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Solution of Ordered Inclusion Problem via Graph Convergence in Real Ordered Hilbert Spaces

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Abstract The objective of this article is to investigate a variant of ordered inclusion problems, namely, $\tilde{H}(\cdot,\cdot)$-compression (α,λ)-XOR-ordered NODSM mapping in real ordered Hilbert spaces by using graph convergence approach. An associated resolvent operator is defined and some of its properties are discussed. An equivalence between resolvent operator convergence and graph convergence is established. An existence result for $\tilde{H}(\cdot,\cdot)$-compression (α,λ)-XOR-ordered NODSM mapping is proved using XOR and XNOR operations technique. Moreover, an ordered inclusion problem in real ordered Hilbert space is considered and an iterative algorithm is presented. Also, the concept of graph convergence is applied to analyze the convergence of the proposed algorithm. Finally, the notion of graph convergence is clarified by an illustrative example.

1 Introduction

The theory of variational inequality is one of the most acclaimed tools in nonlinear analysis. Since its inception in 1964 by Stampachia [35], it has been paid much attention and generalized in various diverse directions due to its wide ranging applications in optimizations, economics, engineering sciences, structural analysis, etc., see, [8, 10, 18, 19, 20]. One of the important generalizations of variational inequality is variational inclusion, which includes variational inequality, equilibrium problems, game theory, optimizations and fixed point theory as special cases.

The notable monotone operators were conceptualized by Zarantonello [39] and Minty [34]. A notable interest has been shown by several researchers to study monotone operators due the following evolution equation

$$\frac{dx}{dt} + A(x) = 0; x(0) = x_0.$$

The above equation is strongly connected with monotone operators and many physical problems can be expressed using this model. The maximal monotone operators provide a framework to develop a suitable resolvent operator to find the approximate solutions of variational inequalities and convex optimization problems. It is noteworthy to mention that the resolvent operator associated with maximal monotone operator is a generalization of projection technique. Due to this fact, researchers shown their interests to study the concepts of maximal monotone operators and their generalized forms, see, [2, 3, 14, 16, 17, 23, 24, 26, 27, 37, 41] and references cited therein. These operators play crucial role in convex analysis, optimization, partial differential equations and differential inclusion problems. It was Li and Huang [29] who instigated the graph convergence and established its relationship to resolvent operator convergence. Using this concept, they showed the convergence of $H(\cdot,\cdot)$-accretive operator in Banach spaces. Later on, the technique of graph convergence was generalized to $H(\cdot,\cdot)$-co-accretive operators and Yosida approximation operators, for more related work, see; [5, 7] and references cited therein.

However, the underlying spaces in the traditional fixed point theory are topological space and the presumed mappings must satisfy a certain type of continuity. Tarski [36] introduced the concept in which there are some ordering relations on the underlying spaces (pre-order, partial order or lattice) and such spaces are not required to be equipped with topological structure. To assure the existence of fixed point, the considered mappings satisfy some order-monotonic conditions and it is unnecessary for them to have any continuity property. Since Banach spaces are the fundamental underlying spaces on linear and nonlinear analysis, therefore a Banach space with an ordering structure is called an ordered Banach space. This important idea has been widely used in solving integral equations [11, 12, 21], vector variational inequalities [28], nonlinear fractional evolution equations [25, 38] and Nesh equilibrium problems [9, 40], etc..

In recent past, the nonlinear mapping fixed point theory in ordered Banach spaces has been studied considerably, see, [13, 15, 22, 38]. In 2008, Li [30] discussed the existence result and convergence of the proposed algorithm for a class of nonlinear ordered variational inequalities. In [1, 4, 31, 32, 33], approximate solutions of general nonlinear ordered variational inequalities (ordered equations) are given in ordered Banach and Hilbert spaces.

With inducement from recent findings in this direction, we introduce a variant form of ordered inclusion problems, namely, $\tilde{H}(\cdot,\cdot)$-compression (α,λ)-XOR-ordered NODSM mapping in real ordered Hilbert spaces. The paper is lay out as follows. In section 2, we recollect basic definitions, notions and tools required for the accomplishment of subsequent sections. Also, we define the resolvent operator and discuss some of its characteristics. Section 3 deals with the definition of graph convergence and its relationship with resolvent operator convergence associated to $\tilde{H}(\cdot,\cdot)$-compression (α,λ)-XOR-ordered NODSM mapping is also discussed. Moreover, an illustrative example is given to justify the concept of graph convergence for $\tilde{H}(\cdot,\cdot)$-compression (α,λ)-XOR-ordered NODSM mapping. Section 4 begins with the formulation of ordered inclusion problem and discuss the existence result for the proposed problem. In the last section, we present an iterative algorithm and report its convergence analysis.

2 Preliminaries and auxiliary results

In this section, we mention basic definitions, perceptions and handy outcomes that are constructive instruments in succeeding analysis which will be deployed throughout the paper.

Let $\mathcal{H}_p$ be a positive real ordered Hilbert space equipped with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let d be the metric induced by the norm $\|\cdot\|$; $CB(\mathcal{H}_p)$ (respectively, $2^{\mathcal{H}_p}$) be the family of all nonempty closed and bounded subsets (respectively, all non empty subsets) of $\mathcal{H}_p$. " $\leq$ " denotes the partial ordering relation defined by the normal cone $\mathcal{C}$ with normal constant $\lambda_{\mathcal{C}}$. For any arbitrary $\tilde{\kappa_1},\tilde{\kappa_2}\in\mathcal{H}_p$, the greatest lower bound and least upper bound of $\{\tilde{\kappa_1},\tilde{\kappa_2}\}$ is represented by glb$\{\tilde{\kappa_1},\tilde{\kappa_2}\}$ and lub$\{\tilde{\kappa_1},\tilde{\kappa_2}\}$, respectively with partial ordering relation $\leq$. AND, OR, XOR and XNOR operators are denoted by $\wedge,\vee,\oplus$ and $\odot$, respectively which are given below by following relations:

(i) $\tilde{\kappa_1}\wedge\tilde{\kappa_2}=\text{glb}\{\tilde{\kappa_1},\tilde{\kappa_2}\}$,

(ii) $\tilde{\kappa_1}\vee\tilde{\kappa_2}=\text{lub}\{\tilde{\kappa_1},\tilde{\kappa_2}\}$,

(iii) $\tilde{\kappa_1}\oplus\tilde{\kappa_2}=(\tilde{\kappa_1}-\tilde{\kappa_2})\vee(\tilde{\kappa_2}-\tilde{\kappa_1})$,

(iv) $\tilde{\kappa_1}\odot\tilde{\kappa_2}=(\tilde{\kappa_1}-\tilde{\kappa_2})\wedge(\tilde{\kappa_2}-\tilde{\kappa_1})$.

Now, we recall some familiar definitions, notions and results which are requisite to accomplish the ambitions of this article.

Definition 2.1. A nonempty closed convex subset $\mathcal{C}$ of $\mathcal{H}_p$ is called a cone

(i) if $\tilde{m}\in\mathcal{C}$, $\alpha>0\Rightarrow\alpha\tilde{m}\in\mathcal{C}$;

(ii) if $\tilde{m}\in\mathcal{C}$ and $-\tilde{m}\in\mathcal{C}\Rightarrow\tilde{m}=0$.

