\Gamma_3} p_\tau(u_\nu)\|v_\tau\| da, \tag{2.55}$$

$$J_{el}(\varphi,u,v) = \int_{\Gamma_3} h_\nu(\varphi-\varphi_F)p_\nu(u_\nu)v_\nu da, \tag{2.56}$$

$$J_{te}(\theta,u,v) = \int_{\Gamma_3} k_\nu(\theta-\theta_F)p_\nu(u_\nu)v_\nu da, \tag{2.57}$$

$$J_e(u,\varphi,\psi) = \int_{\Gamma_3} h_e(u_\nu)\phi(\varphi-\varphi_F)\psi da, \tag{2.58}$$

$$\langle S(t),\mu\rangle_{Q'\times Q} = \int_\Omega q_{th}(t)\mu dx + \int_{\Gamma_3} k_{th}\theta_F(t)\mu da, \tag{2.59}$$

$$\langle \mathcal{Z}\tau,\mu\rangle_{Q'\times Q} = \sum_{i,j=1}^{d} \int_\Omega k_{ij}\frac{\partial\tau}{\partial x_j}\frac{\partial\mu}{\partial x_i} dx + \int_{\Gamma_3} k_{th}\tau\mu da, \tag{2.60}$$

$$\langle \mathcal{R}v,\mu\rangle_{Q'\times Q} = -\int_\Omega (\mathcal{M}.\nabla v)\mu dx + \int_{\Gamma_3} k_\tau(\|v_\tau\|)\mu \, da, \tag{2.61}$$

for all $u,v \in V$, $\varphi,\psi \in W$, $\theta,\mu,\tau \in Q$ and $t \in [0,T]$.
We now turn to the variational formulation of Problem P and, to this end, we assume in what follows that $(u,\sigma,\varphi,D,\theta)$ represents a quintiple of regular functions which satisfy (2.1)-(2.15). Let $v \in U$ and $t \in (0,T)$ be given. We use the Green's formula (2.22) to see that

$$\int_\Omega \sigma(t)\varepsilon(v)dx + \int_\Omega \text{Div}\,\sigma(t).v dx = \int_\Gamma \sigma(t)\nu.v da,$$

and, combining this equality with the equilibrium equation (2.1), we find that

$$\int_\Omega \sigma(t).(\varepsilon(v)-\varepsilon(\dot{u}(t)))dx = \int_\Omega f_0(t)(v-\dot{u}(t))dx + \int_\Gamma \sigma(t)\nu.(v-\dot{u}(t))da. \tag{2.62}$$

We split the surface integral over Γ_1, Γ_2 and Γ_3 and, since $v-\dot{u}(t)$=0 a.e. on Γ_1, $\sigma(t)\nu = f_2(t)$ on Γ_2, we deduce that

$$\begin{aligned}\int_\Omega \sigma(t).(\varepsilon(v)-\varepsilon(\dot{u}(t)))dx &= \int_\Omega f_0(t)(v-\dot{u}(t))dx + \int_{\Gamma_2} f_2(t).(v-\dot{u}(t))da \\ &+ \int_{\Gamma_3} \sigma(t)\nu.(v-\dot{u}(t))da.\end{aligned} \tag{2.63}$$

Next, we turn to the integral over Γ_3. We decompose the vectors and tensors into their normal and tangential components as follows

$$\sigma(t)\nu(v-\dot{u}(t)) = \sigma_\nu(t)(v_\nu - \dot{u}_\nu(t)) + \sigma_\tau(t).(v_\tau - \dot{u}_\tau(t)).$$

We write now

$$\begin{aligned}
&\sigma_\nu(t)(v_\nu - \dot{u}_\nu(t)) \\
&= [\sigma_\nu + \big(h_\nu(\varphi(t)-\varphi_F) + k_\nu(\theta(t)-\theta_F(t))\big)p_\nu(u_\nu(t)) + \xi(t)](v_\nu - g) \\
&+ [\sigma_\nu + \big(h_\nu(\varphi(t)-\varphi_F) + k_\nu(\theta(t)-\theta_F(t))\big)p_\nu(u_\nu(t)) + \xi(t)](g - \dot{u}_\nu(t)) \\
&- [\big(h_\nu(\varphi(t)-\varphi_F) + k_\nu(\theta(t)-\theta_F(t))\big)p_\nu(u_\nu(t)) + \xi(t)](v_\nu - \dot{u}_\nu(t)) \ \text{ on } \Gamma_3,
\end{aligned}$$

then we use the contact conditions (2.12) and the definition (2.25) of the closed subspace U to see that

$$\begin{aligned}
\sigma_\nu(t)(v_\nu - \dot{u}_\nu(t)) \geq -\,[\big(h_\nu(\varphi(t)-\varphi_F) + k_\nu(\theta(t)-\theta_F(t))\big)p_\nu(u_\nu(t)) \\
+ \xi(t)](v_\nu - \dot{u}_\nu(t)) \ \text{ on } \Gamma_3.
\end{aligned} \tag{2.64}$$

We use (2.12), again, and the hypothesis (2.51) on function b to deduce that

$$\Big(\int_0^t b(t-s)u_\nu^+(s)ds\Big)(v_\nu^+ - \dot{u}_\nu^+(t)) \geq \xi(t)(v_\nu - \dot{u}_\nu(t)) \ \text{ on } \Gamma_3. \tag{2.65}$$

