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Semi Implicit Scheme of Fisher Equation Based on Crank-Nicolson Method and Method of Lagging

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Abstract In this article, we present finite difference solutions of Fisher equation subject to initial and boundary conditions. The numerical solutions are computed by semi-implicit time discretization based on Crank-Nicolson scheme and semi-implicit scheme. The stability and consistency of the two numerical methods are shown. Two examples are provided to show stability of the numerical scheme. The solutions which are obtained by the two numerical schemes are compared to each other and compared to the exact solution.

1 Introduction

Fisher equation which is given by

$$\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2} + \alpha u(1-u),\ a \le x \le b,\ t \ge 0, \tag{1.1}$$

where c is diffusion coefficient, α is reactive factor, a and b are real numbers, x is position and t is time, arises in chemistry, heat and mass transfer, biology and ecology[1, 2, 3, 4, 5, 6]. In 1937 Fisher [7] and Kolmogorov et al. [8] investigated independently equation (1.1). Due to this it is called the Fisher- Kolmogorov-Petrovsky-Piscounov (Fisher-KPP) equation. However it is widely known as Fisher equation. Equation (1.1) is a non-linear model for a physical system involving linear diffusion and non-linear growth [9]. As a result, equation (1.1) describes the process of interaction between diffusion and reaction [5].

Many scholars have been developing numerical methods for solving partial differential equations (PDEs). Crank-Nicolson and semi-implicit schemes are the two numerical methods that are applied to compute the solutions of PDEs by discretizing the domain into finite number of regions. The two methods are categorized under finite difference methods. The finite difference techniques are based upon the approximations that permit replacing differential equations by finite difference equations (FDEs). These finite difference approximations are algebraic in form, and the solutions are related to grid points. Finite difference methods have been used for solving PDEs arising in modeling and design[10].

In the past decades, it has been studied the numerical solutions of equation (1.1) subject to initial and boundary conditions (see [5, 11, 12, 13, 14, 15, 16]). In this paper we propose semi-implicit time discretization based on Crank-Nicolson scheme and semi-implicit scheme to compute equation (1.1) subject to initial condition

$$u(x,0) = h(x),\ a \le x \le b \tag{1.2}$$

and boundary conditions

$$u(a,t) = p(t)\ and\ u(b,t) = q(t),\ 0 \le t \le t_{end} \tag{1.3}$$

when the diffusion coefficient $c = 1$, to represent real physical processes [5, 9].

For the purpose of numerical solution we take $a = 0, b = 1$ and $t_{end} = 1$, and divide the intervals $[0, b]$ and $[0, t_{end}]$ into n and m equal parts respectively by the points $x_1, x_2, \cdots, x_{n+1}$

and $t_1, t_2, \cdots, t_{m+1}$ with step length h and k respectively. So the problem is computing the solution $u(x_i, t_j)$ for $i = 2, 3, \cdots, n$ and $j = 2, 3, \cdots, m+1$.

The paper is organized as follows. In sections two, three and four, we discuss semi-implicit time discretization based on Crank-Nicolson scheme, semi-implicit scheme, and truncation error, stability and consistency respectively. In section five, analytic solutions are provided for different conditions. In section six, numerical simulation is presented. Finally, it includes conclusion and abbreviations in section seven and eight respectively.

2 Semi-Implicit Time Discretization Based on Crank-Nicolson Scheme(SICNS)

A common family of implicit schemes is defined by the Crank-Nicolson scheme. It gives for a weighted average of the spatial derivatives at the j^{th} and the $(j+1)^{th}$ time levels. As a result, we discretize equation (1.1) as

$$\begin{aligned} u_i^{j+1} - u_i^j &= \tfrac{k}{2h^2}\left[u_{i-1}^{j+1} - 2u_i^{j+1} + u_{i+1}^{j+1} + u_{i-1}^j - 2u_i^j + u_{i+1}^j\right] \\ &\quad + k\alpha u_i^j(1-u_i^j), \end{aligned} \tag{2.1}$$

where $i = 2, 3, \cdots, n, \quad j = 1, 2, \cdots, m$
or

$$\begin{aligned} -\tfrac{k}{h^2}u_{i-1}^{j+1} + \left[2 + \tfrac{2k}{h^2}\right]u_i^{j+1} - \tfrac{k}{h^2}u_{i+1}^{j+1} &= \tfrac{k}{h^2}u_{i-1}^j + \left(2 - \tfrac{2k}{h^2}\right)u_i^j \\ &\quad + \tfrac{k}{h^2}u_{i+1}^j + 2k\alpha u_i^j(1-u_i^j), \end{aligned} \tag{2.2}$$

where $i = 2, 3, \cdots, n, \quad j = 1, 2, \cdots, m$. We call this scheme semi-implicit time discretization based on Crank-Nicolson scheme. The system of equations (2.2) is a triangular system of the form