Definition 2.2. Let $\mathcal{C}$ be a cone. Then

(i) $\mathcal{C}$ is called a normal cone, if there exists $\lambda_{\mathcal{C}}>0$ such that $0\leq\tilde{\kappa_1}\leq\tilde{\kappa_2}\Rightarrow\|\tilde{\kappa_1}\|\leq\lambda_{\mathcal{C}}\|\tilde{\kappa_2}\|,\forall\tilde{\kappa_1},\tilde{\kappa_2}\in\mathcal{H}_p$;

(ii) $\tilde{\kappa_1}\leq\tilde{\kappa_2}$ if and only if $\tilde{\kappa_1}-\tilde{\kappa_2}\in\mathcal{C},\forall\tilde{\kappa_1},\tilde{\kappa_2}\in\mathcal{H}_p$;

(iii) $\tilde{\kappa_1} \propto \tilde{\kappa_2}$ if and only if $\tilde{\kappa_1} \le \tilde{\kappa_2}$ or $\tilde{\kappa_2} \le \tilde{\kappa_1}$.

Lemma 2.3. *[15] Let $\mathcal{C} \subseteq \mathcal{H}_p$ and $\tilde{\kappa_1}, \tilde{\kappa_2} \in \mathcal{H}_p$. Then the following assertions hold:*

(i) $\|\tilde{0} \oplus \tilde{0}\| = \|\tilde{0}\| = \tilde{0}$;

(ii) $\|\tilde{\kappa_1} \vee \tilde{\kappa_2}\| \le \|\tilde{\kappa_1}\| \vee \|\tilde{\kappa_2}\| \le \|\tilde{\kappa_1}\| + \|\tilde{\kappa_2}\|$;

(iii) $\|\tilde{\kappa_1} \oplus \tilde{\kappa_2}\| \le \|\tilde{\kappa_1} - \tilde{\kappa_2}\| \le \lambda_{\mathcal{C}} \|\tilde{\kappa_1} \oplus \tilde{\kappa_2}\|$;

(iv) if $\tilde{\kappa_1} \propto \tilde{\kappa_2}$, then $\|\tilde{\kappa_1} \oplus \tilde{\kappa_2}\| = \|\tilde{\kappa_1} - \tilde{\kappa_2}\|$.

Lemma 2.4. *[15] If $\tilde{p} \propto \tilde{q}_n$ and $\tilde{q}_n \to \tilde{q}^*$ as $n \to \infty$, then $\tilde{p} \propto \tilde{q}^*$, $\forall n \in N$.*

Lemma 2.5. *[31] Let $\mathcal{C} \subseteq \mathcal{H}_p$ with relation $\le$ induced by $\mathcal{C}$. Then for any $\tilde{a}, \tilde{b}, \tilde{\kappa_1}, \tilde{\kappa_2}, \tilde{\kappa_3} \in \mathcal{H}_p$, the following assertions hold:*

(i) $\tilde{\kappa_1} \odot \tilde{\kappa_1} = 0,\ \tilde{\kappa_1} \odot \tilde{\kappa_2} = \tilde{\kappa_2} \odot \tilde{\kappa_1} = -(\tilde{\kappa_1} \oplus \tilde{\kappa_2}) = -(\tilde{\kappa_2} \oplus \tilde{\kappa_1})$;

(ii) if $\tilde{\kappa_1} \propto 0$ then $-\tilde{\kappa_1} \oplus 0 \le \tilde{\kappa_1} \le \tilde{\kappa_1} \oplus 0$;

(iii) $0 \le \tilde{\kappa_1} \oplus \tilde{\kappa_2}$, if $\tilde{\kappa_1} \propto \tilde{\kappa_2}$;

(iv) $(\lambda\tilde{\kappa_1}) \oplus (\lambda\tilde{\kappa_2}) = |\lambda|(\tilde{\kappa_1} \oplus \tilde{\kappa_2})$, for any real λ;

(v) if $\tilde{\kappa_1} \propto \tilde{\kappa_2}$ then $\tilde{\kappa_1} \oplus \tilde{\kappa_2} = 0$ if and only if $\tilde{\kappa_1} = \tilde{\kappa_2}$;

(vi) $(\tilde{a} + \tilde{b}) \odot (\tilde{\kappa_1} + \tilde{\kappa_2}) \ge (\tilde{a} \odot \tilde{\kappa_1}) + (\tilde{b} \odot \tilde{\kappa_2})$;

(vii) $(\tilde{a} + \tilde{b}) \odot (\tilde{\kappa_1} + \tilde{\kappa_2}) \ge (\tilde{a} \odot \tilde{\kappa_2}) + (\tilde{b} \odot \tilde{\kappa_1})$;

(viii) if $\tilde{\kappa_1}, \tilde{\kappa_2}, \tilde{\kappa_3}$ are comparable, then $(\tilde{\kappa_1} \oplus \tilde{\kappa_2}) \le (\tilde{\kappa_1} \oplus \tilde{\kappa_3}) + (\tilde{\kappa_3} \oplus \tilde{\kappa_2})$;

(ix) $\alpha\tilde{\kappa_1} \oplus \beta\tilde{\kappa_1} = |\alpha - \beta|\tilde{\kappa_1} = (\alpha \oplus \beta)\tilde{\kappa_1}$, if $\tilde{\kappa_1} \propto 0$, for any real α, β.

Definition 2.6. [31] $\tilde{U} : \mathcal{H}_p \to \mathcal{H}_p$ is said to be comparison mapping if for any $\tilde{\kappa_1}, \tilde{\kappa_2} \in \mathcal{H}_p$, $\tilde{\kappa_1} \propto \tilde{\kappa_2}$ implies $\tilde{U}(\tilde{\kappa_1}) \propto \tilde{U}(\tilde{\kappa_2}), \tilde{\kappa_1} \propto \tilde{U}(\tilde{\kappa_1}), \tilde{\kappa_2} \propto \tilde{U}(\tilde{\kappa_2})$.

Definition 2.7. [31] A comparison mapping $\tilde{U} : \mathcal{H}_p \to \mathcal{H}_p$ is said to be

(i) strongly comparison, for any $\tilde{\kappa_1}, \tilde{\kappa_2} \in \mathcal{H}_p, \tilde{\kappa_1} \propto \tilde{\kappa_2}$ if and only if $\tilde{U}(\tilde{\kappa_1}) \propto \tilde{U}(\tilde{\kappa_2})$;

(ii) γ-ordered compression mapping, if there exists a constant $\gamma \in (0, 1)$ such that

$$\tilde{U}(\tilde{\kappa_1}) \oplus \tilde{U}(\tilde{\kappa_2}) \le \gamma(\tilde{\kappa_1} \oplus \tilde{\kappa_2}),\ \forall \tilde{\kappa_1}, \tilde{\kappa_2} \in \mathcal{H}_p.$$

Definition 2.8. $\tilde{U} : \mathcal{H}_p \to \mathcal{H}_p$ is said to be ordered Lipschitz continuous if

$$\|\tilde{U}(\tilde{\kappa_1}) \oplus \tilde{U}(\tilde{\kappa_2})\| \le \lambda\|\tilde{\kappa_1} \oplus \tilde{\kappa_2}\|, \forall \tilde{\kappa_1}, \tilde{\kappa_2} \in \mathcal{H}_p, \tilde{\kappa_1} \propto \tilde{\kappa_2} \text{ and } \lambda > 0.$$