Then we add the inequalities (2.64) and (2.65) and integrate the result on Γ_3 we find that

$$\begin{aligned}
&\int_{\Gamma_3} \sigma_\nu(t)(v_\nu - \dot{u}_\nu(t))da + \int_{\Gamma 3}\big(\int_0^t b(t-s)u_\nu^+(s)ds\big)(v_\nu^+ - \dot{u}_\nu^+(t))da \\
&\geq -\int_{\Gamma_3}[\big(h_\nu(\varphi(t)-\varphi_F) + k_\nu(\theta(t)-\theta_F(t))\big)p_\nu(u_\nu(t))(v_\nu - \dot{u}_\nu(t)).
\end{aligned} \tag{2.66}$$

Finally, it follows from (2.11) that

$$\begin{aligned}
&\sigma_\tau(t).v_\tau \geq -\|\sigma_\tau(t)\|\|v_\tau\| \geq -p_\tau(u_\nu(t))\|v_\tau\|, \\
&-\sigma_\tau(t).\dot{u}_\tau(t) = p_\tau(u_\nu(t))\|\dot{u}_\tau(t)\|,
\end{aligned}$$

and thus,

$$\int_{\Gamma 3} \sigma_\tau(t).(v_\tau - \dot{u}_\tau(t))da \geq -\int_{\Gamma 3} p_\tau(u_\nu(t))(\|v_\tau\| - \|\dot{u}_\tau(t)\|)da. \tag{2.67}$$

We combine (2.63), (2.66) and (2.67) with the definitions (2.53) and (2.55)-(2.57) to deduce that

$$\begin{aligned}
&(\sigma(t), \varepsilon(v) - \varepsilon(\dot{u}(t)))_{\mathcal{H}} + J_{el}(\varphi(t), u(t), v - \dot{u}(t)) \\
&+ J_{te}(\theta(t), u(t), v - \dot{u}(t)) + \Big(\int_0^t b(t-s)u_\nu^+(s)ds, v_\nu^+ - \dot{u}_\nu^+(t)\Big)_{L^2(\Gamma_3)} \\
&+ j(u(t), v) - j(u(t), \dot{u}(t)) \\
&\geq (f(t), v - \dot{u}(t))_V.
\end{aligned} \tag{2.68}$$

We multiply the equation (2.5) by $\mu \in Q$, applying the Green formula (2.19) and taking into account, the definitions (2.59)-(2.61), we obtain

$$\dot{\theta}(t) + \mathcal{Z}\theta(t) = \mathcal{R}\dot{u}(t) + S(t) \ \text{ in } Q'. \tag{2.69}$$

Also, using Green formula (2.19), the definitions (2.58) and (2.54), it is straightforward to see that

$$(D(t), \nabla\psi)_{L^2(\Omega)^d} = J_e(u(t), \varphi(t), \psi) - (q(t), \psi)_W. \tag{2.70}$$

We plug (2.3) in (2.68), (2.4) in (2.70) to obtain the following variational formulation of P in terms of displacement, temperature, and electric potential .

Problem P_V. Find a displacement field $u : [0,T] \to V$, a temperature field $\theta : [0,T] \to Q$ and an electric potential $\varphi : [0,T] \to W$, such that

$$\begin{aligned} &u(t) \in U, \ \ (\mathcal{A}\varepsilon(\dot{u}(t)) + \mathcal{B}\varepsilon(u(t)) + \mathcal{E}^*(\nabla\varphi(t)) - \mathcal{M}\theta(t), \varepsilon(v) - \varepsilon(\dot{u}(t)))_{\mathcal{H}} \\ &+J_{el}(\varphi(t), u(t), v - \dot{u}(t)) + J_{te}(\theta(t), u(t), v - \dot{u}(t)) \\ &+\Big(\int_0^t b(t-s)u_\nu^+(s)ds, v_\nu^+ - \dot{u}_\nu^+(t)\Big)_{L^2(\Gamma_3)} + j(u(t), v) - j(u(t), \dot{u}(t)) \\ &\geq (f(t), v - \dot{u}(t))_V, \ \ \forall v \in U, \end{aligned} \tag{2.71}$$

$$\dot{\theta}(t) + \mathcal{Z}\theta(t) = \mathcal{R}\dot{u}(t) + S(t) \text{ in } Q', \tag{2.72}$$

$$\begin{aligned} &(\mathcal{C}\nabla\varphi(t), \nabla\psi)_{L^2(\Omega)^d} - (\mathcal{E}\varepsilon(u(t)), \nabla\psi)_{L^2(\Omega)^d} - (\mathcal{P}\theta(t), \nabla\psi)_{L^2(\Omega)^d} + J_e(u(t), \varphi(t), \psi) \\ &= (q(t), \psi)_W \ \ \forall \psi \in W, \end{aligned} \tag{2.73}$$

$$u(0) = u_0, \quad \theta(0) = \theta_0. \tag{2.74}$$

To study problem $\mathcal{P}_V$ we make the following smallness assumption.

$$l_e M_{h_e} < \frac{m_{\mathcal{C}}}{C_0^2}, \tag{2.75}$$

where l_e, M_{h_e}, C_0 and $m_{\mathcal{C}}$ are given in (2.49), (2.40), (2.18) and (2.38), respectively.
In the study of Problem P_V we have the following existence and uniqueness result.

