Palestine Journal of Mathematics

Vol. 11(3)(2022) , 549–557

Inequalities for the generalized polar derivative of a polynomial

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Communicated by Simon Donaldson

MSC 2010 Classifications: Primary 26D10, 41A17; Secondary 30A10, 26D05.

Keywords and phrases: Polynomials, Generalized polar derivative, Gauss Lucas Theorem, Inequalities in the complex domain.

Abstract Recently Rather et al. [On the zeros of a class of generalized derivatives. Rend. Circ. Mat. Palermo, II. Ser (2020). https://doi.org/10.1007/s12215-020-00552-z] considered the generalized polar derivative and studied the relative position of zeros of the generalized polar derivative with respect to the zeros of the polynomial.
In this paper, we obtain some lower bound estimates for the generalized polar derivative of certain polynomials, which include various results due to Aziz and Rather, Malik, Turán and Govil as special cases.

1 Introduction

For each positive integer n, let $\mathcal{P}_n$ denote the set of all polynomials of degree n over the field $\mathbb{C}$ of complex numbers, $\partial\mathcal{P}_n$ denote the collection of all monic polynomials in $\mathcal{P}_n$ and $\mathbb{R}^n_+$ be the set of all n-tuples $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$ of non-negative real numbers (not all zeros) with $\gamma_1 + \gamma_2 + \cdots + \gamma_n = \wedge$.
The problem concerning the extremal properties of polynomials attracted interests in the second half of 19th century with some investigation of famous chemist Mendeleev who was interested to find the bound of the derivative of a special type of polynomial. It was Serge Bernstein, who formulated a result (for details see [17]) regarding the estimation of upper bound of the maximum modulus of the derived polynomial in terms of maximum modulus of the polynomial and proved that if $P(z) \in \mathcal{P}_n$, then

$$\underset{|z|=1}{max}|P'(z)| \leq n \underset{|z|=1}{max}|P(z)|. \tag{1.1}$$

This excellent introduction to the topic of polynomial inequalities attracts many researchers to this field and motivates them to find refinements of the result for different types of polynomials. On the other hand Paul Turán [18] was the first who estimated the lower bound for the maximum modulus of derived polynomial in terms of maximum modulus of polynomial. More precisely he proved that if $P(z) \in \mathcal{P}_n$ has all its zeros in $|z| \leq 1$, then

$$\underset{|z|=1}{max}|P'(z)| \geq \frac{n}{2}\underset{|z|=1}{max}|P(z)|. \tag{1.2}$$

The inequality (1.2) is best possible and become equality for the polynomials having all its zeros on $|z| = 1$.
It was Malik[8] who extended the inequality (1.2) by proving that if $P(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$, then

$$\underset{|z|=1}{max}|P'(z)| \geq \frac{n}{1+k}\underset{|z|=1}{max}|P(z)|. \tag{1.3}$$

Equality in (1.3) holds for the polynomial $P(z) = (z + k)^n$.
The case $k \geq 1$ was considered by Govil [7], who showed that if P(z) is a polynomial of degree n having all the zeros in $|z| \leq k$, $k \geq 1$, then

$$\underset{|z|=1}{max}|P'(z)| \geq \frac{n}{1+k^n}\underset{|z|=1}{max}|P(z)|. \tag{1.4}$$

The extremal polynomial is $P(z) = z^n + k^n$.

Let $D_\alpha[P](z)$ denote the polar differentiation (see [9]) of a polynomial $P(z)$ of degree n with respect to a complex number α, then

$$D_\alpha[P](z) = nP(z) + (\alpha - z)P'(z).$$

Note that the polynomial $D_\alpha[P](z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative $P'(z)$ of $P(z)$ in the sense that

$$\lim_{\alpha\to\infty} \frac{D_\alpha[P](z)}{\alpha} = P'(z)$$

uniformly with respect z for $|z| \leq R, R > 0$.
The Bernstein-type inequalities for the class of polynomials with ordinary derivative replaced by polar derivative have attracted number of mathematicians. In this direction, Aziz[2] was the first to establish inequalities concerning the polar derivative of a polynomial in terms of the modulus of the polynomial on the unit disk.
As an extension of inequality (1.1) to the polar derivative, Aziz [2] proved the following result. If $P(z)$ is a polynomial of degree n, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$, we have

$$|D_\alpha[P](z)| \leq n|\alpha z^{n-1}| \max_{|z|=1}|P(z)| \quad for \quad |z| \geq 1.$$