Definition 2.9. $\tilde{P} : \mathcal{H}_p \to CB(\mathcal{H}_p)$ is said to be ordered $\mathcal{D}$-Lipschitz continuous if

$$\mathcal{D}(\tilde{P}(\tilde{\kappa_1}), \tilde{P}(\tilde{\kappa_2})) \le \lambda_{\tilde{P}}\|\tilde{\kappa_1} \oplus \tilde{\kappa_2}\|, \forall \tilde{\kappa_1}, \tilde{\kappa_2} \in \mathcal{H}_p, \tilde{\kappa_1} \propto \tilde{\kappa_2} \text{ and } \lambda_{\tilde{P}} > 0.$$

Definition 2.10. $\tilde{M} : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ said to be comparison mapping, if for any $p_{\tilde{\kappa_1}} \in \tilde{M}(\tilde{\kappa_1}), \tilde{\kappa_1} \propto p_{\tilde{\kappa_1}}$ and if $\tilde{\kappa_1} \propto \tilde{\kappa_2}$, then for any $p_{\tilde{\kappa_1}} \in \tilde{M}(\tilde{\kappa_1})$ and $p_{\tilde{\kappa_2}} \in \tilde{M}(\tilde{\kappa_2}), p_{\tilde{\kappa_1}} \propto p_{\tilde{\kappa_2}},\ \forall \tilde{\kappa_1}, \tilde{\kappa_2} \in \mathcal{H}_p$.

Definition 2.11. A comparison mapping $\tilde{M} : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ is said to be

(i) α-non-ordinary difference mapping if there exists $p_{\tilde{\kappa_1}} \in \tilde{M}(\tilde{\kappa_1})$ and $p_{\tilde{\kappa_2}} \in \tilde{M}(\tilde{\kappa_2})$ such that

$$(p_{\tilde{\kappa_1}} \oplus p_{\tilde{\kappa_2}}) \oplus \alpha(\tilde{\kappa_1} \oplus \tilde{\kappa_2}) = 0, \forall \tilde{\kappa_1}, \tilde{\kappa_2} \in \mathcal{H}_p \text{ and } \alpha > 0;$$

(ii) α-ordered rectangular if there exists $p_{\tilde{\kappa_1}} \in \tilde{M}(\tilde{\kappa_1})$ and $p_{\tilde{\kappa_2}} \in \tilde{M}(\tilde{\kappa_2})$ such that

$$\langle p_{\tilde{\kappa_1}} \odot p_{\tilde{\kappa_2}}, -(\tilde{\kappa_1} \oplus \tilde{\kappa_2})\rangle \ge \alpha\|\tilde{\kappa_1} \oplus \tilde{\kappa_2}\|^2, \forall \tilde{\kappa_1}, \tilde{\kappa_2} \in \mathcal{H}_p \text{ and } \alpha > 0;$$

(iii) λ-XOR-ordered strongly monotone compression mapping, if $\tilde{\kappa_1} \propto \tilde{\kappa_2}$ and for any $p_{\tilde{\kappa_1}} \in \tilde{M}(\tilde{\kappa_1})$ and $p_{\tilde{\kappa_2}} \in \tilde{M}(\tilde{\kappa_2})$, we have

$$\lambda(p_{\tilde{\kappa_1}} \oplus p_{\tilde{\kappa_2}}) \geq (\tilde{\kappa_1} \oplus \tilde{\kappa_2}), \forall \tilde{\kappa_1}, \tilde{\kappa_2} \in \mathcal{H}_p \text{ and } \lambda > 0.$$

Definition 2.12. Let $\tilde{U}, \tilde{V} : \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings. Then $\tilde{H} : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ is said to be

(i) τ_1-ordered compression mapping with respect to $\tilde{U}$, if there exists a constant $\tau_1 \in (0,1)$ satisfying
$$\tilde{H}(\tilde{U}(\tilde{\kappa_1}), \cdot) \oplus \tilde{H}(\tilde{U}(\tilde{\kappa_2}), \cdot) \leq \tau_1(\tilde{\kappa_1} \oplus \tilde{\kappa_2}),\ \forall \tilde{\kappa_1}, \tilde{\kappa_2} \in \mathcal{H}_p;$$

(ii) τ_2-ordered compression mapping with respect to $\tilde{V}$, if there exists a constant $\tau_2 \in (0,1)$ satisfying
$$\tilde{H}(\cdot, \tilde{V}(\tilde{\kappa_1})) \oplus \tilde{H}(\cdot, \tilde{V}(\tilde{\kappa_2})) \leq \tau_2(\tilde{\kappa_1} \oplus \tilde{\kappa_2}),\ \forall \tilde{\kappa_1}, \tilde{\kappa_2} \in \mathcal{H}_p;$$

(iii) mixed comparison mapping with respect to $\tilde{U}$ and $\tilde{V}$, if for all $\tilde{\kappa_1}, \tilde{\kappa_2} \in \mathcal{H}_p, \tilde{\kappa_1} \propto \tilde{\kappa_2}$ then $\tilde{H}(\tilde{U}(\tilde{\kappa_1}), \tilde{V}(\tilde{\kappa_1})) \propto \tilde{H}(\tilde{U}(\tilde{\kappa_2}), \tilde{V}(\tilde{\kappa_2})), \tilde{\kappa_1} \propto \tilde{H}(\tilde{U}(\tilde{\kappa_1}), \tilde{V}(\tilde{\kappa_1}))$ and $\tilde{\kappa_2} \propto \tilde{H}(\tilde{U}(\tilde{\kappa_2}), \tilde{V}(\tilde{\kappa_2}))$;

(iv) mixed strongly comparison mapping with respect to $\tilde{U}$ and $\tilde{V}$, if for all $\tilde{\kappa_1}, \tilde{\kappa_2} \in \mathcal{H}_p$, $\tilde{H}(\tilde{U}(\tilde{\kappa_1}), \tilde{V}(\tilde{\kappa_1})) \propto \tilde{H}(\tilde{U}(\tilde{\kappa_2}), \tilde{V}(\tilde{\kappa_2}))$ if and only if $\tilde{\kappa_1} \propto \tilde{\kappa_2}$, for all $\tilde{\kappa_1}, \tilde{\kappa_2} \in \mathcal{H}_p$.