Theorem 2.2. *Under assumptions* (2.35)-(2.52) *and* (2.75)*, there exists a unique solution* (u, θ, φ) *to problem* P_V*. Moreover, the solution has the regularity*

$$u \in C^1([0,T]; V), \tag{2.76}$$

$$\theta \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; Q), \ \dot{\theta} \in L^2(0,T; Q'), \tag{2.77}$$

$$\varphi \in C([0,T]; W). \tag{2.78}$$

The functions u, σ, φ, D and θ which satisfy (2.3), (2.4) and (2.71)-(2.74) are called weak solutions of the contact problem P. We conclude that under the assumptions (2.35)-(2.52) and (2.75), problem (2.1)–(2.15) has a unique weak solution (u, σ, φ) satisfying (2.76)–(2.78). The regularity of the weak solution is given by (2.76)–(2.78) and, in term of stresses,

$$\sigma \in C([0,T]; \mathcal{H}_1), \tag{2.79}$$

$$D \in C([0,T]; \mathcal{W}_1). \tag{2.80}$$

Indeed, the regularities (2.76), (2.77) and (2.78) of u, θ and φ combined with (2.35)-(2.38) and (2.43) imply $\sigma \in C([0,T]; \mathcal{H})$ and $D \in C([0,T]; L^2(\Omega)^d)$. On the other hand, we use (2.71) and (2.73) to obtain $\operatorname{Div} \sigma + f_0 = 0$, $\operatorname{div} D - q_0 = 0$. Therefore, regularities (2.46)-(2.47) imply (2.79)-(2.80).

3 Proof of main result

The proof of Theorem 2.2 is carried out in several steps. We assume in what follows that (2.35)-(2.52) and (2.75) hold and everywhere below, we denote by C a positive constant which is independent of time and whose value may change from line to line.
Let $\eta \in C([0,T]; V)$ be given. In the first step, we prove the following lemma for the displacement field.

Lemma 3.1. *If* $M_{\mathcal{A}} > \widetilde{C}_0^2 L_\tau$*, then there exists a unique function* $u_\eta \in C^1([0,T]; V)$ *such that*

$$\begin{cases} u_\eta(t) \in U, \ (\mathcal{A}\varepsilon(\dot{u}_\eta(t)), \varepsilon(v) - \varepsilon(\dot{u}_\eta(t)))_{\mathcal{H}} + (\mathcal{B}\varepsilon(u_\eta(t)), \varepsilon(v) - \varepsilon(\dot{u}_\eta(t)))_{\mathcal{H}} \\ +(\eta(t), v - \dot{u}_\eta(t))_V + j(u_\eta(t), v) - j(u_\eta(t), \dot{u}_\eta(t)) \\ \quad \geq (f(t), v - \dot{u}_\eta(t))_V, \ \ \forall v \in U. \\ u_\eta(0) = u_0 \end{cases} \tag{3.1}$$

Moreover, if u_1 *and* u_2 *are the solutions of* (3.1) *corresponding to* η_1, $\eta_2 \in C([0,T];V)$*, then there exists* $C > 0$ *such that*

$$\|\dot{u}_1(t) - \dot{u}_2(t)\|_V \leq C(\|\eta_1(t) - \eta_2(t)\|_V + \|u_1(t) - u_2(t)\|_V) \quad \forall t \in [0,T]. \tag{3.2}$$

Proof. We apply Theorem2.1 where $X = V$, with the inner product $(\cdot,\cdot)_V$ and the associated norm $\|\cdot\|_V$. We use the Riesz representation theorem to define the operators $A : V \to V$, $B : V \to V$ and the function $f_\eta : [0,T] \to V$ by

$$(Au, v)_V = (\mathcal{A}\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \tag{3.3}$$

$$(Bu, v)_V = (\mathcal{B}\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \tag{3.4}$$

$$(f_\eta(t), v)_V = (f(t), v)_V - (\eta(t), v)_V, \tag{3.5}$$

for all $u, v \in V$ and $t \in [0,T]$. Hypothesis (2.35)(b),(c) and (2.36)(b) imply that the operators A and B satisfy the conditions (2.30) and (2.31), respectively with $m_A = m_{\mathcal{A}}, L_A = L_{\mathcal{A}}, m_B = m_{\mathcal{B}}$.
It follows from (2.27) and (2.55) that the functional j satisfies condition (2.32)(a). Moreover we use (2.41), (2.27) and (2.55) to find

$$j(u_1, v_2) - j(u_1, v_1) + j(u_2, v_1) - j(u_2, v_2) \leq \widetilde{C}_0^2 L_\tau \|u_1 - u_2\|_V \|v_1 - v_2\|_V,$$