The result is best possible and equality in above inequality holds for $P(z) = cz^n, \quad c \neq 0$.
Concerning the class of polynomials having all zeros in $|z| \leq k$, Aziz and Rather obtained several sharp results concerning the maximum modulus of $D_\alpha[P](z)$ on $|z| = 1$. Among other things, they [3] established the following extension of inequality (1.3) and (1.4) to the polar derivative of a polynomial.

Theorem 1.1. *If all the zeros of P(z) lie in $|z| \leq k$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$.*

$$\max_{|z|=1}|D_\alpha[P](z)| \geq \frac{n}{1+k}(|\alpha| - k)\max_{|z|=1}|P(z)| \quad for \quad k \leq 1. \tag{1.5}$$

and

$$\max_{|z|=1}|D_\alpha[P](z)| \geq \frac{n}{1+k^n}(|\alpha| - k)\max_{|z|=1}|P(z)| \quad for \quad k \geq 1. \tag{1.6}$$

In literature, there exist several generalizations and refinements of these inequalities (For reference see [5],[11]-[16]).
By the fundamental theorem of algebra (see[9]), every polynomial $P \in \partial P_n$ can be written as $P(z) = \prod_{j=1}^{n}(z - z_j)$, where $z_1, z_2, \ldots, z_n$ are the zeros of $P(z)$ counted multiplicity.
Now for the polynomial $P(z)$, we define incomplete polynomials each of degree $n-1$, associated with the n zeros $z_1, z_2, ..., z_n$ of $P(z)$, are the polynomials $P_k(z), 1 \leq k \leq n$ given by

$$P_k(z) = \prod_{\substack{v=1\\ v\neq k}}^{n}(z - z_v).$$

We can easily notice that the derivative of $P(z)$, normalized to a monic polynomial, is a convex linear combination of incomplete polynomials $P_k(z)$. In fact, its derivative, reduced to monic polynomial, is

$$\frac{1}{n}P'(z) = \frac{1}{n}\sum_{k=1}^{n} P_k(z),$$

where all the coefficients of the convex linear combination are $\frac{1}{n}$.
Diaz Barrero and Egozcue [6] introduced the notion of linear combinations of incomplete polynomials corresponding to the n-tuple $\gamma := (\gamma_1, \gamma_2, \ldots, \gamma_n)$ of non-negative real numbers with $\sum\limits_{j=1}^{n} \gamma_j = 1$ as

$$P^{\gamma}(z) := \sum_{j=1}^{n} \gamma_j P_k(z).$$

Now if we choose $\gamma_j^{,}s$ such that $\sum\limits_{j=1}^{n} \gamma_j = \wedge$, then for $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \mathbb{R}_+^n$, we take

$$P^{\gamma}(z) := \sum_{j=1}^{n} \gamma_j \prod_{\substack{v=1 \\ v \neq j}}^{n} (z - z_v).$$

Noting that for $\gamma = (1, 1, 1, \ldots, 1), P^{\gamma}(z) = P'(z)$. In view of this, we call it generalized derivative of polynomial $P(z)$.
Next we define generalized polar derivative of $P(z)$ as

$$D_{\alpha}^{\gamma}[P](z) := \wedge P(z) + (\alpha - z) P^{\gamma}(z),$$

where $\wedge = \sum\limits_{j=1}^{n} \gamma_j$.
Noting that for $\gamma = (1, 1, 1, \ldots, 1), \quad D_{\alpha}^{\gamma}[P](z) = D_{\alpha} P(z)$.
Recently Rather et al. [10] extended the Gauss Lucas theorem to the class of generalized derivatives by proving the following result.