Definition 2.13. Let $\tilde{U}, \tilde{V} : \mathcal{H}_p \to \mathcal{H}_p$ and $\tilde{H} : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings. $\tilde{M} : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ is said to be $\tilde{H}(\cdot,\cdot)$-compression (α, λ)-ordered NODSM with respect to $\tilde{U}$ and $\tilde{V}$, if $\tilde{H}$ is β_1 and β_2-ordered compression mapping with respect to $\tilde{U}$ and $\tilde{V}$, respectively and $\tilde{M}$ is $\alpha_{\tilde{M}}$-non ordinary difference, λ-XOR-ordered strongly monotone mapping such that

$$[\tilde{H}(\tilde{U}, \tilde{V}) \oplus \lambda\tilde{M}](\mathcal{H}_p) = \mathcal{H}_p, \forall \lambda > 0 \text{ and } 0 < \beta_1, \beta_2 < 1.$$

Definition 2.14. Let $\tilde{U}, \tilde{V} : \mathcal{H}_p \to \mathcal{H}_p; \tilde{H} : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings and $\tilde{M} : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be $\tilde{H}(\cdot,\cdot)$-compression (α, λ)-ordered NODSM mapping with respect to $\tilde{U}$ and $\tilde{V}$. The resolvent operator $R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}} : \mathcal{H}_p \to \mathcal{H}_p$ is defined by

$$R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}(\tilde{u}) = [\tilde{H}(\tilde{U}, \tilde{V}) \oplus \lambda\tilde{M}]^{-1}(\tilde{u}),\ \forall \tilde{u} \in \mathcal{H}_p,\ \lambda, \alpha > 0, 0 < \beta_1, \beta_2 < 1. \tag{2.1}$$

Proposition 2.15. *Let $\tilde{U}, \tilde{V} : \mathcal{H}_p \to \mathcal{H}_p$ and $\tilde{H} : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings such that $\tilde{H}$ is β_1 and β_2-ordered compression mapping with respect to $\tilde{U}$ and $\tilde{V}$, respectively. Let $\tilde{M} : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be an α-ordered rectangular mapping with $\lambda\alpha > \beta_1 + \beta_2$ and $(\tilde{u} \oplus \tilde{v}) \propto 0$. Then the resolvent operator $R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}} : \mathcal{H}_p \to \mathcal{H}_p$ is a single-valued mapping.*

Proof. Given $\tilde{w} \in \mathcal{H}_p, \tilde{u} \propto \tilde{v}$ and $\lambda > 0$, let $\tilde{u}, \tilde{v} \in [\tilde{H}(\tilde{U}, \tilde{V}) \oplus \lambda\tilde{M}]^{-1}(\tilde{w})$. Then,

$$p_{\tilde{u}} = \frac{1}{\lambda}[\tilde{w} \oplus \tilde{H}(\tilde{U}(\tilde{u}), \tilde{V}(\tilde{u}))] \in \tilde{M}(\tilde{u}),$$

and
$$p_{\tilde{v}} = \frac{1}{\lambda}[\tilde{w} \oplus \tilde{H}(\tilde{U}(\tilde{v}), \tilde{V}(\tilde{v}))] \in \tilde{M}(\tilde{v}).$$

Following (i) and (ii) of Lemma 2.5, we have

$$\begin{aligned}
p_{\tilde{u}} \odot p_{\tilde{v}} &= \frac{1}{\lambda}[\tilde{w} \oplus \tilde{H}(\tilde{U}(\tilde{u}), \tilde{V}(\tilde{u}))] \odot \frac{1}{\lambda}[\tilde{w} \oplus \tilde{H}(\tilde{U}(\tilde{v}), \tilde{V}(\tilde{v}))] \\
&= \frac{1}{\lambda}([\tilde{w} \oplus \tilde{H}(\tilde{U}(\tilde{u}), \tilde{V}(\tilde{u}))] \odot [\tilde{w} \oplus \tilde{H}(\tilde{U}(\tilde{v}), \tilde{V}(\tilde{v}))]) \\
&= -\frac{1}{\lambda}([\tilde{w} \oplus \tilde{H}(\tilde{U}(\tilde{u}), \tilde{V}(\tilde{u}))] \oplus [\tilde{w} \oplus \tilde{H}(\tilde{U}(\tilde{v}), \tilde{V}(\tilde{v}))]) \\
&= -\frac{1}{\lambda}([\tilde{w} \oplus \tilde{w}] \oplus [\tilde{H}(\tilde{U}(\tilde{u}), \tilde{V}(\tilde{u})) \oplus \tilde{H}(\tilde{U}(\tilde{v}), \tilde{V}(\tilde{v}))]) \\
&= -\frac{1}{\lambda}(0 \oplus [\tilde{H}(\tilde{U}(\tilde{u}), \tilde{V}(\tilde{u})) \oplus \tilde{H}(\tilde{U}(\tilde{v}), \tilde{V}(\tilde{v}))] \\
&\leq -\frac{1}{\lambda}[\tilde{H}(\tilde{U}(\tilde{u}), \tilde{V}(\tilde{u})) \oplus \tilde{H}(\tilde{U}(\tilde{v}), \tilde{V}(\tilde{v}))].
\end{aligned} \tag{2.2}$$

Since $\tilde{M}$ is α-ordered rectangular mapping, $\tilde{H}$ is β_1 and β_2-ordered compression mapping with respect to $\tilde{U}$ and $\tilde{V}$, respectively. Therefore,

$$\begin{aligned}
\alpha\|\tilde{u}\oplus\tilde{v}\|^2 &\leq \langle p_{\tilde{u}} \odot p_{\tilde{v}}, -(\tilde{u}\oplus\tilde{v})\rangle \\
&\leq \langle -\frac{1}{\lambda}(\tilde{H}(\tilde{U}(\tilde{u}),\tilde{V}(\tilde{u}))\oplus\tilde{H}(\tilde{U}(\tilde{v}),\tilde{V}(\tilde{v}))), -(\tilde{u}\oplus\tilde{v})\rangle \\
&= \frac{1}{\lambda}\langle \tilde{H}(\tilde{U}(\tilde{u}),\tilde{V}(\tilde{u}))\oplus\tilde{H}(\tilde{U}(\tilde{v}),\tilde{V}(\tilde{v})), (\tilde{u}\oplus\tilde{v})\rangle \\
&\leq \frac{1}{\lambda}\langle \tilde{H}(\tilde{U}(\tilde{u}),\tilde{V}(\tilde{u}))\oplus\tilde{H}(\tilde{U}(\tilde{v}),\tilde{V}(\tilde{u})), (\tilde{u}\oplus\tilde{v})\rangle \\
&\quad +\frac{1}{\lambda}\langle \tilde{H}(\tilde{U}(\tilde{v}),\tilde{V}(\tilde{u}))\oplus\tilde{H}(\tilde{U}(\tilde{v}),\tilde{V}(\tilde{v})), (\tilde{u}\oplus\tilde{v})\rangle \\
&\leq \frac{1}{\lambda}\langle \beta_1(\tilde{u}\oplus\tilde{v}), (\tilde{u}\oplus\tilde{v})\rangle + \frac{1}{\lambda}\langle \beta_2(\tilde{u}\oplus\tilde{v}), (\tilde{u}\oplus\tilde{v})\rangle \\
&= \frac{(\beta_1+\beta_2)}{\lambda}\|\tilde{u}\oplus\tilde{v}\|^2,
\end{aligned} \tag{2.3}$$

that is, $\left(\alpha - \dfrac{\beta_1+\beta_2}{\lambda}\right)\|\tilde{u}\oplus\tilde{v}\|^2 \leq 0$. Since $\lambda\alpha > \beta_1+\beta_2$, therefore $\|\tilde{u}\oplus\tilde{v}\| = 0$. Thus $\tilde{u} = \tilde{v}$, i.e., the resolvent operator $R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}$ associated to $\tilde{U}, \tilde{V}, \tilde{H}$ and $\tilde{M}$ is a single-valued mapping.