for all $u_1, u_2, v_1, v_2 \in V$, which shows that the functional j satisfies condition (2.32)(b) with $m = \widetilde{C}_0^2 L_\tau$ on $X = V$. Moreover, using (2.46) it is easy to see that the function f defined by (2.53) satisfies $f \in C([0,T];V)$ and, keeping in mind that $\eta \in C([0,T];V)$, we deduce from (3.5) that $f_\eta \in C([0,T];V)$, i.e., f_η satisfies (2.33). Finally, we note that (2.52) shows that condition (2.34) is satisfied. Using now (3.3)-(3.5) and if $M_{\mathcal{A}} > \widetilde{C}_0^2 L_\tau$ we find that the first part of Lemma 3.1 is a direct consequence of Theorem 2.1.
Now, let $\eta_1, \eta_2 \in C([0,T];V)$ and let $u_{\eta_i} = u_i$, $\dot{u}_{\eta_i} = \dot{u}_i$, for $i = 1, 2$. From (3.1), we obtain

$$(\mathcal{A}\varepsilon(\dot{u}_1(t)) - \mathcal{A}\varepsilon(\dot{u}_2(t)), \varepsilon(\dot{u}_1(t)) - \varepsilon(\dot{u}_2(t)))_{\mathcal{H}} \leq (\mathcal{B}\varepsilon(u_2(t)) - \mathcal{B}\varepsilon(u_1(t)), \varepsilon(\dot{u}_1(t)) - \varepsilon(\dot{u}_2(t)))_{\mathcal{H}}$$
$$+ (\eta_2(t) - \eta_1(t), \dot{u}_1(t) - \dot{u}_2(t))_V + j(u_1(t), \dot{u}_2(t)) - j(u_1(t), \dot{u}_1(t)) + j(u_2(t), \dot{u}_1(t)) - j(u_2(t), \dot{u}_2(t)),$$

for all $t \in [0,T]$. Using (2.36)(b), (2.41)(b) and (2.27), we find that

$$(\mathcal{A}\varepsilon(\dot{u}_1(t)) - \mathcal{A}\varepsilon(\dot{u}_2(t)), \varepsilon(\dot{u}_1(t)) - \varepsilon(\dot{u}_2(t)))_{\mathcal{H}} \leq L_{\mathcal{B}} \|\varepsilon(u_1(t)) - \varepsilon(u_2(t))\|_{\mathcal{H}} \|\varepsilon(\dot{u}_1(t)) - \varepsilon(\dot{u}_2(t))\|_{\mathcal{H}}$$
$$+ \|\eta_1(t) - \eta_2(t)\|_V \|\dot{u}_1(t) - \dot{u}_2(t)\|_V + \widetilde{C}_0^2 L_\tau \|u_1(t) - u_2(t)\|_V \|\dot{u}_1(t) - \dot{u}_2(t)\|_V,$$

this inequality combined with (2.35)(c) and (2.26) leads to (3.2) which concludes the proof of Lemma 3.1. □

In the second step we use the displacement field u_η obtained in Lemma 3.1 to prove the following lemma for the temperature field.

Lemma 3.2. *There exists an unique* θ_η *satisfying* (2.77) *such that*

$$\begin{cases} \dot{\theta}_\eta(t) + \mathcal{Z}\theta_\eta(t) = \mathcal{R}\dot{u}_\eta(t) + S(t) \text{ in } Q', \\ \theta_\eta(0) = \theta_0. \end{cases} \tag{3.6}$$

Moreover, if θ_1 *and* θ_2 *are the solutions of* (3.6) *corresponding to* η_1, η_2*, then there exists* $C > 0$ *such that*

$$\|\theta_1(t) - \theta_2(t)\|^2_{L^2(\Omega)} \leq C \int_0^t \|\dot{u}_1(s) - \dot{u}_2(s)\|_V^2 ds. \tag{3.7}$$

Proof. The inclusion mapping of $(Q, \|\cdot\|_Q)$ into $(L^2(\Omega), \|\cdot\|_{L^2(\Omega)})$ is continuous and dense. The operator $\mathcal{Z}$ is linear and coercive. Using the Friedrich's poincaré inequality, we have

$$\langle \mathcal{Z}\tau, \tau \rangle_{Q' \times Q} \geq C\|\tau\|_Q^2. \tag{3.8}$$

Through (2.60) and (2.44) for all $\tau, \omega \in Q$, we have

$$\langle \mathcal{Z}\tau, \omega \rangle_{Q' \times Q} \leq \sum_{i,j=1}^{d} \|k_{i,j}\|_{L^\infty(\Omega)} \|\tau_{,i}\|_{L^2(\Omega)} \|\omega_{,i}\|_{L^2(\Omega)} + k_{th} \|\tau\|_{L^2(\Gamma_3)} \|\omega\|_{L^2(\Gamma_3)}.$$

Using (2.20), we find

$$\langle \mathcal{Z}\tau, \omega \rangle_{Q' \times Q} \leq C \|\tau\|_Q \|\omega\|_Q. \tag{3.9}$$

