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Statistical Relative Uniform Convergence of Double Sequence of Positive Linear Functions

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Abstract In this article we define the notion of statistical relative uniform convergence, statistical relative uniform Cauchy, regular statistical relative uniform convergence and regular statistical relative uniform Cauchy of double sequence of functions over compact subset D of real numbers. We study the relation between the notions and the decomposition theorem is proved. We have derived some results on algebraic properties of these notions and other related results.

1 Introduction

Sequence spaces has been studied from various aspects by [26], [9] and many others. A double sequence is a double infinite array of numbers by (x_{nk}). The notion of double sequence was introduced by Pringsheim [19]. It is also found in Bromwich [3]. Hardy [13] introduced the notion of regular convergence of double sequence. The double sequence has been investigated from different aspects by Moricz [18], Basarir and Sonalcan [1], Tripathy and Sarma [25], Yurdal and Erdinc [22] and others. The notion of statistical convergence was introduced by Fast [10] and Schoenberg [21] independently. Later on it was studied from sequence space point of view and linked with summability theory by Fridy [11], Gokhan et al. [12], Tripathy [23, 24], Mohiuddine and Alamri [17], Kadak and Mohiuddine [15], Belen and Mohiuddine [2], Hazarika et al. [14] and others. The idea depends on the notion of asymptotic density of subsets of N, the set of natural numbers.
A subset D of N is said to have density $\rho(D)$ if

$$\rho(D)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\chi_D(k) \text{ exists, where } \chi_D \text{ is the characteristics function of } D.$$

A subset D of $N\times N$ is said to have density $\delta(D)$ if

$$\delta(D)=\lim_{p,q\to\infty}\frac{1}{pq}\sum_{n\le p}\sum_{k\le q}\chi_D(n,k) \text{ exists, } \delta(D^c)=\delta(N\times N-D)=1-\delta(D).$$

If a sequence (x_{nk}) fails to satisfy a property over a subset of $N\times N$ of density zero, then (x_{nk}) is said to satisfy property P for almost all n and k, written as $a.a.n$ and k.

A double sequence (x_{nk}) is said to converge in Pringsheim's sense if,
$\lim_{n,k\to\infty} x_{nk} = L$, both n and k tend to ∞ independent of one another.

A double sequence (x_{nk}) is said to converge regularly if it converges in Pringsheim's sense and in addition the following limit exist:

$$\lim_{n\to\infty} x_{nk} = L_k, \text{ for each } k\in N;$$

and

$$\lim_{k\to\infty} x_{nk} = M_n, \text{ for each } n\in N.$$

The notion of uniform convergence of sequence of functions relative to scale function was given by E. H. Moore. Chittenden [4] gave a formulation of the definition given by Moore as follows:

Definition 1.1. A sequence (f_n) of real, single-valued functions f_n of a real variable x, ranging over a compact subset D of real numbers, converges relatively uniformly on D, in case there exist functions g and σ, defined on D, and for every $\varepsilon > 0$, there exists an integer n_o(dependent on ε) such that for every $n \geq n_o$,
the inequality $\mid g(x) - f_n(x) \mid < \varepsilon \mid \sigma(x) \mid$, holds for every element x of D.

The function σ of the above definition is called a scale function. The sequence (f_n) is said to converge uniformly relative to the scale function σ.
The notion was further studied by [5] [6] [7] [8] [16] [20] and many others.

2 Preliminaries and definitions

In this section we procure some existing definitions, also introduce some new notions. Also some basic properties have been listed as results.

Definition 2.1. A double sequence of functions (f_{nk}) is said to be statistically uniformly convergent to $f(x)$ on a compact subset D, if for every $\varepsilon > 0$,

$$\lim_{p,q\to\infty} \frac{1}{pq} \mid \{n \leq p; k \leq q :\mid f_{nk}(x) - f(x) \mid \geq \varepsilon\} \mid = 0, \text{ for all } x \in D.$$

i.e., for all $x \in D, \mid f_{nk}(x) - f(x) \mid < \varepsilon$, for $a.a.n$ and k.
Symbolically, we write $f_{nk} \to f(stat)$ uniformly on D.

Definition 2.2. A double sequence of functions (f_{nk}) defined on a compact subset D of real numbers converges statistically relatively uniformly in Pringsheim's sense to a limit function $f(x)$, if there exists a scale function $\sigma(x)$ such that for every $\varepsilon > 0$,

$$\lim_{p,q\to\infty} \frac{1}{pq} \mid \{n \leq p; k \leq q :\mid f_{nk}(x) - f(x) \mid \geq \varepsilon \mid \sigma(x) \mid\} \mid, \text{ for all } x \in D.$$

i.e., for all $x \in D, \mid f_{nk}(x) - f(x) \mid < \varepsilon \mid \sigma(x) \mid, \quad a.a.n$ and k.

It is denoted by $f_{nk} \to f(stat)$ relative uniformly on D.

Remark 2.1. When $f = \theta$, the zero function, we get the definition of statistical relative uniform null from the Definition 2.2.

Lemma 2.1. *A double sequence of functions $(f_{nk}(x))$, is statistically uniformly convergent on D implies that $(f_{nk}(x))$ is statistically relatively uniformly convergent on D with respect to a scale function $\sigma(x)$ but the converse is not necessarily true.*

Proof. First part of the statement is obvious, since the scale function is a constant function on D. The converse part follows from the example given below.