$$\begin{bmatrix} B & A & & & & & \\ A & B & A & & & & \\ & A & B & A & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & A & B & A \\ & & & & & A & B \end{bmatrix} \begin{bmatrix} u_2^{j+1} \\ u_3^{j+1} \\ \cdot \\ \cdot \\ \cdot \\ u_{n-1}^{j+1} \\ u_n^{j+1} \end{bmatrix} = \begin{bmatrix} V_2 \\ V_3 \\ \cdot \\ \cdot \\ \cdot \\ V_{n-1} \\ V_n \end{bmatrix} + \begin{bmatrix} Cu_1^j \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ Cu_{n+1}^j \end{bmatrix}$$

$$- \begin{bmatrix} Au_1^{j+1} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ Au_{n+1}^{j+1} \end{bmatrix} + \begin{bmatrix} F_2^j \\ F_3^j \\ \cdot \\ \cdot \\ \cdot \\ F_{n-1}^j \\ F_n^j \end{bmatrix},$$

where

$$A = -\frac{k}{h^2},\ B = 2 + \frac{2k}{h^2},\ F_i^j = \alpha k u_i^j(1-u_i^j),\ i = 2, 3, \cdots, n,\ j = 2, 3, \cdots, m+1$$

and

$$\begin{bmatrix} V_2 \\ V_3 \\ \cdot \\ \cdot \\ \cdot \\ V_{n-1} \\ V_n \end{bmatrix} = \begin{bmatrix} D & C & & & & & \\ C & D & C & & & & \\ & C & D & C & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & C & D & C \\ & & & & & C & D \end{bmatrix} \begin{bmatrix} u_2^j \\ u_3^j \\ \cdot \\ \cdot \\ \cdot \\ u_{n-1}^j \\ u_n^j \end{bmatrix},$$

where $C = \frac{k}{h^2}, D = 2 - \frac{2k}{h^2},\ j = 2, 3, \cdots, m+1$.

In the same manner, the difference equations of equations (1.2) and (1.3) are respectively

$$u_i^1 = h(x_i),\ i = 1, 2, \cdots, n+1$$

and

$$u_1^j = p_j \; and \; u_{n+1}^j = q_j,\ j = 1, 2, \cdots, m+1.$$

3 Semi-Implicit Scheme(SIS)

In this method, derivatives are calculated at $j+1$ time level. We use central difference formula in space and forward difference formula for time. For converting the nonlinear term in equation (1.1) to linear term we use method of lagging. In method of lagging one is calculated at j time level and other is calculated at $j+1$ time level[5].

As a result, equation (1.1) becomes

$$\begin{aligned} u_i^{j+1} - u_i^j &= \tfrac{k}{h^2}\left[u_{i-1}^{j+1} - 2u_i^{j+1} + u_{i+1}^{j+1}\right] + k\alpha u_i^{j+1}(1 - u_i^j) \\ & i = 2, 3, \cdots, n, \;\; j = 1, 2, \cdots, m \end{aligned} \tag{3.1}$$

or

$$-\frac{k}{h^2}u_{i-1}^{j+1} + \left[1 + \frac{2k}{h^2} - k\alpha(1 - u_i^j)\right] u_i^{j+1} - \frac{k}{h^2}u_{i+1}^{j+1} = u_i^j. \tag{3.2}$$

The system of equations (3.2) is a triangular system of the form

$$\begin{bmatrix} B_2 & A & & & & & \\ A & B_3 & A & & & & \\ & A & B_4 & A & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & A & B_{n-1} & A \\ & & & & & A & B_n \end{bmatrix} \begin{bmatrix} u_2^{j+1} \\ u_3^{j+1} \\ \cdot \\ \cdot \\ \cdot \\ u_{n-1}^{j+1} \\ u_n^{j+1} \end{bmatrix} = \begin{bmatrix} u_2^j \\ u_3^j \\ \cdot \\ \cdot \\ \cdot \\ u_{n-1}^j \\ u_n^j \end{bmatrix} - \begin{bmatrix} Au_1^{j+1} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ Au_{n+1}^{j+1} \end{bmatrix},$$

where

$$A = -\frac{k}{h^2},\ B_i = 1 + \frac{2k}{h^2} - k\alpha(1 - u(i,j)),\ i = 2, 3, \cdots, n, j = 2, 3, \cdots, m+1.$$