On the other hand, from the definitions of $\mathcal{R}$, $\mathcal{S}$ and the regularities of q_{th}, k_{th} and u_η given in (2.48), (2.50) and Lemma 3.1, we deduce that

$$F_\eta = \mathcal{R}\dot{u}_\eta + S \in L^2(0,T;Q'). \tag{3.10}$$

From (2.52), we recall that $\theta_0 \in L^2(\Omega)$ then from the inequalities (3.8), (3.9) and the regularity (3.10), it follows that the operator $\mathcal{Z}$ is hemicontinuous and monotone, then by using classical arguments of functional analysis concerning parabolic equations (see e.g. [19], p 48) we can easily prove the existence and the uniqueness of θ_η satisfying

$$\begin{cases} \theta_\eta \in C([0,T];L^2(\Omega)) \cap L^2(0,T;Q),\ \dot{\theta}_\eta \in L^2(0,T;Q'), \\ \dot{\theta}_\eta(t) + \mathcal{Z}\theta_\eta(t) = F_\eta(t) \text{ in } Q', \\ \theta_\eta(0) = \theta_0. \end{cases} \tag{3.11}$$

Now for $\eta_1, \eta_2 \in C([0,T];V)$, we have for $t \in [0,T]$:

$$\begin{aligned} &\langle \dot{\theta}_{\eta_1}(t) - \dot{\theta}_{\eta_2}(t), \theta_{\eta_1}(t) - \theta_{\eta_2}(t) \rangle_{L^2(\Omega)} \\ &+ \langle \mathcal{Z}\theta_{\eta_1}(t) - \mathcal{Z}\theta_{\eta_2}(t), \theta_{\eta_1}(t) - \theta_{\eta_2}(t) \rangle_{Q' \times Q} \\ &= \langle \mathcal{R}\dot{u}_{\eta_1}(t) - \mathcal{R}\dot{u}_{\eta_2}(t), \theta_{\eta_1}(t) - \theta_{\eta_2}(t) \rangle_{Q' \times Q}. \end{aligned} \tag{3.12}$$

Then by integrating the last equality over $(0,t)$, (3.7) follows by using (2.60), (2.61), (2.42), (2.43), (2.44), (2.45) and (2.50). □

In the next step we use the solutions u_η and θ_η, obtained in Lemmas 3.1 and 3.2 respectively to demonstrate the following lemma for the electrical potential.

Lemma 3.3. *There exists a unique solution* $\varphi_\eta \in C(0,T;W)$ *such that*

$$\begin{aligned} &(\mathcal{C}\nabla\varphi_\eta(t), \nabla\psi)_{L^2(\Omega)^d} - (\mathcal{E}\varepsilon(u_\eta(t)), \nabla\psi)_{L^2(\Omega)^d} \\ &- (\mathcal{P}\theta_\eta(t), \nabla\psi)_{L^2(\Omega)^d} + J_e(u_\eta(t), \varphi_\eta(t), \psi) \\ &= (q(t), \psi)_W,\ \forall \psi \in W, \forall t \in [0,T]. \end{aligned} \tag{3.13}$$

Moreover, if φ_{η_1} *and* φ_{η_2} *are the solutions of* (3.13) *corresponding to* η_1, $\eta_2 \in C([0,T];V)$ *then, there exists* $C > 0$*, such that*

$$\|\varphi_{\eta_1}(t) - \varphi_{\eta_2}(t)\|_W \leq C(\|u_{\eta_1}(t) - u_{\eta_2}(t)\|_V + \|\theta_{\eta_1}(t) - \theta_{\eta_1}(t)\|_{L^2(\Omega)}). \tag{3.14}$$

Proof. Let $t \in [0,T]$. We use the Riesz representation theorem to define the operator $\mathcal{L}_\eta(t)$: $W \to W$ by

$$\begin{aligned} (\mathcal{L}_\eta(t)\varphi, \psi)_W =& (\mathcal{C}\nabla\varphi(t), \nabla\psi)_{L^2(\Omega)^d} - (\mathcal{E}\varepsilon(u_\eta(t)), \nabla\psi)_{L^2(\Omega)^d} \\ &- (\mathcal{P}\theta_\eta(t), \nabla\psi)_{L^2(\Omega)^d} + J_e(u_\eta(t), \varphi(t), \psi), \end{aligned} \tag{3.15}$$

for all $\varphi, \psi \in W$. Let $\varphi_1, \varphi_2 \in W$, then we use (2.17), (2.38)(d) and notation (2.58) to deduce that

$$\begin{aligned} &(\mathcal{L}_\eta(t)\varphi_1 - \mathcal{L}_\eta(t)\varphi_2, \varphi_1 - \varphi_2)_W \\ &\geq m_{\mathcal{C}} \|\varphi_1 - \varphi_2\|_W^2 + \int_{\Gamma_3} h_e(u_{\eta\nu}(t)) \big(\phi(\varphi_1 - \varphi_F) - \phi(\varphi_2 - \varphi_F)\big)(\varphi_1 - \varphi_2)\, da. \end{aligned}$$

Therefore, using the bound (2.40)(c), Lipschitz-continuity of the function ϕ and the trace theorem (2.18) we obtain

$$(\mathcal{L}_\eta(t)\varphi_1 - \mathcal{L}_\eta(t)\varphi_2, \varphi_1 - \varphi_2)_W \geq m_{\mathcal{C}} \|\varphi_1 - \varphi_2\|_W^2 - M_{h_e} l_e C_0^2 \|\varphi_1 - \varphi_2\|_W^2. \tag{3.16}$$

It follows from inequality (3.16) and the smallness assumption (2.75) that there exists $C > 0$ such that

$$(\mathcal{L}_\eta(t)\varphi_1 - \mathcal{L}_\eta(t)\varphi_2, \varphi_1 - \varphi_2)_W \geq C \|\varphi_1 - \varphi_2\|_W^2. \tag{3.17}$$