Theorem 1.2. *Every convex set containing all the zeros of $P(z)$ also contains the zeros of $P^{\gamma}(z)$ for all $\gamma \in \mathbb{R}_n^+$.*

Moreover, they [10] also obtained the following extension of the Laguerre's theorem to the class of generalized polar derivatives. In fact, they proved:

Theorem 1.3. *If all the zeros of the polynomial $P(z) \in \mathcal{P}_n$, lie in a circular region $\mathcal{C}$ and if ξ is a zero of $D_{\alpha}^{\gamma}[P](z)$ for some $\gamma \in \mathbb{R}_+^n$, then at most one of the points ξ and α may lie outside of $\mathcal{C}$.*

2 Main Results

In this section, we extend Theorem 1.1 to the class of generalized polar derivatives of the polynomial. We begin by proving the following result:

Theorem 2.1. *If all the zeros of a polynomial $P(z) \in \mathcal{P}_n$ lie in $|z| \leq k$ where $k \leq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$,*

$$\underset{|z|=1}{max} |D_{\alpha}^{\gamma}[P](z)| \geq \frac{\wedge}{1+k} (|\alpha| - k) \underset{|z|=1}{max} |P(z)|. \tag{2.1}$$

Remark 2.2. For the n-tuple $\gamma = (1, 1, 1, ..., 1)$, the inequality (2.1) reduces to the inequality (1.5).

Remark 2.3. If we divide both sides of inequality (2.1) by $|\alpha|$ and let $|\alpha| \to \infty$, and for n-tuple $\gamma = (1, 1, 1, ..., 1)$, the inequality reduces to inequality (1.3).

In the above theorem (2.1), if we put $k = 1$, we obtain the following corollary.

Corollary 2.4. *If all the zeros of a polynomial $P(z) \in \mathcal{P}_n$ lie in $|z| \leq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$,*

$$\underset{|z|=1}{max} |D_{\alpha}^{\gamma}[P](z)| \geq \frac{\wedge}{2} (|\alpha| - 1) \underset{|z|=1}{max} |P(z)|. \tag{2.2}$$

Remark 2.5. If we divide both sides of inequality (2.2) by $|\alpha|$ and let $|\alpha| \to \infty$, and for n-tuple $\gamma = (1, 1, 1, ..., 1)$, the inequality results to inequality (1.2).

Next, we also present the following result.

Theorem 2.6. *If all the zeros of a polynomial $P(z) \in \mathcal{P}_n$ lie in $|z| \leq k$ where $k \geq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$,*

$$\max_{|z|=1} |D_\alpha^\gamma[P](z)| \geq \frac{\wedge}{1+k^n}(|\alpha| - k)\max_{|z|=1}|P(z)|. \tag{2.3}$$

Remark 2.7. For the n-tuple $\gamma = (1, 1, 1, ..., 1)$, the inequality (2.3) reduces to the inequality (1.6).

Remark 2.8. If we divide both sides of inequality (2.3) by $|\alpha|$ and let $|\alpha| \to \infty$, and for n-tuple $\gamma = (1, 1, 1, ..., 1)$, then it reduces to inequality (1.4).

Now if we put k=1 in theorem (2.6), we again get corollary (2.4).

3 Lemmas

For the proof of above theorems, we need the following lemmas.

Lemma 3.1. *If $P(z)$ is a polynomial of degree n and $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$, then $for \quad |z| = 1$*

$$|Q^\gamma(z)| = |\wedge P(z) - zP^\gamma(z)|$$
$$and \quad |P^\gamma(z)| = |\wedge Q(z) - zQ^\gamma(z)|.$$

We can easily verify the above lemma by taking $z = e^{i\theta}; 0 \leq \theta < 2\pi$.

Lemma 3.2. *If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < k, k \geq 1$, then*

$$k|P^\gamma(z)| \leq |Q^\gamma(z)| \quad for \quad |z| = 1,$$

where $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$.