In the same manner, the difference equation of (1.2) and (1.3) are respectively

$$u_i^1 = h(x_i),\ i = 1, 2, \cdots, n+1$$

and

$$u_1^j = p_j \; and \; u_{n+1}^j = q_j,\ j = 1, 2, \cdots, m+1.$$

4 Truncation Error, Stability and Consistency

4.1 Truncation Error

It occurs when the solution of partial differential equation is approximating by the numerical method. Assume u is smooth function at (x_i, t_j). Using Taylor series method,

$$u_i^{j+1} = u_i^j + k u_{t,i}^j + \frac{k^2}{2} u_{tt,i}^j + \frac{k^3}{6} u_{ttt,i}^j + \frac{k^4}{24} u_{tttt,i}^j + \cdots . \tag{4.1}$$

$$\begin{aligned} u_{i-1}^{j+1} &= u_i^j - h u_{x,i}^j + k u_{t,i}^j + \tfrac{h^2}{2} u_{xx,i}^j - kh u_{tx,i}^j + \tfrac{k^2}{2} u_{tt,i}^j - \\ &\quad \tfrac{h^3}{6} u_{xxx,i}^j + \tfrac{kh^2}{2} u_{txx,i}^j - \tfrac{hk^2}{2} u_{ttx,i}^j + \tfrac{h^4}{24} u_{xxxx,i}^j + \\ &\quad \tfrac{k^3}{6} u_{ttt,i}^j - \tfrac{kh^3}{6} u_{txxx,i}^j + \tfrac{k^2h^2}{2} u_{ttxx,i}^j - \tfrac{hk^3}{6} u_{xttt,i}^j + \cdots . \end{aligned} \tag{4.2}$$

$$\begin{aligned} u_{i+1}^{j+1} &= u_i^j + h u_{x,i}^j + k u_{t,i}^j + \tfrac{h^2}{2} u_{xx,i}^j + kh u_{tx,i}^j + \tfrac{k^2}{2} u_{tt,i}^j + \\ &\quad \tfrac{h^3}{6} u_{xxx,i}^j + \tfrac{kh^2}{2} u_{txx,i}^j + \tfrac{hk^2}{2} u_{ttx,i}^j + \tfrac{h^4}{24} u_{xxxx,i}^j + \\ &\quad \tfrac{k^3}{6} u_{ttt,i}^j + \tfrac{kh^3}{6} u_{txxx,i}^j + \tfrac{k^2h^2}{2} u_{ttxx,i}^j + \tfrac{hk^3}{6} u_{xttt,i}^j + \cdots . \end{aligned} \tag{4.3}$$

$$u_{i-1}^j = u_i^j - h u_{x,i}^j + \frac{h^2}{2} u_{xx,i}^j - \frac{h^3}{6} u_{xxx,i}^j + \frac{h^4}{24} u_{xxxx,i}^j + \cdots . \tag{4.4}$$

$$u_{i+1}^j = u_i^j + h u_{x,i}^j + \frac{h^2}{2} u_{xx,i}^j + \frac{h^3}{6} u_{xxx,i}^j + \frac{h^4}{24} u_{xxxx,i}^j + \cdots . \tag{4.5}$$

Semi-Implicit Time Discretization Based on Crank-Nicolson Scheme

Substituting equations (4.1), (4.2), (4.3), (4.4), (4.5) into equation (2.1), we get

$$SICNS\ (equation(2.1)) - PDE\ (equation(1.1)) = O(k, h^2).$$

This indicates that the semi-implicit time discretization based on Crank-Nicolson scheme we have applied here for fisher equation is first order accurate in time and second order accurate in space.

Semi-Implicit scheme

Substituting equations (4.1), (4.2), (4.3), (4.4), (4.5) into equation (3.1) we get

$$SIS\ (equation(3.1)) - PDE\ (equation(1.1)) = O(k, h^2).$$

Hence, semi-implicit scheme we have used here for fisher equation is first order accurate in time and second order accurate in space.

4.2 Stability

Numerical errors which are generated during the solutions of discretized equations should not be magnified. This refereed as stability. The Von-Neumann stability analysis of finite difference schemes for non-linear problem (reaction-diffusion model) have been discussed in [17, 18, 19]. According to the Von-Neumann stability analysis, we assume the solution of equation (1.1) as

$$u_i^j = \xi^j e^{\mathbf{i} i r h}, \tag{4.6}$$

where $\mathbf{i} = \sqrt{-1}$, wave number r and amplification factor $\xi = \xi(r)$. And its stability condition is

$$|\xi(r)| \leq 1.$$

We will assume

$$\kappa\alpha u_i^j(1-u_i^j)=k\alpha u_i^j(1-constant) \text{ and } \kappa\alpha u_i^{j+1}(1-u_i^j)=k\alpha u_i^{j+1}(1-constant)$$

to linearize (2.1) and (3.1) respectively [5, 19].

Semi-Implicit Time Discretization Based on Crank-Nicolson Scheme

In order to use this stability condition we first linearize equation (2.1) as follows.

$$\begin{aligned} u_i^{j+1}-u_i^j &= \tfrac{k}{2h^2}\left[u_{i-1}^{j+1}-2u_i^{j+1}+u_{i+1}^{j+1}+u_{i-1}^j-2u_i^j+u_{i+1}^j\right] \\ &\quad + k\alpha u_i^j(1-constant), \end{aligned} \tag{4.7}$$

where $i=2,3,\cdots,n,\quad j=1,2,\cdots,m$. Putting equation (4.6) into equation (4.7) we get

$$\xi(r)=\frac{1-\frac{2k}{h^2}\sin^2\frac{rh}{2}+k\alpha(1-constant)}{1+\frac{2k}{h^2}\sin^2\frac{rh}{2}}. \tag{4.8}$$

If $|\xi(r)|\le 1$, the linear stability requirement is

$$-\frac{2}{k}\le\alpha(1-constant)\le\frac{4}{h^2}.$$

So, the semi-implicit time discretization based on Crank-Nicolson scheme we applied here in equation (2.1) is conditionally stable.