Example 2.1. Let $0 < a < 1$ be a real number and $D = [a, 1]$. Consider the sequence of real valued functions $(f_{nk}(x))$, $f_{nk} : [a, 1] \to R$, for all $n, k \in N$, defined on D by

$$f_{nk}(x) = \begin{cases} x, & \text{for } n = k = i^2, i \in N; \\ 0 & \text{otherwise.} \end{cases}$$

Then, we find that $(f_{nk}(x))$ is statistically relative uniform convergent on $[a, 1]$ w.r.t. the scale function

$$\sigma(x) = \frac{1}{x}, \text{ for all } x \in [a, 1].$$

However, the function $(f_{nk}(x))$ is not statistically uniformly convergent on the interval $[a, 1]$.

Based on the above definitions, we state some relation between statistical uniform convergence and statistical relative uniform convergence of double sequence of functions without proof.

Result 2.1. Statistical uniform convergence of double sequence of functions relative to constant scale function different from zero is equivalent to statistical uniform convergence of double sequence of functions.

Result 2.2. Statistical uniform convergence of double sequence of functions relative to σ implies statistical uniform convergence relative to every function τ such that $\mid \tau(x) \mid \geq \mid \sigma(x) \mid$, for all $x \in D$.

Result 2.3. Statistical uniform convergence of double sequence of functions relative to a scale function σ such that $A \leq \mid \sigma \mid \leq B$, where A and B are positive implies statistical uniform convergence of double sequence of functions.

Result 2.4. If a double sequence of functions is not statistically uniformly convergent but converges statistically relatively uniformly to a scale function σ, then the scale function is not bounded.

Result 2.5. If σ is bounded from zero and infinity, then the statistical relative uniform convergence is equivalent to statistical uniform convergence of double sequence of functions.

Definition 2.3. A double sequence of functions (f_{nk}) is said to be statistically relatively uniformly Cauchy on a compact subset D of real numbers, for every $\varepsilon > 0$, there exist $s = s(\varepsilon), t = t(\varepsilon)$ and function $\sigma(x)$ defined on D such that

$$\lim_{p,q\to\infty} \frac{1}{pq} \mid \{n \leq p; k \leq q : \mid f_{nk}(x) - f_{st}(x) \mid \geq \varepsilon \mid \sigma(x) \mid\} \mid = 0.$$

i.e., $\mid f_{nk}(x) - f_{st}(x) \mid \leq \varepsilon|\sigma(x)|$, $a.a.n$ and k.

Definition 2.4. A double sequence of functions (f_{nk}) is said to be regularly statistically relatively uniformly convergent on a compact subset D, if there exist functions $g(x), g_k(x), f_n(x), \sigma(x), \xi_n(x)$ and $\eta_k(x)$ such that, for every $\varepsilon > 0$ and for all $x \in D$,

$$\lim_{p,q\to\infty} \frac{1}{pq} \mid \{n \leq p; k \leq q : \mid f_{nk}(x) - g(x) \mid \geq \varepsilon \mid \sigma(x) \mid\} \mid = 0, \text{ for all } n, k \in N;$$

$$\lim_{q\to\infty} \frac{1}{q} \mid \{k \leq q : \mid f_{nk}(x) - f_n(x) \mid \geq \varepsilon \mid \xi_n(x) \mid\} \mid = 0, \text{ for each } n \in N;$$

$$\lim_{p\to\infty} \frac{1}{p} \mid \{n \leq p : \mid f_{nk}(x) - g_k(x) \mid \geq \varepsilon \mid \eta_k(x) \mid\} \mid = 0, \text{ for each } k \in N.$$

Remark 2.2. When $g = g_k = f_n = \theta$, the zero function, we get the definition of regular null statistical relative uniform from the Definition 2.4.

Definition 2.5. A double sequence of functions (f_{nk}) defined on a compact subset D is said to be regularly statistically relatively uniformly Cauchy, if for every $\varepsilon > 0$, there exist integers $s = s(\varepsilon), t = t(\varepsilon)$ and scale functions $\sigma(x), \xi_n(x), \eta_k(x)$ defined on D such that for all $x \in D$,

$$\mid f_{nk}(x) - f_{st}(x) \mid \leq \varepsilon \mid \sigma(x) \mid, \; a.a.n \text{ and } k, \text{ for all } n, k \in N;$$

$$\mid f_{nk}(x) - f_{nt}(x) \mid \leq \varepsilon \mid \xi_n(x) \mid, \; a.a.k, \text{ for each } n \in N;$$

$$\mid f_{nk}(x) - f_{sk}(x) \mid \leq \varepsilon \mid \eta_k(x) \mid, \; a.a.n, \text{ for each } k \in N.$$

Lemma 2.2. *A double sequence of functions $(f_{nk}(x))$ is statistically regularly relatively uni-*

formly convergent on D implies that it is statistically relative uniform convergent on D but not necessarily conversely.

The converse part follows from the following example.

Example 2.2. Let $0 < a < 1$ be a real number and $D = [a, 1]$. Consider the sequence of real valued functions $(f_{nk}(x))$, $f_{nk} : [a, 1] \to R$, for all $n, k \in N$, defined on D by

$$f_{nk}(x) = \begin{cases} x, & \text{for } n = 1 \text{ and } k \text{ is odd}, k \in N; \\ 0 & \text{otherwise.} \end{cases}$$

$(f_{nk}(x))$ is statistically relatively uniformly convergent on $[a, 1]$ with respect to the scale function

$$\sigma(x) = \frac{1}{x}, \text{ for all } x \in [a, 1].$$

However, one cannot get a scale function that makes first row of $(f_{nk}(x))$ convergent.