Proof. Since $P(z)$ is a polynomial of degree n having all its zeros in $|z| \geq k; k \geq 1$, it follows that all the zeros of the polynomial $F(z) = P(kz)$ lie in $|z| \geq 1$.
Now we can easily verify that if H(z) is a polynomial of degree n having all its zeros in $|z| \geq 1$, then

$$|H^\gamma(z)| \leq |\wedge H(z) - zH^\gamma(z)| \quad for \quad |z| = 1. \tag{3.1}$$

Applying (3.1) to the polynomial F(z), we get

$$|F^\gamma(z)| \leq |\wedge F(z) - zF^\gamma(z)| \quad for \quad |z| = 1. \tag{3.2}$$

We show inequality (3.2) also holds for $|z| < 1$.
Since F(z) is a polynomial of degree n having all its zeros in $|z| \geq 1$, it follows by Theorem C with $\alpha = 0$ and $\mathbb{C}$ replaced by $|z| \geq 1$, that the polynomial $\wedge F(z) - zF^\gamma(z)$ has all its zeros in $|z| \geq 1$.
Since for $|z| = 1$,

$$|F^\gamma(z)| \leq |\wedge F(z) - zF^\gamma(z)|,$$

it follows that the function

$$G(z) = \frac{F^\gamma(z)}{\wedge F(z) - zF^\gamma(z)}$$

is analytic in $|z| \leq 1$ and

$$|G(z)| \leq 1 \quad for \quad |z| = 1.$$

Hence by Maximum Modulus theorem, we conclude that

$$|G(z)| \leq 1 \quad for \quad |z| \leq 1.$$

Equivalently, we have

$$|F^{\gamma}(z)| \leq | \wedge F(z) - zF^{\gamma}(z)| \quad for \quad |z| \leq 1.$$

Thus inequality (3.2) also holds for $|z| < 1$.
Since $k \geq 1$, we take in particular

$$z = \frac{e^{i\theta}}{k} \quad 0 \leq \theta < 2\pi.$$

Then for $|z| = \frac{1}{k} \leq 1$ and from inequality (3.2), we get

$$\left|F^{\gamma}\left(\frac{e^{i\theta}}{k}\right)\right| \leq \left| \wedge F\left(\frac{e^{i\theta}}{k}\right) - \frac{e^{i\theta}}{k}F^{\gamma}\left(\frac{e^{i\theta}}{k}\right)\right| \quad ; \quad 0 \leq \theta < 2\pi.$$

Which implies,

$$\left|kP^{\gamma}\left(e^{i\theta}\right)\right| \leq \left| \wedge P\left(e^{i\theta}\right) - e^{i\theta}P^{\gamma}\left(e^{i\theta}\right)\right| \quad ; \quad 0 \leq \theta < 2\pi.$$

On using lemma 3.1, we get for $|z| = 1$

$$k|P^{\gamma}(z)| \leq | \wedge P(z) - zP^{\gamma}(z)| = |Q^{\gamma}(z)|.$$

$$\Rightarrow k|P^{\gamma}(z)| \leq |Q^{\gamma}(z)| \quad for \quad |z| = 1, \quad where \quad Q(z) = z^n\overline{P(\frac{1}{\overline{z}})},$$

that completes the proof of the lemma. □

Lemma 3.3. *If $P(z)$ is a polynomial of degree n having all zeros in $|z| \leq k$ where $k \leq 1$,then*

$$k|P^{\gamma}(z)| \geq |Q^{\gamma}(z)| \quad for \quad |z| = 1,$$

where $Q(z) = z^n\overline{P(\frac{1}{\overline{z}})}$.

Proof. Since $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, where $k \leq 1$. Hence all the zeros of the polynomial $Q(z)$ lie in $|z| \geq \frac{1}{k}$.

$$\Rightarrow Q(z) \neq 0 \quad for \quad |z| < \frac{1}{k}, \quad where \quad \frac{1}{k} \geq 1.$$

Therefore by applying lemma (3.2) to the polynomial Q(z), we get

$$\frac{1}{k}|Q^{\gamma}(z)| \leq |P^{\gamma}(z)| \quad for \quad |z| = 1.$$

$$\Rightarrow k|P^{\gamma}(z)| \geq |Q^{\gamma}(z)| \quad for \quad |z| = 1.$$

which proves lemma (3.3). □

Lemma 3.4. *If $P(z)$ is a polynomial of degree n, then*

$$|P^{\gamma}(z)| + |Q^{\gamma}(z)| \geq \wedge|P(z)| \quad for \quad |z| = 1,$$

where $Q(z) = z^n\overline{P(\frac{1}{\overline{z}})}$

Proof. Since by lemma 3.1,

$$|P^{\gamma}(z)| = |\wedge Q(z) - zQ^{\gamma}(z)| \quad for \quad |z| = 1$$
$$\geq |\wedge Q(z)| - |zQ^{\gamma}(z)| \quad for \quad |z| = 1.$$

$$\Rightarrow |P^{\gamma}(z)| + |Q^{\gamma}(z)| \geq \wedge |Q(z)| \quad for \quad |z| = 1.$$