Proposition 2.16. *Let $\tilde{U}, \tilde{V} : \mathcal{H}_p \to \mathcal{H}_p$ and $\tilde{H} : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings such that $\tilde{H}$ is mixed strongly comparison mapping, β_1 and β_2-ordered compression mapping with respect to $\tilde{U}$ and $\tilde{V}$, respectively. Let $\tilde{M} : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be an $\tilde{H}(\cdot,\cdot)$-compression (α,λ)-XOR-NODSM mapping. Then the resolvent operator $R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}} : \mathcal{H}_p \to \mathcal{H}_p$ is a comparison mapping.*

Proof. Given $\tilde{u}, \tilde{v} \in \mathcal{H}_p$, assume that $\tilde{u} \propto \tilde{v}$ and

$$p_{\tilde{u}} = \frac{1}{\lambda}[\tilde{u}\oplus\tilde{H}(\tilde{U}(R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}(\tilde{u})),\tilde{V}(R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}(\tilde{u})))] \in \tilde{M}(R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}(\tilde{u})), \tag{2.4}$$

and

$$p_{\tilde{v}} = \frac{1}{\lambda}[\tilde{v}\oplus\tilde{H}(\tilde{U}(R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}(\tilde{v})),\tilde{V}(R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}(\tilde{v})))] \in \tilde{M}(R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}(\tilde{v})). \tag{2.5}$$

For the sake of computation, assume

$$T(\tilde{u}) = R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}(\tilde{u}) \text{ and } T(\tilde{v}) = R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}(\tilde{v}).$$

Using the (α,λ)-XOR-oredered strong monotonicity of $\tilde{M}$, (2.4) and (2.5), we have

$$\begin{aligned}
(\tilde{u}\oplus\tilde{v}) &\leq \lambda(p_{\tilde{u}}\oplus p_{\tilde{v}}) \\
(\tilde{u}\oplus\tilde{v}) &\leq [\tilde{u}\oplus\tilde{H}(\tilde{U}(T(\tilde{u})),\tilde{V}(T(\tilde{u})))]\oplus[\tilde{v}\oplus\tilde{H}(\tilde{U}(T(\tilde{v})),\tilde{V}(T(\tilde{v})))] \\
(\tilde{u}\oplus\tilde{v}) &\leq (\tilde{u}\oplus\tilde{v})\oplus(\tilde{H}(\tilde{U}(T(\tilde{u})),\tilde{V}(T(\tilde{u}))))\oplus(\tilde{H}(\tilde{U}(T(\tilde{v})),\tilde{V}(T(\tilde{v})))) \\
0 &\leq (\tilde{H}(\tilde{U}(T(\tilde{u})),\tilde{V}(T(\tilde{u}))))\oplus(\tilde{H}(\tilde{U}(T(\tilde{v})),\tilde{V}(T(\tilde{v})))).
\end{aligned}$$

The above inequality gives

$$\begin{aligned}
0 \leq\ & [\tilde{H}(\tilde{U}(T(\tilde{u})),\tilde{V}(T(\tilde{u}))) - \tilde{H}(\tilde{U}(T(\tilde{v})),\tilde{V}(T(\tilde{v})))] \\
& \vee [\tilde{H}(\tilde{U}(T(\tilde{v})),\tilde{V}(T(\tilde{v}))) - \tilde{H}(\tilde{U}(T(\tilde{u})),\tilde{V}(T(\tilde{u})))].
\end{aligned}$$

Thus, we have

$$\text{either } 0 \leq [\tilde{H}(\tilde{U}(T(\tilde{u})),\tilde{V}(T(\tilde{u}))) - \tilde{H}(\tilde{U}(T(\tilde{v})),\tilde{V}(T(\tilde{v})))]$$
$$\text{or } 0 \leq [\tilde{H}(\tilde{U}(T(\tilde{v})),\tilde{V}(T(\tilde{v}))) - \tilde{H}(\tilde{U}(T(\tilde{u})),\tilde{V}(T(\tilde{u})))].$$

i.e.,

$$\tilde{H}(\tilde{U}(T(\tilde{v})),\tilde{V}(T(\tilde{v}))) \leq \tilde{H}(\tilde{U}(T(\tilde{u})),\tilde{V}(T(\tilde{u}))) \tag{2.6}$$

$$\text{or } \tilde{H}(\tilde{U}(T(\tilde{u})), \tilde{V}(T(\tilde{u}))) \leq \tilde{H}(\tilde{U}(T(\tilde{v})), \tilde{V}(T(\tilde{v}))). \tag{2.7}$$

From (2.6) and (2.7), we have

$$\tilde{H}(\tilde{U}(T(\tilde{u})), \tilde{V}(T(\tilde{u}))) \propto \tilde{H}(\tilde{U}(T(\tilde{v})), \tilde{V}(T(\tilde{v})))$$

i.e.,

$$\tilde{H}(\tilde{U}(R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}(\tilde{u})), \tilde{V}(R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}T(\tilde{u}))) \propto \tilde{H}(\tilde{U}(R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}(\tilde{v})), \tilde{V}(R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}(\tilde{v}))).$$

Since $\tilde{H}$ is a mixed strongly comparison mapping with respect to $\tilde{U}$ and $\tilde{V}$, then $R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}(\tilde{u}) \propto R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}(\tilde{v})$, that is, $R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}$ is a comparison mapping. □

Lemma 2.17. *[6] Let $\tilde{U}, \tilde{V} : \mathcal{H}_p \to \mathcal{H}_p$ and $\tilde{H} : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings such that $\tilde{H}$ is mixed strongly comparison mapping with respect to $\tilde{U}$ and $\tilde{V}$, β_1 and β_2-ordered compression mapping with respect to $\tilde{U}$ and $\tilde{V}$, respectively. If $\tilde{M} : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ is $\tilde{H}(\cdot,\cdot)$-compression (α,λ)-XOR-NODSM, α-ordered rectangular mapping with $\lambda\alpha > (\beta_1+\beta_2), 0 < \beta_1, \beta_2 < 1$, then the resolvent operator $R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}} : \mathcal{H}_p \to \mathcal{H}_p$ is $\dfrac{1}{\lambda\alpha-(\beta_1+\beta_2)}$-Lipschitz continuous. i.e.,*

$$\|R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}(\tilde{u}) \oplus R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}(\tilde{v})\| \leq \Omega\|\tilde{u} \oplus \tilde{v}\|, \forall \tilde{u}, \tilde{v} \in \mathcal{H}_p,$$

where $\Omega = \dfrac{1}{\lambda\alpha-(\beta_1+\beta_2)}$.

3 Graph convergence for $\tilde{H}(\cdot,\cdot)$-compression (α,λ)-XOR-NODSM mapping

This section begins with the definition of the graph convergence. We establish the relationship between the graph convergence and resolvent operator convergence associated to $\tilde{H}(\cdot,\cdot)$-compression (α,λ)-XOR-NODSM mapping.