Hence, $(f_{nk}(x))$ is not regularly statistically relatively uniformly convergent.

Definition 2.6. A double sequence of functions (f_{nk}) is said to be strong double r- *Cesáro* summable to $f(x)$ if

$$\lim_{pq\to\infty} \frac{1}{pq} \sum_{n=1}^{p} \sum_{k=1}^{q} | f_{nk}(x) - f(x) |^r = 0.$$

If $r = 1$, then, the above expression reduces to

$$\lim_{pq\to\infty} \frac{1}{pq} \sum_{n=1}^{p} \sum_{k=1}^{q} | f_{nk}(x) - f(x) | = 0.$$

Definition 2.7. A double sequence of functions $(f_{nk}(x))$ is said to be statistical regular relative bounded, if there exist positive real number M and scale functions $\sigma(x), \xi_n(x), \eta_k(x)$ such that

$$\lim_{p,q\to\infty} \frac{1}{pq} | \{n \le p; k \le q :| f_{nk}(x) |> M | \sigma(x) |\} |= 0;$$

$$\lim_{q\to\infty} \frac{1}{q} | \{k \le q; \text{ for each } n \in N :| f_{nk}(x) |> M | \xi_n(x) |\} |= 0;$$

$$\lim_{p\to\infty} \frac{1}{p} | \{n \le p; \text{ for each } k \in N :| f_{nk}(x) |> M | \eta_k(x) |\} |= 0.$$

The above equations hold for every $x \in D$.

3 Main results

In this section we establish the results of this article.

Theorem 3.1. *A double sequence of functions (f_{nk}), defined on a compact domain D, converges statistically relatively uniformly on D if and only if it is statistically relatively uniformly Cauchy.*

Proof. Let (f_{nk}) be statistically relatively uniformly convergent on a compact subset D of real numbers. Then, there exists a scale function $\sigma(x)$ such that for every $\varepsilon > 0$ and for all $x \in D$,

$$\lim_{p,q\to\infty} \frac{1}{pq} |\{n \le p; k \le q :| f_{nk}(x) - f_{st}(x) |\ge \tfrac{\varepsilon}{2} | \sigma(x) |\}| = 0.$$

$$| f_{nk}(x) - f(x) |< \tfrac{\varepsilon}{2},\ a.a.n \text{ and } k.$$

$$| f_{st}(x) - f(x) |< \tfrac{\varepsilon}{2},\ a.a.s \text{ and } t.$$

$$| f_{nk}(x) - f_{st}(x) |=| f_{nk}(x) - f(x) + f(x) - f_{st}(x) |$$

$$\leq | f_{nk}(x) - f(x) | + | f(x) - f_{st}(x) |$$

$$< \tfrac{\varepsilon}{2} | \sigma(x) | + \tfrac{\varepsilon}{2} | \sigma(x) |, \ a.a.n \text{ and } k$$

$$< \varepsilon | \sigma(x) |, \ a.a.n \text{ and } k.$$

i.e., $\lim_{p,q\to\infty} \frac{1}{pq}|\{n \leq p; k \leq q :| f_{nk}(x) - f_{st}(x) |\geq \varepsilon | \sigma(x) |\}| = 0$, for all $x \in D$.

Hence, (f_{nk}) is statistically relatively uniformly Cauchy.

Conversely, Let (f_{nk}) be statistically relatively uniformly Cauchy on a compact subset D of real numbers. Then, for every $\varepsilon > 0$, there exist integers
$s = s(\varepsilon), t = t(\varepsilon)$ and function $\sigma(x)$ defined on D such that,

$$\lim_{p,q \to \infty} \frac{1}{pq}|\{n \leq p; k \leq q :| f_{nk}(x) - f_{st}(x) |\geq \varepsilon | \sigma(x) |\} |= 0, \text{ for all } x \in D.$$

$$| f_{nk}(x) - f_{st}(x) |\leq \varepsilon, \ a.a.n \text{ and } k, \text{ for all } x \in D. \tag{3.1}$$

Since, (f_{nk}) is statistically relatively uniformly Cauchy for single sequences (row wise and column wise respectively) and Cauchy implies convergent, we have,

$$| f_{nk}(x) - f_n(x) |< \frac{\varepsilon}{2} | \sigma(x) |, \ a.a.k, \text{ for all } x \in D. \tag{3.2}$$

$$| f_{nk}(x) - g_k(x) |< \frac{\varepsilon}{2} | \sigma(x) |, \ a.a.n, \text{ for all } x \in D. \tag{3.3}$$

By equation (3.1), the scale function will be same for both (3.2) and (3.3).
We have from (3.2),

$$| f_k(x) - f_n(x) |< \varepsilon | \sigma(x) |, \ a.a.n, \text{ for all } x \in D.$$