Semi-Implicit Scheme

We now first change equation (3.1) into linear form as follows.

$$\begin{aligned} u_i^{j+1}-u_i^j &= \tfrac{k}{h^2}\,[u_{i-1}^{j+1}-2u_i^{j+1}+u_{i+1}^{j+1}]+k\alpha u_i^{j+1}(1-constant), \\ & i=2,3,\cdots,n,\quad j=1,2,\cdots,m \end{aligned} \tag{4.9}$$

Substituting equation (4.6) into equation (4.9), we have

$$\xi(k)=\frac{1}{1+\frac{4k}{h^2}\sin^2\frac{rh}{2}-k\alpha(1-constant)}.$$

If $constant\ge 1$, the scheme is unconditionally stable no matter how the values of h and k are. If $constant<1$, the scheme is conditionally stable and the linear stability requirement is

$$\frac{1}{h^2\alpha(1-constant)}\ge\frac{1}{4}.$$

So, the semi-implicit scheme we consider here is conditionally stable.

4.3 Consistency

A numerical method is said to be consistent if the difference between a partial differential equation (PDE) and its corresponding finite difference equation (FDE) approaches zero as the number of subdivision increases. In both methods we discussed here

$$\lim_{h,k\to 0} PDE-FDE=0.$$

5 Analytic Solutions

Case 1: $\alpha = 1,\ p(t) = \frac{\lambda e^t}{1-\lambda(1-e^t)},\ q(t) = \frac{\lambda e^t}{1-\lambda(1-e^t)}$ and $h(x) = \lambda$.
The analytic solution of (1.1),(1.2) and (1.3) as described in [9], is given by $u(x,t) = \frac{\lambda e^t}{1-\lambda(1-e^t)}$.
Case 2: $\alpha = 6,\ p(t) = \frac{1}{(1+e^{-5t})^2},\ q(t) = \frac{1}{(1+e^{b-5t})^2}$ and $h(x) = 1/(1+e^x)^2$.
The analytic solution of (1.1),(1.2) and (1.3) as described in [9], is given by $u(x,t) = \frac{1}{(1+e^{x-5t})^2}$.

In [9], the analytic solution was computed by Homotopy perturbation method. It is the one that provides series solution to linear and nonlinear PDEs [20, 21, 22, 23]. As a result, in general it is approximate method. If we obtain the closed form of the series, the solution will be analytic solution[9, 14]. It has been applied for solving PDEs arising in modeling of flow in porous media[24] and the transmission of nerve impulses[25]. In [26] and [27], it was computed the solution of non-linear fractional PDE and non-linear system of second order boundary value problems respectively.

6 Numerical simulation

In this section we use MatLab software for the purpose of simulation for different values of n and m.

6.1 Solutions using SICNS, SIS and HPM

Here we display the numerical and analytic discrete solutions for $t = 0.4$ and $t = 1$ in each case.

Case 1: $\alpha = 1,\ p(t) = \frac{\lambda e^t}{1-\lambda(1-e^t)},\ q(t) = \frac{\lambda e^t}{1-\lambda(1-e^t)}$ and $h(x) = \lambda = 2$.

Table (1) and (2) describe the solution of (1.1) subject to (1.2) and (1.3) at $t = 0.4$ and $t = 1$ respectively by SICNS, SIS and HPM. Figures (1) is graphical representation of the solution to the problem (1.1), (1.2), (1.3) at $t = 0.4$ and $t = 1$. Figure (2) shows the solution graph of the problem (1.1), (1.2), (1.3) for $x \in [0, 1]$ and $t \in [0, 1]$.

Table 1. Solutions at $t = 0.4$ for $\alpha = 1$ and $n = 10$

	$m = 10$			$m = 1000$		
x	SICNS	SIS	HPM	SICNS	SIS	HPM
0	1.50412134	1.50412134	1.50412134	1.50412134	1.50412134	1.50412134
0.1	1.50088051	1.50232444	1.50412134	1.50408508	1.50410457	1.50412134
0.2	1.49774786	1.50094284	1.50412134	1.50405665	1.50409157	1.50412134
0.3	1.49544645	1.49996515	1.50412134	1.50403623	1.5040823	1.50412134
0.4	1.49409199	1.49938238	1.50412134	1.50402393	1.50407674	1.50412134
0.5	1.49364875	1.49918878	1.50412134	1.50401982	1.5040749	1.50412134
0.6	1.49409199	1.49938238	1.50412134	1.50402393	1.50407674	1.50412134
0.7	1.49544645	1.49996515	1.50412134	1.50403623	1.5040823	1.50412134
0.8	1.49774786	1.50094284	1.50412134	1.50405665	1.50409157	1.50412134
0.9	1.50088051	1.50232444	1.50412134	1.50408508	1.50410457	1.50412134
1	1.50412134	1.50412134	1.50412134	1.50412134	1.50412134	1.50412134