On the other hand, we use (2.38), (2.40), (3.15), (2.58) and (2.17) to have

$$(\mathcal{L}_\eta(t)\varphi_1 - \mathcal{L}_\eta(t)\varphi_2, \psi)_W \leq L_{\mathcal{C}}\|\varphi_1 - \varphi_2\|_W \|\psi\|_W + \int_{\Gamma_3} M_{h_e} l_e |\varphi_1 - \varphi_2|\, |\psi|\, da \quad \forall \psi \in W, \tag{3.18}$$

with $L_{\mathcal{C}} = \sup_{i,j} \|c_{ij}\|_{L^\infty(\Omega)}$. It follows from (3.18) and (2.18) that

$$(\mathcal{L}_\eta(t)\varphi_1 - \mathcal{L}_\eta(t)\varphi_2, \psi)_W \leq (L_{\mathcal{C}} + M_{h_e} l_e C_0^2)\|\varphi_1 - \varphi_2\|_W \|\psi\|_W \quad \forall \psi \in W,$$

thus,

$$\|\mathcal{L}_\eta(t)\varphi_1 - \mathcal{L}_\eta(t)\varphi_2\|_W \leq (L_{\mathcal{C}} + M_{h_e} l_e C_0^2)\|\varphi_1 - \varphi_2\|_W. \tag{3.19}$$

Inequalities (3.17) and (3.19) show that the operator $\mathcal{L}_\eta(t)$ is a strongly monotone Lipschitz continuous operator on W and, therefore, there exists a unique element $\varphi_\eta(t) \in W$ such that

$$\mathcal{L}_\eta(t)\varphi_\eta(t) = q(t). \tag{3.20}$$

We combine now (3.15) and (3.20) to find that $\varphi_\eta(t) \in W$ is the unique solution of the nonlinear variational equation (3.13).

Next, we show that $\varphi_\eta \in C([0,T]; W)$. To this end, let $t_1, t_2 \in [0,T]$ and for the sake of simplicity, we write $\varphi_\eta(t_i) = \varphi_i$, $u_{\eta\nu}(t_i) = u_i$, $\theta_\eta(t_i) = \theta_i$, $q(t_i) = q_i$, for $i = 1, 2$. Using (3.13), (2.37), (2.26), (2.38), (2.58) and (2.17), we find

$$\begin{aligned} m_{\mathcal{C}} \|\varphi_1 - \varphi_2\|_W^2 \leq\ & M_{\mathcal{E}}\|u_1 - u_2\|_V \|\varphi_1 - \varphi_2\|_W + \sqrt{d} M_{\mathcal{P}} \|\theta_1 - \theta_2\|_{L^2(\Omega)} \|\varphi_1 - \varphi_2\|_W \\ & + \int_{\Gamma_3} |h_e(u_1)\phi(\varphi_1 - \varphi_F) - h_e(u_2)\phi(\varphi_2 - \varphi_F)| |\varphi_1 - \varphi_2|\, da \\ & + \|q_1 - q_2\|_W \|\varphi_1 - \varphi_2\|_W, \end{aligned} \tag{3.21}$$

where $M_{\mathcal{E}} = \sup_{i,j,k} \|e_{ijk}\|_{L^\infty(\Omega)}$ and $M_{\mathcal{P}} = \sup_i \|p_i\|_{L^\infty(\Omega)}$. We use the bounds $|h_e(u_i)| \leq M_{h_e}$, $|\phi(\varphi_1 - \varphi_2)| \leq \bar{l}_e$, the Lipschitz continuity of the functions h_e and ϕ, and inequalities (2.18) and (2.27) to obtain

$$\begin{aligned} & \int_{\Gamma_3} |h_e(u_1)\phi(\varphi_1 - \varphi_F) - h_e(u_2)\phi(\varphi_2 - \varphi_F)| |\varphi_1 - \varphi_2|\, da \\ & \leq M_{h_e} l_e \int_{\Gamma_3} |\varphi_1 - \varphi_2|^2\, da + L_{h_e} \bar{l}_e \int_{\Gamma_3} |u_1 - u_2|\, |\varphi_1 - \varphi_2|\, da \\ & \leq M_{h_e} l_e\, C_0^2 \|\varphi_1 - \varphi_2\|_W^2 + L_{h_e} \bar{l}_e C_0 \widetilde{C}_0 \|u_1 - u_2\|_V \|\varphi_1 - \varphi_2\|_W. \end{aligned}$$

Inserting the last inequality in (3.21) yields

$$\begin{aligned} m_{\mathcal{C}} \|\varphi_1 - \varphi_2\|_W \leq\ & (M_{\mathcal{E}} + L_{h_e} \bar{l}_e C_0 \widetilde{C}_0)\|u_1 - u_2\|_V + \sqrt{d} M_{\mathcal{P}} \|\theta_1 - \theta_2\|_{L^2(\Omega)} \\ & + M_{h_e} l_e\, C_0^2 \|\varphi_1 - \varphi_2\|_W + \|q_1 - q_2\|_W. \end{aligned} \tag{3.22}$$

It follows from inequality (3.22) and assumption (2.75) that

$$\|\varphi_1 - \varphi_2\|_W \leq C(\|u_1 - u_2\|_V + \|\theta_1 - \theta_2\|_{L^2(\Omega)} + \|q_1 - q_2\|_W). \tag{3.23}$$

We also note that assumptions (2.47), combined with definition (2.54) imply that $q \in C([0,T]; W)$. Since $u_\eta \in C^1([0,T]; V)$ and $\theta_\eta \in C([0,T]; L^2(\Omega))$ inequality (3.23) implies that $\varphi_\eta \in C([0,T]; W)$.