This implies (f_n) is statistical relatively uniformly Cauchy w.r.t. scale function $\sigma(x)$.
Every Cauchy sequence of real or complex number is convergent. Thus, there exists a function $f(x)$ such that for every $\varepsilon > 0$ and for all $x \in D$,

$$| f_n(x) - f(x) |< \frac{\varepsilon}{2}|\sigma(x)|, \ a.a.n.$$

Hence,

$$\begin{aligned} | f_{nk}(x) - f(x) | &=| f_{nk}(x) - f_n(x) + f_n(x) - f(x) | \\ &\leq| f_{nk}(x) - f_n(x) | + | f_n(x) - f(x) | \\ &< \frac{\varepsilon}{2} | \sigma(x) | + \frac{\varepsilon}{2} | \sigma(x) |, \ \ a.a.n \text{ and } k, \text{ for all } x \in D. \end{aligned}$$

$$| f_{nk}(x) - f(x) |< \varepsilon | \sigma(x) |, \ a.a.n \text{ and } k, \text{ for all } x \in D. \tag{3.4}$$

Similarly, we can show that (g_k) is a statistical relative uniform Cauchy w.r.t. scale function $\sigma(x)$. Then, there exists a function $g(x)$ on D such that for all $x \in D$,

$$| g_k(x) - g(x) |< \frac{\varepsilon}{2}|\sigma(x)|, \ a.a.k.$$

Hence,

$$\begin{aligned} | f_{nk}(x) - g(x) | &=| f_{nk}(x) - g_k(x) + g_k(x) - g(x) | \\ &\leq| f_{nk}(x) - g_k(x) | + | g_k(x) - g(x) | \\ &< \frac{\varepsilon}{2} | \sigma(x) | + \frac{\varepsilon}{2} | \sigma(x) |, \ a.a.n \text{ and } k, \text{ for all } x \in D \end{aligned}$$

$$| f_{nk}(x) - g(x) |< \varepsilon \mid \sigma(x) \mid, \; a.a.n \text{ and } k, \; \text{for all } x \in D. \tag{3.5}$$

From equation (3.4) and (3.5), we get $(f_{nk}(x))$ converges statistically relatively uniformly for $a.a.n$ and k, for all $x \in D$.

Next, for $a.a.n$ and k we have,

$$\begin{aligned}
| f(x) - g(x) |=& | f(x) - f_n(x) + f_n(x) - f_{nk}(x) + f_{nk}(x) - g_k(x) + g_k(x) - g(x) | \\
&\leq | f(x) - f_n(x) | + | f_n(x) - f_{nk}(x) | \\
&\quad + | f_{nk}(x) - g_k(x) | + | g_k(x) - g(x) | \\
&< \tfrac{\varepsilon}{2} \mid \sigma(x) \mid + \tfrac{\varepsilon}{2} \mid \sigma(x) \mid + \tfrac{\varepsilon}{2} \mid \sigma(x) \mid + \tfrac{\varepsilon}{2} \mid \sigma(x) \mid, \\
&< 2\varepsilon \mid \sigma(x) \mid.
\end{aligned}$$

This implies that $f(x) = g(x)$, for all $x \in D$, relative to scale function $\sigma(x)$.

Theorem 3.2. *A double sequence of function (f_{nk}) defined on a compact domain D, converges regularly statistically relatively uniformly if and only if (f_{nk}) is regularly statistically relatively uniformly Cauchy.*

Proof. Let $(f_{nk}(x))$ be regularly statistically relatively uniformly convergent.Then for every $\varepsilon > 0$ and for all $x \in D$

$$\begin{aligned}
| f_{nk}(x) - f(x) |&< \frac{\varepsilon}{2} \mid \sigma(x) \mid, a.a.n \text{ and } k, \\
| f_{nk}(x) - f_n(x) |&< \frac{\varepsilon}{2} \mid \xi_n(x) \mid, a.a.k, \; \text{for each } n \in N, \\
| f_{nk}(x) - g_k(x) |&< \frac{\varepsilon}{2} \mid \eta_k(x) \mid, a.a.n, \; \text{for each } k \in N.
\end{aligned}$$

Similarly, we choose integers $s = s(\varepsilon)$ and $t = t(\varepsilon)$ so that the double sequence of functions $(f_{st}(x))$ is regularly statistically relatively uniformly convergent.
Then, we have, for all $x \in D$,

$$\begin{aligned}
| f_{nk}(x) - f_{st}(x) |=& | f_{nk}(x) - f(x) + f(x) - f_{st}(x) | \\
&\leq | f_{nk}(x) - f(x) | + | f_{st}(x) - f(x) | \\
&\leq \tfrac{\varepsilon}{2} \mid \sigma(x) \mid + \tfrac{\varepsilon}{2} \mid \sigma(x) \mid \\
&< \varepsilon \mid \sigma(x) \mid, \; a.a.n \text{ and } k.
\end{aligned}$$

$$| f_{nk}(x) - f_{st}(x) |< \varepsilon \mid \sigma(x) \mid, \; a.a.n \text{ and } k, \; \text{for all } n, k \in N. \tag{3.6}$$

$$\begin{aligned}
| f_{nk}(x) - f_{nt}(x) |=& | f_{nk}(x) - f_n(x) + f_n(x) - f_{nt}(x) | \\
&\leq | f_{nk}(x) - f_n(x) | + | f_{nt}(x) - f_n(x) | \\
&\leq \tfrac{\varepsilon}{2} \mid \xi_n(x) \mid + \tfrac{\varepsilon}{2} \mid \xi_n(x) \mid \\
&< \varepsilon \mid \xi_n(x) \mid, a.a.k, \; \text{for all } x \in D \text{ and for each } n \in N.
\end{aligned}$$