The graph of a multi-valued mapping $\tilde{M} : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ is defined by

$$graph(\tilde{M}) = \{(\tilde{u}, \tilde{v}) : \tilde{v} \in \tilde{M}(\tilde{u})\}.$$

Definition 3.1. Let $\tilde{U}, \tilde{V} : \mathcal{H}_p \to \mathcal{H}_p; \tilde{H} : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings and $\tilde{M}_n, \tilde{M} : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be $\tilde{H}(\cdot,\cdot)$-compression (α,λ)-XOR-NODSM mappings, $n = 0, 1, 2, \cdots$. The sequence $\{\tilde{M}_n\}$ is said to be graph convergence to $\tilde{M}$, indicated by $\tilde{M}_n \underset{\to}{G} \tilde{M}$ if for each element $(\tilde{u}, \tilde{v}) \in graph(\tilde{M})$, there exists $\{(\tilde{u}_n, \tilde{v}_n)\} \in graph(\tilde{M}_n)$ such that

$$\tilde{u}_n \to \tilde{u} \text{ and } \tilde{v}_n \to \tilde{v} \text{ as } n \to \infty.$$

Theorem 3.2. *Let $\tilde{U}, \tilde{V} : \mathcal{H}_p \to \mathcal{H}_p$ and $\tilde{H} : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings such that $\tilde{H}$ is β_1-ordered compression mapping with respect to $\tilde{U}$, β_2-ordered compression mapping with respect to $\tilde{V}$ and $\tilde{H}$ is comparison mapping with respect to $\tilde{U}$ and $\tilde{V}$, $R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}_n}(\tilde{u}) \propto R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}_n}(\tilde{v})$ and $R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}(\tilde{u}) \propto R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}(\tilde{v})$. Let $\tilde{M}_n, \tilde{M} : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be $\tilde{H}(\cdot,\cdot)$-compression (α,λ)-XOR-NODSM mappings for $n = 0, 1, 2, \cdots$. Then $\tilde{M}_n \underset{\to}{G} \tilde{M}$, if and only if*

$$R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}_n}(\tilde{u}) \to R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}(\tilde{u}), \forall \tilde{u} \in \mathcal{H}_p, \lambda > 0, 0 < \beta_1, \beta_2 < 1,$$

where; $R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}_n}(\tilde{u}) = [\tilde{H}(\tilde{U}, \tilde{V}) \oplus \lambda\tilde{M}_n]^{-1}(\tilde{u})$ *and* $R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}(\tilde{u}) = [\tilde{H}(\tilde{U}, \tilde{V}) \oplus \lambda\tilde{M}]^{-1}(\tilde{u})$.

Proof. Suppose that $\tilde{M}_n \underset{\to}{G} \tilde{M}$, then for any given $\tilde{w} \in \mathcal{H}_p$, let

$$\tilde{u}_n = R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}_n}(\tilde{w}) \text{ and } \tilde{u} = R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}(\tilde{w}).$$

Then $\tilde{u} = R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}(\tilde{w}) = [\tilde{H}(\tilde{U},\tilde{V}) \oplus \lambda\tilde{M}]^{-1}(\tilde{w})$, thus

$$\frac{1}{\lambda}[\tilde{w} \oplus \tilde{H}(\tilde{U}(\tilde{u}),\tilde{V}(\tilde{u}))] \in \tilde{M}(\tilde{u}),$$

which implies that

$$\Big(\tilde{u}, \frac{1}{\lambda}[\tilde{w} \oplus \tilde{H}(\tilde{U}(\tilde{u}),\tilde{V}(\tilde{u}))]\Big) \in graph(\tilde{M}).$$

Thus, by the graph convergence it follows that there exists a sequence $\{(\tilde{u}'_n, \tilde{v}'_n)\} \in graph(\tilde{M}_n)$ such that

$$\tilde{u}'_n \to \tilde{u} \text{ and } \tilde{v}'_n \to \frac{1}{\lambda}[\tilde{w} \oplus \tilde{H}(\tilde{U}\tilde{u},\tilde{V}\tilde{u})] \ \text{ as } n \to \infty. \tag{3.1}$$

Since $\tilde{v}'_n \in \tilde{M}_n(\tilde{u}'_n)$, we have

$$\tilde{H}(\tilde{U}(\tilde{u}'_n),\tilde{V}(\tilde{u}'_n)) \oplus \lambda\tilde{v}'_n \in [\tilde{H}(\tilde{U},\tilde{V}) \oplus \lambda\tilde{M}_n](\tilde{u}'_n)$$

and thus,

$$\tilde{u}'_n = R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}_n}[\tilde{H}(\tilde{U}(\tilde{u}'_n),\tilde{V}(\tilde{u}'_n)) \oplus \lambda\tilde{v}'_n].$$

Now,

$$\begin{aligned}\|\tilde{u}_n - \tilde{u}\| &\le \|\tilde{u}_n - \tilde{u}'_n\| + \|\tilde{u}'_n - \tilde{u}\|\\ &= \|\tilde{u}_n \oplus \tilde{u}'_n\| + \|\tilde{u}'_n \oplus \tilde{u}\|\\ &= \|R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}_n}(\tilde{w}) \oplus R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}_n}[\tilde{H}(\tilde{U}(\tilde{u}'_n),\tilde{V}(\tilde{u}'_n)) \oplus \lambda\tilde{v}'_n]\| + \|\tilde{u}'_n \oplus \tilde{u}\|.\end{aligned}$$

Utilizing the Lipschitz continuity of $R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}_n}$, we have

$$\begin{aligned}\|\tilde{u}_n - \tilde{u}\| &\le \Omega\|\tilde{w} \oplus [\tilde{H}(\tilde{U}(\tilde{u}'_n),\tilde{V}(\tilde{u}'_n)) \oplus \lambda\tilde{v}'_n]\| + \|\tilde{u}'_n \oplus \tilde{u}\|\\ &\le \Omega\|\tilde{w} \oplus [\tilde{H}(\tilde{U}(\tilde{u}'_n),\tilde{V}(\tilde{u}'_n)) - \lambda\tilde{v}'_n]\| + \|\tilde{u}'_n \oplus \tilde{u}\|\\ &\le \Omega\|[\tilde{w} \oplus \tilde{H}(\tilde{U}(\tilde{u}),\tilde{V}(\tilde{u}))] - \lambda\tilde{v}'_n\|\\ &\quad + \Omega\|\tilde{H}(\tilde{U}(\tilde{u}'_n),\tilde{V}(\tilde{u}'_n)) - \tilde{H}(\tilde{U}(\tilde{u}),\tilde{V}(\tilde{u}))\| + \|\tilde{u}'_n \oplus \tilde{u}\|\\ &\le \Omega\|[\tilde{w} \oplus \tilde{H}(\tilde{U}(\tilde{u}),\tilde{V}(\tilde{u}))] - \lambda\tilde{v}'_n\| + \Omega\|\tilde{H}(\tilde{U}(\tilde{u}'_n),\tilde{V}(\tilde{u}'_n)) - \tilde{H}(\tilde{U}(\tilde{u}),\tilde{V}(\tilde{u}'_n))\|\\ &\quad + \Omega\|\tilde{H}(\tilde{U}(\tilde{u}),\tilde{V}(\tilde{u}'_n)) - \tilde{H}(\tilde{U}(\tilde{u}),\tilde{V}(\tilde{u}))\| + \|\tilde{u}'_n \oplus \tilde{u}\|\\ &\le \Omega\|[\tilde{w} \oplus \tilde{H}(\tilde{U}(\tilde{u}),\tilde{V}(\tilde{u}))] - \lambda\tilde{v}'_n\| + \Omega\|\tilde{H}(\tilde{U}(\tilde{u}'_n),\tilde{V}(\tilde{u}'_n)) \oplus \tilde{H}(\tilde{U}(\tilde{u}),\tilde{V}(\tilde{u}'_n))\|\\ &\quad + \Omega\|\tilde{H}(\tilde{A}(\tilde{u}),\tilde{V}(\tilde{u}'_n)) \oplus \tilde{H}(\tilde{U}(\tilde{u}),\tilde{V}(\tilde{u}))\| + \|\tilde{u}'_n \oplus \tilde{u}\|.\end{aligned}$$