On using the fact that for $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$,

$$|Q(z)| = |P(z)| \quad for \quad |z| = 1,$$

we get,

$$|P^{\gamma}(z)| + |Q^{\gamma}(z)| \geq \wedge |P(z)| \quad for \quad |z| = 1.$$

Which proves lemma (3.4) □

From a simple consequence of Maximum Modulus theorem, we can get the following result.

Lemma 3.5. *If $P(z)$ is a polynomial of degree n, then for $R \geq 1$*

$$\underset{|z|=R}{max}|P(z)| \leq R^n \underset{|z|=1}{max}|P(z)|.$$

Lemma 3.6. *If $P(z) \in \mathcal{P}_n$, having all its zeros in the disk $|z| \leq k$ where $k \geq 1$, then*

$$\underset{|z|=k}{max}|P(z)| \geq \frac{2k^n}{1+k^n} \underset{|z|=1}{max}|P(z)|. \tag{3.3}$$

Lemma (3.6) was due to A.Aziz [1].

4 Proof of Theorems

Proof of Theorem 2.1. By using triangle's inequality, we get

$$|D_{\alpha}^{\gamma}[P](z)| = |\wedge P(z) + (\alpha - z)P^{\gamma}(z)|$$
$$= |\wedge P(z) + \alpha P^{\gamma}(z) - zP^{\gamma}(z)|$$
$$\geq |\alpha||P^{\gamma}(z)| - |\wedge P(z) - zP^{\gamma}(z)|.$$

On using lemma (3.1) and lemma (3.3) this implies, for $|z| = 1$

$$|D_{\alpha}^{\gamma}[P](z)| \geq |\alpha||P^{\gamma}(z)| - |Q^{\gamma}(z)|$$
$$\geq |\alpha||P^{\gamma}(z)| - k|P^{\gamma}(z)|$$
$$= (|\alpha| - k)|P^{\gamma}(z)|.$$

That is

$$|D_{\alpha}^{\gamma}[P](z)| \geq (|\alpha| - k)|P^{\gamma}(z)|. \tag{4.1}$$

Now from lemma (3.4), we have

$$|P^{\gamma}(z)| + |Q^{\gamma}(z)| \geq \wedge |P(z)| \quad for \quad |z| = 1.$$

On using lemma (3.3), we get

$$|P^{\gamma}(z)| + k|P^{\gamma}(z)| \geq \wedge |P(z)| \qquad for \quad |z| = 1.$$

Which implies

$$(1 + k)|P^{\gamma}(z)| \geq \wedge |P(z)| \qquad for \quad |z| = 1.$$

Thus,

$$|P^{\gamma}(z)| \geq \frac{\wedge}{1+k}|P(z)| \qquad for \quad |z|=1 \tag{4.2}$$

On using the above inequality (4.2) in the inequality (4.1), we get for $|z|=1$

$$\max_{|z|=1}|D_{\alpha}^{\gamma}[P](z)| \geq (|\alpha|-k)\frac{\wedge}{1+k}\max_{|z|=1}|P(z)|.$$

Which proves the theorem.

□

Proof of Theorem 2.6. Since the polynomial $P(z)$ has all its zeros in $|z| \leq k$, where $k \geq 1$. Hence the polynomial $F(z)=P(kz)$ has all its zeros in $|z| \leq 1$.
Applying corollary (2.4) to the polynomial $F(z)$ and noting that $\left|\frac{\alpha}{k}\right| \geq 1$, we get

$$\max_{|z|=1}\left|D_{\frac{\alpha}{k}}^{\gamma}[F](z)\right| \geq \frac{\wedge}{2}\left(\frac{|\alpha|}{k}-1\right)\max_{|z|=1}|F(z)|.$$