$$| f_{nk}(x) - f_{nt}(x) |< \varepsilon \mid \xi_n(x) \mid, \; a.a.k, \; \text{for all } x \in D \text{ and for each } n \in N. \tag{3.7}$$

$$| f_{nk}(x) - f_{sk}(x) |=| f_{nk}(x) - f_k(x) + f_k(x) - f_{sk}(x) |$$

$$\leq \mid f_{nk}(x) - f_k(x) \mid + \mid f_{sk}(x) - f_k(x) \mid$$

$$\leq \frac{\varepsilon}{2} \mid \eta_k(x) \mid + \frac{\varepsilon}{2} \mid \eta_k(x) \mid$$

$$< \varepsilon \mid \eta_k(x) \mid, \ a.a.n, \text{ for all } x \in D \text{ and for each } k \in N.$$

$$\mid f_{nk}(x) - f_{sk}(x) \mid < \varepsilon \mid \eta_k(x) \mid, \ a.a.n, \text{ for all } x \in D \text{ and for each } k \in N. \quad (3.8)$$

From Eq.(3.6), Eq.(3.7) and Eq.(3.8), we obtained that the double sequence of functions $(f_{nk}(x))$ is regularly statistically relatively uniformly Cauchy.

Conversely, Let $(f_{nk}(x))$ be regularly statistically relatively uniformly Cauchy.
Then, for every $\varepsilon > 0$, there exist $s = s(\varepsilon), t = t(\varepsilon)$ and scale functions $\sigma(x), \xi_n(x), \eta_k(x)$ defined on D such that for all $x \in D$,

(i) $\mid f_{nk}(x) - f_{st}(x) \mid \leq \varepsilon \mid \sigma(x) \mid, a.a.n$ and k;

(ii) $\mid f_{nk}(x) - f_{nt}(x) \mid \leq \varepsilon \mid \xi_n(x) \mid, a.a.k$ for each $n \in N$;

(iii) $\mid f_{nk}(x) - f_{sk}(x) \mid \leq \varepsilon \mid \eta_k(x) \mid, a.a.n$ for each $k \in N$.
First condition has been shown in Theorem 3.1. We now show that conditions (ii) and (iii) hold true.
Let condition (ii) holds true.
i.e., $(f_{nk}(x))$ is statistically relatively uniformly Cauchy row-wise w.r.t. the scale function $\xi_n(x)$. Since the Cauchy condition holds, by Cauchy's general principle of convergence, $(f_{nk}(x))$ converges statistically relatively point-wise to the function f_n defined over D, for every $x \in D$ and for each $n \in N$.
Therefore, $stat - \lim f_{nk}(x) = f_n$, for each $n \in N$.
Let $\varepsilon > 0$ be given and for all $x \in D$, by condition (ii)

$$\mid f_{nk}(x) - f_{nt}(x) \mid \leq \varepsilon \mid \xi_n(x) \mid, a.a.k, \text{ for each } n \in N.$$

Taking limit $t \to \infty$ we get,

$$\mid f_{nk}(x) - f_n(x) \mid \leq \varepsilon \mid \xi_n(x) \mid, a.a.k, \text{ for each } n \in N.$$

Similarly,

$$\mid f_{nk}(x) - f_k(x) \mid \leq \varepsilon \mid \eta_k(x) \mid, a.a.n, \text{ for each } k \in N.$$

Hence, $(f_{nk}(x))$ is regularly statistically relatively uniformly convergent.

Theorem 3.3. *Let $1 \leq r < \infty$. If a double sequence of functions $f = (f_{nk}(x))$ defined on a compact subset D is strongly double r- $Ces\acute{a}ro$ summable to $f(x)$, then, it is statistically point-wise convergent to $f(x)$. If a bounded double sequence of function is statistically point-wise convergent to $f(x)$, then, it is strongly double r- $Ces\acute{a}ro$ summable to $f(x)$.*

Proof. Let the double sequence of functions $(f_{nk}(x))$ be strongly double r- $Ces\acute{a}ro$ summable to $f(x)$. Then, for every $\varepsilon > 0$ and for each $x \in D$, we have,

$$\frac{1}{pq}\sum_{n=1}^{p}\sum_{k=1}^{q} \mid f_{nk}(x) - f(x) \mid^r \geq \frac{1}{pq} \mid \{(n,k) : n \leq p; k \leq \mid f_{nk}(x) - f(x) \mid^r \geq \varepsilon\} \mid \varepsilon^r.$$

Taking limit as $p, q \to \infty$, $(f_{nk}(x))$ is statistically point-wise convergent to $f(x)$.

Conversely, Let $(f_{nk}(x))$ be bounded and statistically point-wise convergent to $f(x)$.
We know that $||f||_\infty = \sup_{n,k \in N} |f_{nk}(x)| < \infty$.

Let $K = ||f||_\infty + f(x)$.

Then, for a given $\varepsilon > 0$, there exists p_o, q_o such that for each $x \in D$,

$$\frac{1}{pq}\left\{n \leq p; k \leq q :| f_{nk}(x) - f(x) |\geq (\frac{\epsilon}{2})^{\frac{1}{r}}\right\} < \frac{\epsilon}{2K^r}, \text{ for all } p > p_o \text{ and } q > q_o.$$

Let $f_{pq} = \left\{(n,k) : n \leq p; k \leq q \text{ and } \mid f_{nk}(x) - f(x) \mid^r \geq \frac{\varepsilon}{2}\right\}$.