Case 2: $\alpha = 6,\ p(t) = \frac{1}{(1+e^{-5t})^2},\ q(t) = \frac{1}{(1+e^{1-5t})^2}$ and $h(x) = 1/(1+e^x)^2$.
Table (3) and (4) describe the solution of (1.1) subject to (1.2) and (1.3) at $t = 0.4$ and $t = 1$ respectively by SICNS, SIS and HPM. Figures (3) is graphical representation of the solution to the problem (1.1), (1.2), (1.3) at $t = 0.4$ and $t = 1$. Figure (4) shows the solution graph of the problem (1.1), (1.2), (1.3) for $x \in [0, 1]$ and $t \in [0, 1]$.

Table 2. Solutions at $t = 1$ for $\alpha = 1$ and $n = 10$

	$m = 10$			$m = 1000$		
x	SICNS	SIS	HPM	SICNS	SIS	HPM
0	1.22539967	1.22539967	1.22539967	1.22539967	1.22539967	1.22539967
0.1	1.22455150	1.22472256	1.22539967	1.22539028	1.22539345	1.22539967
0.2	1.22369467	1.22419632	1.22539967	1.22538294	1.22538862	1.22539967
0.3	1.22315483	1.22382064	1.22539967	1.22537767	1.22538516	1.22539967
0.4	1.22286284	1.22359533	1.22539967	1.22537450	1.22538309	1.22539967
0.5	1.22276931	1.22352023	1.22539967	1.22537344	1.22538240	1.22539967
0.6	1.22286284	1.22359533	1.22539967	1.22537450	1.22538309	1.22539967
0.7	1.22315483	1.22382064	1.22539967	1.22537767	1.22538516	1.22539967
0.8	1.22369467	1.22419632	1.22539967	1.22538294	1.22538862	1.22539967
0.9	1.22455150	1.22472256	1.22539967	1.22539028	1.22539345	1.22539967
1	1.22539967	1.22539967	1.22539967	1.22539967	1.22539967	1.22539967