Replacing $F(z)$ by $P(kz)$, we get

$$\begin{aligned}\max_{|z|=1}\left|D_{\frac{\alpha}{k}}^{\gamma}[P](kz)\right| &\geq \frac{\wedge}{2}\left(\frac{|\alpha|}{k}-1\right)\max_{|z|=1}|P(kz)| \\ &= \frac{\wedge}{2}\left(\frac{|\alpha|-k}{k}\right)\max_{|z|=k}|P(z)|.\end{aligned}$$

With the help of lemma (3.6), this implies

$$\begin{aligned}\max_{|z|=1}\left|D_{\frac{\alpha}{k}}^{\gamma}[P](kz)\right| &\geq \frac{\wedge}{2}\left(\frac{|\alpha|-k}{k}\right)\max_{|z|=k}|P(z)| \\ &\geq \frac{\wedge}{2}\left(\frac{|\alpha|-k}{k}\right)\frac{2k^n}{1+k^n}\max_{|z|=1}|P(z)|.\end{aligned}$$

Which gives

$$\max_{|z|=1}\left|D_{\frac{\alpha}{k}}^{\gamma}[P](kz)\right| \geq \wedge\left(\frac{|\alpha|-k}{1+k^n}\right)k^{n-1}\max_{|z|=1}|P(z)|. \tag{4.3}$$

Also we have

$$\begin{aligned}\max_{|z|=1}\left|D_{\frac{\alpha}{k}}^{\gamma}[P](kz)\right| &= \max_{|z|=1}\left|\wedge P(kz)+\left(\frac{\alpha}{k}-z\right)P^{\gamma}(kz)\right| \\ &= \max_{|z|=1}\left|\wedge P(kz)+\left(\frac{\alpha-kz}{k}\right)P(kz)\sum_{j=1}^{n}\frac{\gamma_j}{z-\frac{z_j}{k}}\right| \\ &= \max_{|z|=1}\left|\wedge P(kz)+\left(\frac{\alpha-kz}{k}\right)kP(kz)\sum_{j=1}^{n}\frac{\gamma_j}{kz-z_j}\right| \\ &= \max_{|z|=1}\left|\wedge P(kz)+(\alpha-kz)P(kz)\sum_{j=1}^{n}\frac{\gamma_j}{kz-z_j}\right| \\ &= \max_{|z|=1}|G(kz)| \\ &= \max_{|z|=k}|G(z)|,\end{aligned}$$

where $G(z) = \wedge P(z) + (\alpha - z)P(z)\sum_{j=1}^{n}\frac{\gamma_j}{z-z_j}$ is a polynomial of degree atmost $n-1$.
On using lemma (3.5), this gives

$$\begin{aligned}
\max_{|z|=1}\left|D^{\gamma}_{\frac{\alpha}{k}}[P](kz)\right| &= \max_{|z|=k}|G(z)| \\
&\leq k^{n-1}\max_{|z|=1}|G(z)| \\
&= k^{n-1}\max_{|z|=1}\left|\wedge P(z) + (\alpha - z)P(z)\sum_{j=1}^{n}\frac{\gamma_j}{z-z_j}\right| \\
&= k^{n-1}\max_{|z|=1}|D^{\gamma}_{\alpha}[P](z)|.
\end{aligned}$$

Hence,

$$\max_{|z|=1}\left|D^{\gamma}_{\frac{\alpha}{k}}[P](kz)\right| \leq k^{n-1}\max_{|z|=1}|D^{\gamma}_{\alpha}[P](z)|. \tag{4.4}$$

On combining inequalities (4.3) and (4.4), we get

$$k^{n-1}\max_{|z|=1}|D^{\gamma}_{\alpha}[P](z)| \geq \wedge\left(\frac{|\alpha|-k}{1+k^n}\right)k^{n-1}\max_{|z|=1}|P(z)|.$$

This implies,

$$\max_{|z|=1}|D^{\gamma}_{\alpha}[P](z)| \geq \wedge\left(\frac{|\alpha|-k}{1+k^n}\right)\max_{|z|=1}|P(z)|.$$

Which proves the result. □

Acknowledgements. The authors would like to thank the anonymous referee for comments and suggestions.

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Received: September 10, 2021
Accepted: February 1, 2022