We have, for all $p \geq p_o$ and $q \geq q_o$,

$$\frac{1}{pq}\sum_{n=1}^{p}\sum_{k=1}^{q} \mid f_{nk}(x) - f(x) \mid^r =$$

$$\frac{1}{pq}\left\{\sum_{(n,k)\in f_{pq}}\sum \mid f_{nk}(x) - f(x) \mid^r\right\} + \frac{1}{pq}\left\{\sum_{\substack{(n,k)\in f_{pq} \\ n\leq p_o; k\leq q_o}}\sum \mid f_{nk}(x) - f(x) \mid^r\right\}$$

$$< \tfrac{1}{pq}\left\{pq.\tfrac{\varepsilon}{2K^r}.K^r + pq.\tfrac{\varepsilon}{2}\right\} < \varepsilon.$$

This implies that $\frac{1}{pq}\sum\limits_{n=1}^{p}\sum\limits_{k=1}^{q} \mid f_{nk}(x) - f(x) \mid^r < \varepsilon$.
Hence, $(f_{nk}(x))$ is strongly double r- $Cesáro$ summable to $f(x)$.

Corollary 3.4. *Let $1 \leq r < \infty$. If a bounded double sequence of positive real functions is relatively statistically uniform convergent to $f(x)$, then, it is strong double r- $Cesáro$ summable to $f(x)$.*

Corollary 3.5. *A bounded double sequence of functions which is statistically relatively uniformly convergent to a function $f(x)$, defined over D w.r.t. a scale function $\sigma(x)$ is Cesaro summable to $f(x)$ w.r.t. a scale function $\sigma(x)$ but not conversely.*

The converse is not true is shown in the following example.

Example 3.1. Let $0 < a < 1$ be a real number and $D = [a, 1]$. Consider the sequence of real valued functions $(f_{nk}(x))$, $f_{nk} : [a, 1] \to R$, for all $n, k \in N$, defined on D by

$$f_{nk}(x) = \begin{cases} x, & \text{for } n = \text{odd }, k \in N; \\ -x & \text{for } n = \text{even }, k \in N. \end{cases}$$

$(f_{nk}(x))$ is Cesaro summable to $\frac{1}{2}$ w.r.t. the scale function $\sigma(x)$ defined as following:

$$\sigma(x) = \frac{1}{x}, \text{when } x \in [a, 1].$$

But $(f_{nk}(x))$ is not statistical uniform convergent relative to the scale function $\sigma(x)$.

We state and prove the decomposition theorem for statistically relative uniform convergence of double sequence of positive linear functions w.r.t. a scale function $\sigma(x)$.

Theorem 3.6. *The following statements are equivalent:*

(i) The double sequence of linear functions (f_{nk}) is statistically relatively uniformly convergent to $f(x)$.

(ii) The double sequence of linear functions (f_{nk}) is statistically relatively uniformly Cauchy.

(iii) There exists a double sequence of linear functions (g_{nk}) which is relatively uniformly convergent w.r.t the scale function $\sigma(x)$ such that $(f_{nk}) \equiv (g_{nk})$, $a.a.n$ and k for all $x \in D$.

Proof. The equivalence of (i) and (ii) is shown in Theorem 3.1.
Let the double sequence of linear functions $(f_{nk}(x))$ be statistically relatively uniformly Cauchy w.r.t. scale function $\sigma(x)$.
We choose s_o, t_o, so that the interval $I = [f_{s_o t_o}(x) - 1, f_{s_o t_o}(x) + 1]$ contains $f_{nk}(x)$, $a.a.n$ and k w.r.t $\sigma(x)$, for all $x \in D$.
Next, we choose s_1, t_1 so that the interval $I^{'} = [f_{s_1 t_1}(x) - 1/2, f_{s_1 t_1}(x) + 1/2]$ contains $f_{nk}(x)$, $a.a.n$ and k w.r.t $\sigma(x)$, for all $x \in D$.

Let $I_1 = I \bigcap I^{'}$ contains $f_{nk}(x)$, $a.a.n$ and k w.r.t $\sigma(x)$, for all $x \in D$.

It is clear from the above assumption that I_1 is a closed interval of length less than or equal to 1 that contains $f_{nk}(x)$, $a.a.n$ and k w.r.t $\sigma(x)$, for all $x \in D$.

Next, we choose s_2, t_2 so that the interval $I^{''} = [f_{s_2 t_2}(x) - 1/4, f_{s_2 t_2}(x) + 1/4]$ contains $f_{nk}(x)$, $a.a.n$ and k w.r.t $\sigma(x)$, for all $x \in D$.

Let $I_2 = I_1 \bigcap I^{''}$ contains $f_{nk}(x)$, $a.a.n$ and k w.r.t $\sigma(x)$, for all $x \in D$.

It is clear from the above assumption that I_2 is a closed interval of length less than or equal to 1/2. By using induction hypothesis, we have a nest of closed intervals $\{I_m\}_{m=1}^{\infty}$, such that for all m, $I_m \supseteq I_{m+1}$, the length of I_m is less than or equal to 2^{1-m} and I_m contains $f_{nk}(x)$, $a.a.n$ and k w.r.t $\sigma(x)$, for all $x \in D$.

By nested interval theorem, $\bigcap_{m=1}^{\infty} I_m$ will contain only one function say $f(x)$ defined on D.