Now, let $\eta_1, \eta_2 \in C([0,T];V)$ and let $\varphi_{\eta_i} = \varphi_i$, $u_{\eta_i} = u_i$, $\theta_{\eta_i} = \theta_i$, for $i = 1,2$. We use (3.13) and arguments similar to those used in the proof of (3.22) to obtain

$$m_{\mathcal{C}} \|\varphi_1(t) - \varphi_2(t)\|_W \leq (M_{\mathcal{E}} + L_{h_e}\bar{l}_e C_0 \widetilde{C}_0)\|u_1(t) - u_2(t)\|_V + \sqrt{d} M_{\mathcal{P}} \|\theta_1(t) - \theta_2(t)\|_{L^2(\Omega)} + M_{h_e} l_e C_0^2 \|\varphi_1(t) - \varphi_2(t)\|_W \ \ \forall t \in [0,T],$$

this inequality combined with assumption (2.75) leads to (3.14), which concludes the proof. □

Now, for every $\eta \in C([0,T];V)$, we denote by u_η, θ_η and φ_η the solutions provided in Lemmas 3.1, 3.2 and 3.3 respectively. Moreover, we apply the Riesz representation theorem to define the function $\mathcal{T}\eta : [0,T] \to V$ defined by

$$(\mathcal{T}\eta(t), v)_V = (\mathcal{E}^*(\nabla\varphi_\eta(t)) - \mathcal{M}\theta_\eta(t), \varepsilon(v))_{\mathcal{H}} + \Big(\int_0^t b(t-s) u^+_{\eta\nu}(s) ds, v^+_\nu\Big)_{L^2(\Gamma_3)} + J_{el}(\varphi_\eta(t), u_\eta(t), v) + J_{te}(\theta_\eta(t), u_\eta(t), v), \ \forall v \in U, \ \forall t \in [0,T]. \tag{3.24}$$

Lemma 3.4. *For each $\eta \in C([0,T];V)$ the function $\mathcal{T}\eta : [0,T] \to V$ belongs to $C([0,T];V)$. Moreover, there exists a unique $\eta^* \in C([0,T];V)$ such that $\mathcal{T}\eta^* = \eta^*$.*

Proof. Let $\eta \in C([0,T];V)$ and $t_1, t_2 \in [0,T]$ with $t_1 < t_2$. Using (3.24), we obtain

$$(\mathcal{T}\eta(t_1) - \mathcal{T}\eta(t_2), v)_V = (\mathcal{E}^*(\nabla\varphi_\eta(t_1)) - \mathcal{E}^*(\nabla\varphi_\eta(t_2)), \varepsilon(v))_{\mathcal{H}} - (\mathcal{M}\theta_\eta(t_1) - \mathcal{M}\theta_\eta(t_2), \varepsilon(v))_{\mathcal{H}}$$
$$+ \Big(\int_0^{t_1} b(t_1 - s) u^+_{\eta\nu}(s) ds - \int_0^{t_2} b(t_2 - s) u^+_{\eta\nu}(s) ds, v^+_\nu\Big)_{L^2(\Gamma_3)} + J_{el}(\varphi_\eta(t_1), u_\eta(t_1), v)$$
$$- J_{el}(\varphi_\eta(t_2), u_\eta(t_2), v) + J_{te}(\theta_{\eta_1}(t), u_\eta(t_1), v) - J_{te}(\theta_\eta(t_2), u_\eta(t_2), v) \ \ \forall v \in V$$

Using (2.37), (2.39), (2.41), (2.43), (2.51), (2.17), (2.18), (2.26) and (2.27), it follows that

$$|(\mathcal{T}\eta(t_1) - \mathcal{T}\eta(t_2), v)_V| \leq C\Big(\|\varphi_\eta(t_1) - \varphi_\eta(t_2)\|_W + \|\theta_\eta(t_1) - \theta_\eta(t_2)\|_{L^2(\Omega)} + \|u_\eta(t_1) - u_\eta(t_2)\|_V + \int_{t_2}^{t_1} \|u_\eta(s)\|_V ds\Big)\|v\|_V.$$

Then we take $v = \mathcal{T}\eta(t_1) - \mathcal{T}\eta(t_2)$ in the previous inequality to find that

$$\|\mathcal{T}\eta(t_1) - \mathcal{T}\eta(t_2)\|_V \leq C\Big(\|\varphi_\eta(t_1) - \varphi_\eta(t_2)\|_W + \|\theta_\eta(t_1) - \theta_\eta(t_2)\|_{L^2(\Omega)} + \|u_\eta(t_1) - u_\eta(t_2)\|_V + \int_{t_2}^{t_1} \|u_\eta(s)\|_V ds\Big). \tag{3.25}$$