Since $\tilde{H}$ is β_1 and β_2-ordered compression mapping with respect to $\tilde{U}$ and $\tilde{V}$, respectively, then

$$\begin{aligned}\|\tilde{u}_n - \tilde{u}\| &\le \Omega\|[\tilde{w} \oplus \tilde{H}(\tilde{U}(\tilde{u}),\tilde{V}(\tilde{u}))] - \lambda\tilde{v}'_n\| + \Omega(\beta_1 \oplus \beta_2)\|\tilde{u}'_n \oplus \tilde{u}\| + \|\tilde{u}'_n \oplus \tilde{u}\|\\ &= \Omega\|[\tilde{w} \oplus \tilde{H}(\tilde{U}(\tilde{u}),\tilde{V}(\tilde{u}))] - \lambda\tilde{v}'_n\| + [1 + \Omega(\beta_1 \oplus \beta_2)]\|\tilde{u}'_n \oplus \tilde{u}\|\\ &= \Omega\|[\tilde{w} \oplus \tilde{H}(\tilde{U}(\tilde{u}),\tilde{V}(\tilde{u}))] - \lambda\tilde{v}'_n\| + [1 + \Omega(\beta_1 \oplus \beta_2)]\|\tilde{u}'_n - \tilde{u}\|.\end{aligned}$$

From (3.1), we know that $\tilde{u}'_n \to \tilde{u}$ and $\lambda\tilde{v}'_n \to \Big[\tilde{w} \oplus \tilde{H}(\tilde{U}\tilde{u},\tilde{V}\tilde{u})\Big]$ as $n \to \infty$, then

$$\|\tilde{u}_n - \tilde{u}\| \to 0 \ \text{ as } n \to \infty,$$

which implies that

$$R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}_n}(\tilde{w}) \to R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}(\tilde{w}).$$

Conversely, suppose that

$$R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}_n}(\tilde{w}) \to R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}(\cdot,\cdot)}(\tilde{w}), \ \ \forall \tilde{w} \in \mathcal{H}_p, \lambda > 0.$$

Let $(\tilde{u}, \tilde{v}) \in graph(\tilde{M})$, then $\tilde{v} \in \tilde{M}(\tilde{u})$. Thus we have

$$\tilde{H}(\tilde{U}(\tilde{u}), \tilde{V}(\tilde{u})) \oplus \lambda\tilde{v} \in [\tilde{H}(\tilde{U}, \tilde{V}) \oplus \lambda\tilde{M}](\tilde{u})$$

and hence

$$\tilde{u} = R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}[\tilde{H}(\tilde{U}(\tilde{u}), \tilde{V}(\tilde{u})) \oplus \lambda\tilde{v}]. \tag{3.2}$$

Let

$$\tilde{u}_n = R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}_n}[\tilde{H}(\tilde{U}(\tilde{u}), \tilde{V}(\tilde{u})) \oplus \lambda\tilde{v}], \tag{3.3}$$

Then we get

$$\frac{1}{\lambda}[\tilde{H}(\tilde{U}(\tilde{u}), \tilde{V}(\tilde{u})) \oplus \tilde{H}(\tilde{U}(\tilde{u}_n), \tilde{V}(\tilde{u}_n)) \oplus \lambda\tilde{v}] \in \tilde{M}_n(\tilde{u}_n).$$

Let $\tilde{v}_n' = \frac{1}{\lambda}[\tilde{H}(\tilde{U}(\tilde{u}), \tilde{V}(\tilde{u})) \oplus \tilde{H}(\tilde{U}(\tilde{u}_n), \tilde{V}(\tilde{u}_n)) \oplus \lambda\tilde{v}]$. Then

$$\begin{aligned}\|\tilde{v}_n' - \tilde{v}\| &= \|\tfrac{1}{\lambda}[\tilde{H}(\tilde{U}(\tilde{u}), \tilde{V}(\tilde{u})) \oplus \tilde{H}(\tilde{U}(\tilde{u}_n), \tilde{V}(\tilde{u}_n)) \oplus \lambda\tilde{y}] - \tilde{y}\| \\ &\le \tfrac{1}{\lambda}\|[\tilde{H}(\tilde{U}(\tilde{u}), \tilde{V}(\tilde{u})) \oplus \tilde{H}(\tilde{U}(\tilde{u}_n), \tilde{V}(\tilde{u}_n))] \oplus (\tilde{y} \oplus \tilde{y})\| \\ &\le \tfrac{1}{\lambda}\|[\tilde{H}(\tilde{U}(\tilde{u}), \tilde{V}(\tilde{u})) \oplus \tilde{H}(\tilde{U}(\tilde{u}_n), \tilde{V}(\tilde{u}_n))\|.\end{aligned}$$

Again using the fact that $\tilde{H}$ is β_1 and β_2-ordered compression mapping with respect to $\tilde{U}$ and $\tilde{V}$, respectively, then

$$\begin{aligned}\|\tilde{v}_n' - \tilde{v}\| &\le \tfrac{1}{\lambda}(\beta_1 + \beta_2)\|\tilde{u}_n \oplus \tilde{u}\| \\ &\le \tfrac{1}{\lambda}(\beta_1 + \beta_2)\|\tilde{u}_n - \tilde{u}\|.\end{aligned} \tag{3.4}$$

Also from (3.2) and (3.3), we have

$$\begin{aligned}\|\tilde{u}_n - \tilde{u}\| &= \|R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}_n}[\tilde{H}(\tilde{U}(\tilde{u}), \tilde{V}(\tilde{u})) \oplus \lambda\tilde{v}] - R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}[\tilde{H}(\tilde{U}(\tilde{u}), \tilde{V}(\tilde{u})) \oplus \lambda\tilde{v}]\| \\ &= \|(R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}_n} - R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}})[\tilde{H}(\tilde{U}(\tilde{u}), \tilde{V}(\tilde{u})) \oplus \lambda\tilde{v}]\|.\end{aligned}$$

Since $R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}_n} \to R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}$, we have $\|\tilde{u}_n - \tilde{u}\| \to 0$ as $n \to \infty$. Thus from (3.4), we have $\|\tilde{v}_n' - \tilde{v}\| \to 0$ as $n \to \infty$. Hence $\tilde{M}_n \xrightarrow{G} \tilde{M}$. This completes the proof. □

The graph convergence for $\tilde{H}(\cdot,\cdot)$-compression (α, λ)-XOR-NODSM mapping is verified by the following illustration.