Since $f_{nk}(x) \in I_m$, $a.a.n$ and k w.r.t $\sigma(x)$, for all $x \in D$, we choose increasing positive integer sequences $\{T_m\}_{m=1}^{\infty}$ and $\{Q_m\}_{m=1}^{\infty}$, for all m such that

$$\frac{1}{pq} \mid \{(n,k) \in N \times N : n \leq p; k \leq q \text{ and } f_{nk}(x) \notin I_m\} \mid < \frac{1}{m} \text{ if } p > T_m \text{ and } q > Q_m.$$

Now, we define a double sequence of linear functions $(g_{nk}(x))$, $g_{nk}(x) = f_{nk}(x)$ if $n \leq T_1$ and $k \leq Q_1$.
Also for all (n,k) with $T_j < n \leq T_{j+1}$ and $Q_j < k \leq Q_{j+1}$,

$$\text{let} \quad g_{nk}(x) = \begin{cases} f_{nk}(x), & \text{if } |f_{nk}(x) - f(x)| < 1/m; \\ f(x), & \text{otherwise.} \end{cases}$$

Let $\varepsilon > 0$ be given and we choose j such that $\varepsilon < \frac{1}{j}$, w.r.t. $\sigma(x)$. Using the above construction we get, for $n > T_j$ and $k > Q_j$,

$$|g_{nk}(x) - f(x)| < \varepsilon |\sigma(x)|, \text{ for all } x \in D.$$

To show that $g_{nk}(x) = f_{nk}(x), a.a.n$ and k w.r.t. $\sigma(x)$, we assume that
$T_j < n \leq T_{j+1}$ or $Q_j < k \leq Q_{j+1}$.
We have,

$$\frac{1}{pq} \mid \{(n,k), n \leq p; k \leq q : f_{nk}(x) \neq g_{nk}(x)\} \mid \subseteq$$
$$\frac{1}{pq} \mid \{(n,k), n \leq p; k \leq q : f_{nk}(x) \notin I_j\} \mid .$$

$$\frac{1}{pq} \mid \{(n,k), n \leq p; k \leq q : f_{nk}(x) \neq g_{nk}(x)\} \mid \leq$$
$$\frac{1}{pq} \mid \{(n,k), n \leq p; k \leq q : f_{nk}(x) \notin I_j\} \mid < \frac{1}{j}.$$

Taking limit as $p, q \to \infty$, in the above expression we get,
$f_{nk}(x) \equiv g_{nk}(x)$, $a.a.n$ and k.
Hence, (ii) implies (iii).
Now, we have to show that $(iii) \Rightarrow (i)$.
Assume that condition (iii) holds, so $f_{nk}(x) \equiv g_{nk}(x)$, $a.a.n$ and and k and $(g_{nk}(x))$ converges relatively uniformly.
Let $\varepsilon > 0$ be given. Then,

$$\frac{1}{pq}\{(n,k) : n < p; k < q :\mid f_{nk}(x) - f(x) \mid \leq \varepsilon \mid \sigma(x) \mid\}$$
$$\subseteq \frac{1}{pq}\{(n,k) : n < p; k < q : f_{nk}(x) \neq g_{nk}(x)\} +$$
$$\frac{1}{pq}\{(n,k) : |g_{nk}(x) - f(x)| \geq \varepsilon|\sigma(x)|\}.$$
$$\Rightarrow \frac{1}{pq} \mid \{(n,k) : n < p; k < q :\mid f_{nk}(x) - f(x) \mid \geq \varepsilon \mid \sigma(x) \mid\} \mid$$
$$\leq \frac{1}{pq} \mid \{(n,k) : n < p; k < q : f_{nk}(x) \neq g_{nk}(x)\} \mid +$$
$$\frac{1}{pq}|\{(n,k) : |g_{nk}(x) - f(x)| \geq \varepsilon|\sigma(x)|\}|.$$
$$\Rightarrow \lim_{p,q\to\infty} \frac{1}{pq} \mid \{(n,k) : n < p; k < q :\mid f_{nk}(x) - f(x) \mid \geq \varepsilon \mid \sigma(x) \mid\} \mid = 0.$$

$\Rightarrow (f_{nk})$ is statistically relatively uniformly convergent w.r.t. the scale function $\sigma(x)$.
Hence, (iii) implies (i) is proved.

We state the following results without proof, these can be easily established.

Result 3.1. The sum, subtraction and product of the statistically uniformly convergence of double sequence of linear functions w.r.t. the same scale function is a statistically uniformly convergence of double sequence of linear functions.

Result 3.2. The sum, subtraction and product of the statistically uniformly Cauchy double sequence of linear functions w.r.t. the same scale function is a statistically uniformly Cauchy of double sequence of linear functions.

Result 3.3. Let two double sequences of functions (f_{nk}) and (h_{nk}) converge statistically uniformly to function $f(x)$ w.r.t. scale function $\sigma(x)$ and we have $f_{nk}(x) \leq g_{nk}(x) \leq h_{nk}(x)$, then, the double sequence (g_{nk}) also converges statistically uniformly to function $f(x)$ w.r.t. the scale function $\sigma(x)$.