It follows from (3.25) and the regularities of u_η, θ_η and φ_η expressed in (2.76), (2.77) and (2.78) respectively, that $\mathcal{T}\eta \in C([0,T];V)$.
Now let $\eta_1, \eta_2 \in C([0,T];V)$ and denote by u_i, θ_i and φ_i the functions u_{η_i}, θ_{η_i} and φ_{η_i} obtained in Lemmas 3.1, 3.2 and 3.3, for $i = 1,2$. Let $t \in [0,T]$. Arguments similar to those used in the proof of (3.25) yield

$$\|\mathcal{T}\eta_1(t) - \mathcal{T}\eta_2(t)\|_V^2 \leq C\Big(\|u_1(t) - u_2(t)\|_V^2 + \int_0^t \|u_1(s) - u_2(s)\|_V^2 ds + \|\varphi_1(t) - \varphi_2(t)\|_W^2 + \|\theta_1(t) - \theta_2(t)\|_{L^2(\Omega)}^2\Big), \tag{3.26}$$

and, keeping in mind (3.7) and (3.14), we find

$$\|\mathcal{T}\eta_1(t) - \mathcal{T}\eta_2(t)\|_V^2 \leq C\Big(\|u_1(t) - u_2(t)\|_V^2 + \int_0^t \|u_1(s) - u_2(s)\|_V^2 + \int_0^t \|\dot{u}_1(s) - \dot{u}_2(s)\|_V^2\Big). \tag{3.27}$$

On the other hand, since $u_i(t) = u_0 + \int_0^t \dot{u}_i(s)\,ds$, we have

$$\|u_1(t) - u_2(t)\|_V^2 \leq \int_0^t \|\dot{u}_1(s) - \dot{u}_2(s)\|_V^2\,ds, \tag{3.28}$$

and using this inequality in (3.2) yields

$$\|\dot{u}_1(t) - \dot{u}_2(t)\|_V^2 \leq C\Big(\|\eta_1(t) - \eta_2(t)\|_V^2 + \int_0^t \|\dot{u}_1(s) - \dot{u}_2(s)\|_V^2\,ds\Big).$$

It follows now from a Gronwall-type argument that

$$\int_0^t \|\dot{u}_1(s) - \dot{u}_2(s)\|_V^2\,ds \leq C\int_0^t \|\eta_1(s) - \eta_2(s)\|_V^2\,ds. \tag{3.29}$$

Combining (3.27)-(3.29) leads to

$$\|\mathcal{T}\eta_1(t) - \mathcal{T}\eta_2(t)\|_V^2 \leq C\int_0^t \|\eta_1(s) - \eta_2(s)\|_V^2\,ds.$$

Reiterating this inequality n times we are led to

$$\|\mathcal{T}^n\eta_1(t) - \mathcal{T}^n\eta_2(t)\|_V^2 \leq \; C^n \underbrace{\int_0^t\int_0^s ... \int_0^m}_{n \text{ integrals}} \|\eta_1(r) - \eta_2(r)\|_V^2\,dr...ds,$$

which implies that

$$\|\mathcal{T}^n\eta_1 - \mathcal{T}^n\eta_2\|_{C([0,T];V)}^2 \leq \frac{C^n\,T^n}{n!}\|\eta_1 - \eta_2\|_{C([0,T];V)}^2. \tag{3.30}$$

Since $\lim_{n\to\infty} \frac{C^n\,T^n}{n!} = 0$, it follows that there exists a positive integer n such that $\frac{C^n\,T^n}{n!} < 1$ and, therefore, (3.30) shows that the operator $\mathcal{T}^n$ is a contraction on the Banach space $C([0,T];V)$ and, so, there exists a unique element $\eta^* \in C([0,T];V)$ such that $\Lambda\eta^* = \eta^*$. □

We have now all the ingredient to prove the Theorem 2.2 which we complete now.
Existence. Let $\eta^* \in C([0,T];V)$ be the fixed point of the operator $\mathcal{T}$, and let u_{η^*}, θ_{η^*} and φ_{η^*} the solutions provided in Lemmas 3.1, 3.2 and 3.3 respectively, for $\eta = \eta^*$.
It follows from (3.24) that

$$\begin{aligned}(\eta^*(t), v)_V = (\mathcal{E}^*(\nabla\varphi_{\eta^*}(t)) - \mathcal{M}\theta_{\eta^*}(t), \varepsilon(v))_{\mathcal{H}} + \Big(\int_0^t b(t-s)u^+_{\eta^*\nu}(s)ds, v^+_\nu\Big)_{L^2(\Gamma_3)} \\ + J_{el}(\varphi_{\eta^*}(t), u_{\eta^*}(t), v) + J_{te}(\theta_{\eta^*}(t), u_{\eta^*}(t), v),\; \forall v \in U,\; \forall t \in [0,T],\end{aligned}$$

and, therefore, (3.1), (3.6) and (3.13) imply that $(u_{\eta^*}, \theta_{\eta^*}, \varphi_{\eta^*})$ is a solution of problem P_V.
Uniqueness. The uniqueness of the solution follows from the uniqueness of the fixed point of the operator $\mathcal{T}$ defined by (3.24).

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