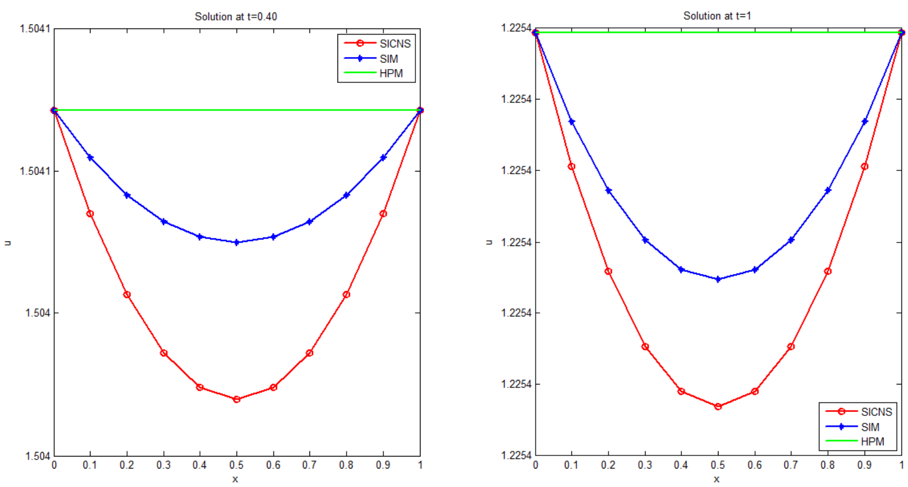

Figure 1. Solutions at $t = 0.4$ and $t = 1$ for $\alpha = 1$

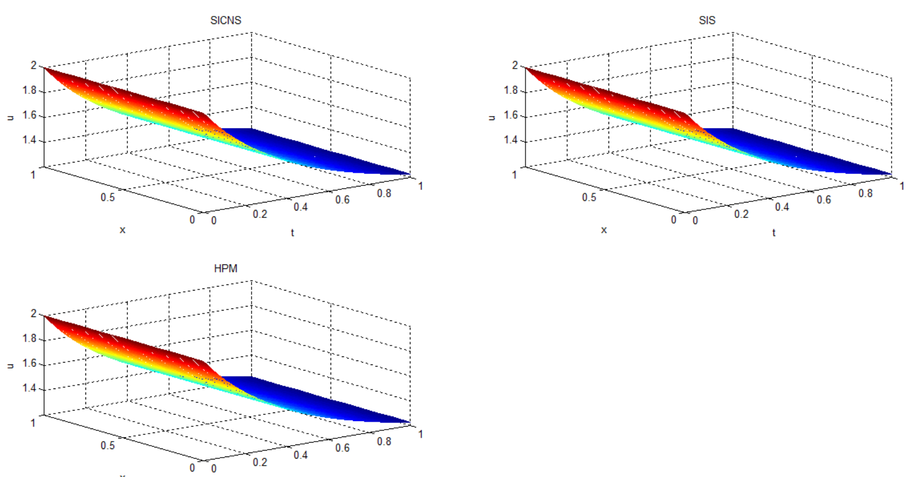

Figure 2. Solutions for $\alpha = 1$

Table 3. Solutions at $t = 0.4$ for $\alpha = 6$ and $n = 10$

	$m = 10$			$m = 1000$		
x	SICNS	SIS	HPM	SICNS	SIS	HPM
0	0.77580349	0.77580349	0.77580349	0.77580349	0.77580349	0.77580349
0.1	0.75869964	0.77289178	0.75671127	0.75673712	0.75685967	0.75671127
0.2	0.73880168	0.76541026	0.73641959	0.73645891	0.73668727	0.73641959
0.3	0.71693898	0.75339775	0.71492899	0.71497275	0.71528604	0.71492899
0.4	0.69351373	0.73681535	0.69225459	0.69229676	0.69266980	0.69225459
0.5	0.66882089	0.71556149	0.66842802	0.66846505	0.66886832	0.66842802
0.6	0.64314093	0.68949244	0.64349899	0.64352927	0.64392921	0.64349899
0.7	0.61674067	0.65844935	0.61753662	0.61755983	0.61791942	0.61753662
0.8	0.58983478	0.62229286	0.59063034	0.59064664	0.59092643	0.59063034
0.9	0.56249912	0.58094620	0.56289023	0.56289928	0.56305891	0.56289023
1	0.53444665	0.53444665	0.53444665	0.53444665	0.53444665	0.53444665

Table 4. Solutions at $t = 1$ for $\alpha = 6$ and $n = 10$

	$m = 10$			$m = 1000$		
x	SICNS	SIS	HPM	SICNS	SIS	HPM
0	0.98665909	0.98665909	0.98665909	0.98665909	0.98665909	0.98665909
0.1	0.98639814	0.98681787	0.98527155	0.98528220	0.98528483	0.98527155
0.2	0.98563931	0.98654046	0.98374149	0.98376090	0.98376573	0.98374149
0.3	0.98460238	0.98581877	0.98205464	0.98208084	0.98208742	0.98205464
0.4	0.98321761	0.98463008	0.98019543	0.98022629	0.98023409	0.98019543
0.5	0.98141656	0.98293650	0.97814683	0.97818004	0.97818847	0.97814683
0.6	0.97914963	0.98068393	0.97589023	0.97592326	0.97593165	0.97589023
0.7	0.97637011	0.97780079	0.97340537	0.97343537	0.97344299	0.97340537
0.8	0.97303397	0.97419624	0.97067021	0.97069398	0.97070000	0.97067021
0.9	0.96909538	0.96975821	0.96766077	0.96767471	0.96767822	0.96766077
1	0.96435108	0.96435108	0.96435108	0.96435108	0.96435108	0.96435108

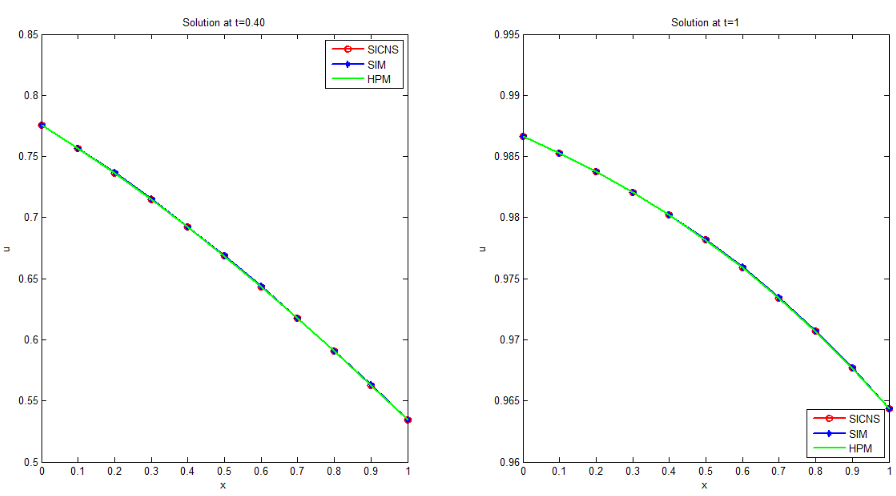

Figure 3. Solutions at $t = 0.4$ and $t = 1$ for $\alpha = 6$

Figure (4) shows the solution graph of the problem (1.1),(1.2), (1.3) for $x \in [0,1]$ and $t \in [0,1]$.