Theorem 3.7. *A statistical regular relative bounded double sequence of functions on compact domain D of real numbers converge statistically regularly relatively uniformly iff M_{nk}, M_n, M_k are statistically null w.r.t. scale functions $\sigma(x), \xi_n(x), \eta_k(x)$ respectively, where*

$$M_{nk} = \sup_{x \in D} \mid f_{nk}(x) - f(x) \mid;$$
$$M_n = \sup_{x \in D} \mid f_{nk}(x) - f_n(x) \mid;$$
$$\textit{and } M_k = \sup_{x \in D} \mid f_{nk}(x) - f_k(x) \mid .$$

Proof. Let M_{nk}, M_n, M_k are statistically null relative to scale functions $\sigma(x)$, $\xi_n(x), \eta_k(x)$ respectively.
Then, for a given $\varepsilon > 0$, there exist integers p, q such that for $n \leq p$ and $k \leq q$ and for all $x \in D$,

$$\lim_{p,q\to\infty} \frac{1}{pq} \mid \{n \leq p; k \leq q :\mid M_{nk} \mid\geq \varepsilon \mid \sigma(x) \mid\} \mid= 0;$$
$$\lim_{q\to\infty} \frac{1}{q} \mid \{k \leq q;\ \text{for each } n \in N :\mid M_n \mid\geq \varepsilon \mid \xi_n(x) \mid\} \mid= 0;$$
$$\lim_{p\to\infty} \frac{1}{p} \mid \{n \leq p;\ \text{for each } k \in N :\mid M_k \mid\geq \varepsilon \mid \eta_k(x) \mid\} \mid= 0.$$

i.e.,

$$\lim_{p,q\to\infty} \frac{1}{pq} \mid \{n \leq p; k \leq q : \sup_{x\in D} \mid f_{nk}(x) - f(x) \mid\geq \varepsilon \mid \sigma(x) \mid\} \mid= 0;$$
$$\lim_{q\to\infty} \frac{1}{q} \mid \{k \leq q;\ \text{for each } n \in N : \sup_{x\in D} \mid f_{nk}(x) - f_n(x) \mid\geq \varepsilon \mid \xi_n(x) \mid\} \mid= 0;$$
$$\lim_{p\to\infty} \frac{1}{p} \mid \{n \leq p;\ \text{for each } k \in N : \sup_{x\in D} \mid f_{nk}(x) - f_k(x) \mid\geq \varepsilon \mid \eta_k(x) \mid\} \mid= 0.$$

For every $\varepsilon > 0$, and for all $x \in D$,

$$\lim_{p,q\to\infty} \frac{1}{pq} \mid \{n \leq p; k \leq q :\mid f_{nk}(x) - f(x) \mid\geq \varepsilon \mid \sigma(x) \mid\} \mid= 0;$$
$$\lim_{q\to\infty} \frac{1}{q} \mid \{k \leq q;\ \text{for each } n \in N :\mid f_{nk}(x) - f_n(x) \mid\geq \varepsilon \mid \xi_n(x) \mid\} \mid= 0;$$
$$\lim_{p\to\infty} \frac{1}{p} \mid \{n \leq p;\ \text{for each } k \in N :\mid f_{nk}(x) - f_k(x) \mid\geq \varepsilon \mid \eta_k(x) \mid\} \mid= 0.$$

From the above expressions we obtain that $(f_{nk}(x))$ converges statistically regularly relatively uniformly to $f(x)$ w.r.t. scale functions $\sigma(x), \zeta_n(x), \eta_k(x)$.

Conversely, Let us assume that M_{nk}, M_n, M_k exist for $n, k \in N$.
We have to prove that, M_{nk}, M_n, M_k converge statistically null relatively uniformly.
Since $(f_{nk}(x))$ converges statistically regularly relatively uniformly, we have for every $\varepsilon > 0$ and for all $x \in D$,

$$\lim_{p,q\to\infty} \frac{1}{pq} \mid \{n \leq p; k \leq q :\mid f_{nk}(x) - f(x) \mid\geq \varepsilon \mid \sigma(x) \mid\} \mid= 0;$$
$$\lim_{q\to\infty} \frac{1}{q} \mid \{k \leq q;\ \text{for each } n \in N :\mid f_{nk}(x) - f_n(x) \mid\geq \varepsilon \mid \xi_n(x) \mid\} \mid= 0;$$
$$\lim_{p\to\infty} \frac{1}{p} \mid \{n \leq p;\ \text{for each } k \in N :\mid f_{nk}(x) - f_k(x) \mid\geq \varepsilon \mid \eta_k(x) \mid\} \mid= 0.$$

We can write, for every $\varepsilon > 0$ and for all $x \in D$,

$$\lim_{p,q\to\infty} \frac{1}{pq} \mid \{n \leq p; k \leq q : \sup_{x\in D} \mid f_{nk}(x) - f(x) \mid\geq \varepsilon \mid \sigma(x) \mid\} \mid= 0;$$
$$\lim_{q\to\infty} \frac{1}{q} \mid \{k \leq q;\ \text{for each } n \in N : \sup_{x\in D} \mid f_{nk}(x) - f_n(x) \mid\geq \varepsilon \mid \xi_n(x) \mid\} \mid= 0; \tag{3.9}$$
$$\lim_{p\to\infty} \frac{1}{p} \mid \{n \leq p;\ \text{for each } k \in N : \sup_{x\in D} \mid f_{nk}(x) - f_k(x) \mid\geq \varepsilon \mid \eta_k(x) \mid\} \mid= 0.$$

From equation (3.9), we see that M_{nk}, M_n and M_k converge statistically null relatively uniformly.

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