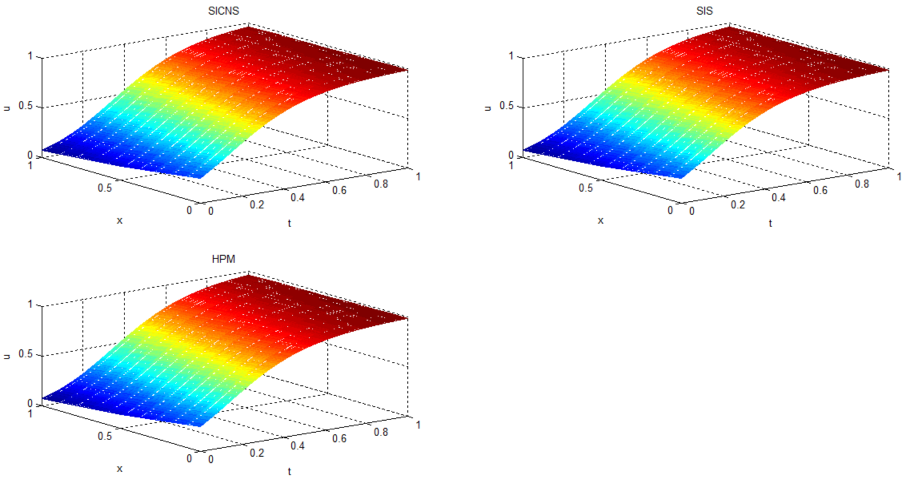

Figure 4. Solutions for $\alpha = 6$

As we have seen in the tables as well as in the figures the two numerical methods make good approximations to the exact solution for each case.

6.2 Numerical Errors

Here, numerical errors (absolute errors) are computed for the two methods in order to compare each other.
Case 1: $\alpha = 1,\ p(t) = \frac{\lambda e^t}{1-\lambda(1-e^t)},\ q(t) = \frac{\lambda e^t}{1-\lambda(1-e^t)}$ and $h(x) = \lambda = 2$.
Table (5) and (6) explain absolute errors computed at $t = 0.4$ and $t = 1$ by SICNS and SIS. Figure (5) shows numerical errors for $x \in [0,1]$ and $t \in [0,1]$.

Table 5. Numerical errors at $t = 0.4$ for $\alpha = 1$ and $n = 10$

	$m = 10$		$m = 1000$	
x	SICNS	SIS	SICNS	SIS
0	0	0	0	0
0.1	0.00324083	0.00179691	0.00003627	0.00001677
0.2	0.00637348	0.00317851	0.00006470	0.00002978
0.3	0.00867490	0.00415619	0.00008512	0.00003905
0.4	0.01002936	0.00473897	0.00009742	0.00004460
0.5	0.01047260	0.00493256	0.00010153	0.00004645
0.6	0.01002936	0.00473897	0.00009742	0.00004460
0.7	0.00867490	0.00415619	0.00008512	0.00003905
0.8	0.00637348	0.00317851	0.00006470	0.00002978
0.9	0.00324083	0.00179691	0.00003627	0.00001677
1	0	0	0	0

Case 2: $\alpha = 6,\ p(t) = \frac{1}{(1+e^{-5t})^2},\ q(t) = \frac{1}{(1+e^{1-5t})^2}$ and $h(x) = 1/(1+e^x)^2$.
Table (7) and (8) explains absolute errors computed at $t = 0.4$ and $t = 1$ by SICNS and SIS. Figure (6) shows numerical errors for $x \in [0,1]$ and $t \in [0,1]$.

Table 6. Numerical errors at $t = 1$ for $\alpha = 1$ and $n = 10$

	$m = 10$		$m = 1000$	
x	SICNS	SIS	SICNS	SIS
0	0	0	0	0
0.1	0.00084817	0.00067711	0.00000939	0.00000622
0.2	0.00170500	0.00120335	0.00001673	0.00001106
0.3	0.00224485	0.00157903	0.00002200	0.00001451
0.4	0.00253683	0.00180435	0.00002517	0.00001659
0.5	0.00263036	0.00187944	0.00002623	0.00001728
0.6	0.00253683	0.00180435	0.00002517	0.00001659
0.7	0.00224485	0.00157903	0.00002200	0.00001451
0.8	0.00170500	0.00120335	0.00001673	0.00001106
0.9	0.00084817	0.00067711	0.00000939	0.00000622
1	0	0	0	0

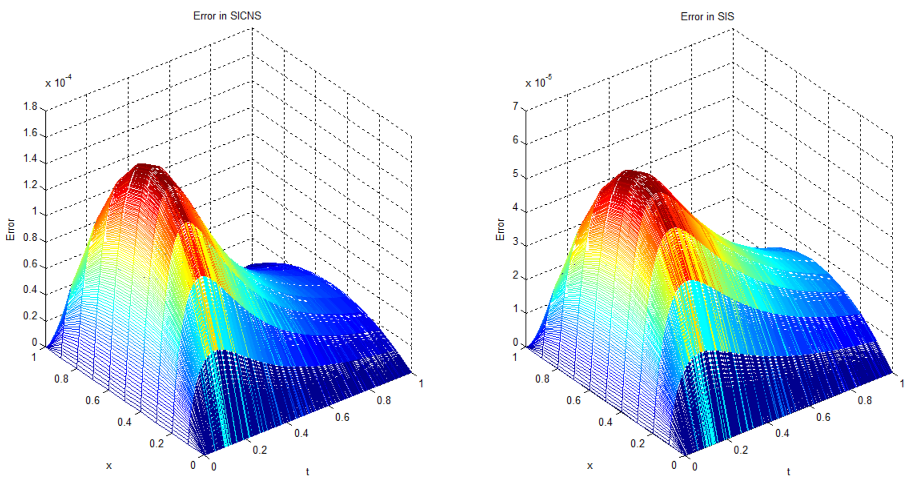

Figure 5. Numerical errors for $\alpha = 1$

Table 7. Numerical errors at $t = 0.4$ for $\alpha = 6$ and $n = 10$

	$m = 10$		$m = 1000$	
x	SICNS	SIS	SICNS	SIS
0	0	0	0	0
0.1	0.00198837	0.01618051	0.00002586	0.00014841
0.2	0.00238209	0.02899067	0.00003932	0.00026767
0.3	0.00200999	0.03846876	0.00004377	0.00035706
0.4	0.00125914	0.04456076	0.00004217	0.00041521
0.5	0.00039287	0.04713346	0.00003703	0.00044030
0.6	0.00035806	0.04599345	0.00003028	0.00043022
0.7	0.00079595	0.04091273	0.00002321	0.00038280
0.8	0.00079556	0.03166252	0.00001629	0.00029609
0.9	0.00039110	0.01805598	0.00000906	0.00016868
1	0	0	0	0

Table 8. Numerical errors at $t = 1$ for $\alpha = 6$ and $n = 10$

	$m = 10$		$m = 1000$	
x	SICNS	SIS	SICNS	SIS
0	0	0	0	0
0.1	0.00112659	0.00154632	0.00001065	0.00001327
0.2	0.00189783	0.00279898	0.00001941	0.00002425
0.3	0.00254773	0.00376413	0.00002619	0.00003277
0.4	0.00302217	0.00443465	0.00003086	0.00003865
0.5	0.00326973	0.00478967	0.00003322	0.00004164
0.6	0.00325941	0.00479370	0.00003303	0.00004142
0.7	0.00296474	0.00439541	0.00003000	0.00003762
0.8	0.00236376	0.00352603	0.00002378	0.00002980
0.9	0.00143461	0.00209744	0.00001394	0.00001745
1	0	0	0	0

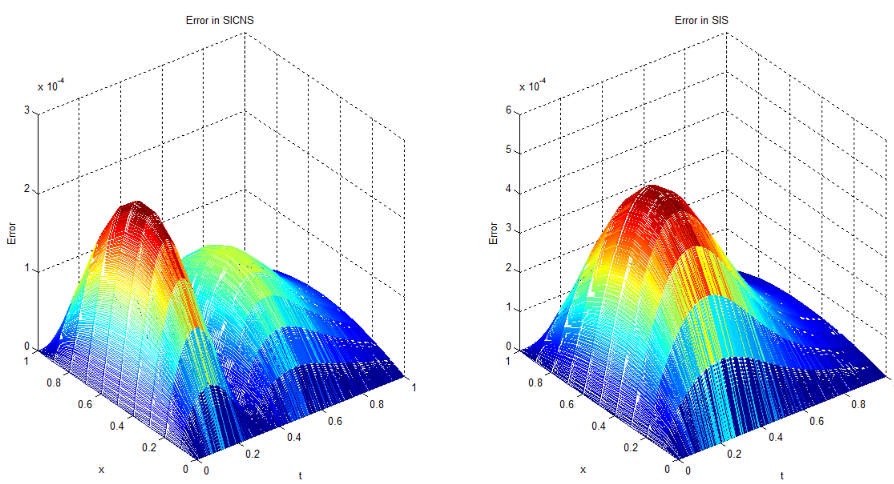

Figure 6. Numerical errors for $\alpha = 6$

7 Conclusion

In this work, we have successfully applied semi-implicit time discretization based on Crank-Nicolson scheme and semi-implicit to solve Fisher equation. The two numerical methods are second order accurate in space and first order accurate in time, and satisfactory for a wide range of time steps. Furthermore, the two numerical methods are conditionally stable.

8 Abbreviations

HPM	=	Homotopy perturbation method
SICNS	=	Semi-implicit time discretization based on Crank-Nicolson scheme
SIS	=	Semi-implicit scheme
PDE	=	Partial differential equation
FDE	=	Finite difference equation

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