Example 3.3. Let $\mathcal{H}_p = \mathbb{R}^2_+ = \mathbb{R}^+ \times \mathbb{R}^+$ with usual inner product and norm and let $\mathcal{C} = [0,1] \times [0,1]$ be a normal cone. Let $\tilde{U}, \tilde{V} : \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings defined by

$$\tilde{U}(x_1, x_2) = \left(\frac{x_1}{2}, \frac{x_2}{3}\right), \ \forall (x_1, x_2) \in \mathcal{H}_p \times \mathcal{H}_p,$$

$$\tilde{V}(x_1, x_2) = \left(\frac{x_1}{3}, \frac{x_2}{4}\right), \ \forall (x_1, x_2) \in \mathcal{H}_p \times \mathcal{H}_p.$$

Let $\tilde{H} : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be a bi-mapping defined by

$$\tilde{H}(\tilde{U}(x), \tilde{V}(x)) = \tilde{U}(x) \oplus \tilde{V}(x), \ \forall x = (x_1, x_2) \in \mathcal{H}_p \times \mathcal{H}_p.$$

Then for any $u = (u_1, u_2), x, y \in \mathcal{H}_p \times \mathcal{H}_p, x \propto y$, we have

$$\begin{aligned}\tilde{H}(\tilde{U}(x), u) \oplus \tilde{H}(\tilde{U}(y), u) &= (\tilde{U}(x) \oplus u) \oplus (\tilde{U}(y) \oplus u) \\ &= \left[\left(\frac{x_1}{2}, \frac{x_2}{3}\right) \oplus (u_1, u_2)\right] \oplus \left[\left(\frac{y_1}{2}, \frac{y_2}{3}\right) \oplus (u_1, u_2)\right] \\ &\le \frac{1}{2}[(x_1, x_2) \oplus (y_1, y_2)] \\ &= \frac{1}{2}(x \oplus y).\end{aligned}$$

$$\text{i.e.,}\quad \tilde{H}(\tilde{U}(x),u)\oplus \tilde{H}(\tilde{U}(y),u)\le \frac{1}{2}(x\oplus y).$$

Hence, $\tilde{H}(\tilde{U},\tilde{V})$ is $\frac{1}{2}$-ordered compression mapping with respect to $\tilde{U}$. Similarly, one can show that $\tilde{H}(\tilde{U},\tilde{V})$ is $\frac{1}{3}$-ordered compression mapping with respect to $\tilde{V}$.

Let $\tilde{M},\tilde{M}_n:\mathcal{H}_p\to 2^{\mathcal{H}_p}$ be multi-valued mappings defined by

$$\tilde{M}(x)=\{3x_1,3x_2\},\forall x=(x_1,x_2)\in\mathcal{H}_p,$$

$$\tilde{M}_n(x)=\{3x_1+\frac{1}{n},3x_2+\frac{3}{2n}\},\forall x=(x_1,x_2)\in\mathcal{H}_p.$$

Then it is easy to substantiate that $\tilde{M}$ is a comparison mapping. Let $p_x=(3x_1,3x_2)\in\tilde{M}(x)$ and $p_y=(3y_1,3y_2)\in\tilde{M}(y)$, then

$$\begin{aligned}(p_x\oplus p_y)+3(x,y)&=((3x_1,3x_2)\oplus(3y_1,3y_2))\oplus 3(x\oplus y)\\&=3[(x\oplus y)\oplus(x\oplus y)]=0.\end{aligned}$$

Hence, $\tilde{M}$ is 3-non-ordinary difference mapping. Also

$$(p_x\oplus p_y)+3(x,y)=((3x_1,3x_2)\oplus(3y_1,3y_2))=3[(x\oplus y)],$$

which implies that $\frac{1}{3}(p_x\oplus p_y)\ge(x\oplus y)$. Thus, $\tilde{M}$ is a $\frac{1}{3}$-XOR-ordered strongly monotone mapping. Also for any $x=(x_1,x_2)\in\mathcal{H}_p,\lambda=1$, we have

$$[\tilde{H}(\tilde{U},\tilde{V})+\lambda\tilde{M}](\mathcal{H}_p)=\mathcal{H}_p.$$

Hence, $\tilde{M}$ is $\tilde{H}(\cdot,\cdot)$-compression (α,λ)-ordered NODSM mapping with respect to $\tilde{U}$ and $\tilde{V}$.

Now, we show that $\tilde{M}_n \underset{\rightarrow}{G} \tilde{M}$. Let $u=\left(\frac{u_1}{3},\frac{u_2}{4}\right)$ and $u_n=\left(\frac{u_1}{3}+\frac{1}{n},\frac{u_2}{4}+\frac{2}{n}\right)$. Then $v_n=\tilde{M}_n(u_n)=\left(u_1+\frac{4}{n},\frac{3u_2}{4}+\frac{15}{2n}\right)$ and $v=\tilde{M}(u)=\left(u_1,\frac{3u_2}{4}\right)$. Thus we have

$$\lim_{n\to\infty}v_n=v \text{ and } \lim_{n\to\infty}u_n=u,\quad \text{as } n\to\infty.$$

Hence, $\tilde{M}_n \underset{\rightarrow}{G} \tilde{M}$ as $n\to\infty$.

$$\begin{aligned}R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}(x)&=[\tilde{H}(\tilde{U},\tilde{V})\oplus\lambda\tilde{M}]^{-1}(x)\\&=[(\tilde{U}(x)\oplus\tilde{V}(x))\oplus\tilde{M}(x)]^{-1}(x)\\&=\left(\tfrac{23}{6}x_1,\tfrac{25}{6}x_2\right)^{-1}=\left(\tfrac{6}{23}x_1,\tfrac{6}{25}x_2\right).\end{aligned}\tag{3.5}$$

Thus, we have

$$\begin{aligned}\|R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}(x)\oplus R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}(y)\|&=\left\|\left(\frac{6}{23}x_1,\frac{6}{25}x_2\right)\oplus\left(\frac{6}{23}x_1,\frac{6}{25}x_2\right)\right\|\\&\le\frac{6}{23}\|x\oplus y\|.\end{aligned}$$

Therefore, the resolvent operator $R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}$ is $\frac{6}{23}$-Lipschitz continuous mapping.

Next, we establish the relation between graph convergence and resolvent operator convergence of $\tilde{H}(\cdot,\cdot)$-compression (α,λ)-ordered NODSM mapping.

Now for $\lambda=1$, the resolvent operator is given by

$$\begin{aligned}R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}_n}(x)&=[\tilde{H}(\tilde{U},\tilde{V})\oplus\lambda\tilde{M}_n]^{-1}(x)\\&=[(\tilde{U}(x)\oplus\tilde{V}(x))\oplus\tilde{M}_n(x)]^{-1}(x),\\&=\left(\tfrac{23}{6}x_1+\tfrac{1}{n},\tfrac{25}{6}x_2+\tfrac{3}{2n}\right)^{-1}\\&=\left(\tfrac{6}{23}x_1-\tfrac{6}{23n},\tfrac{6}{25}x_2-\tfrac{9}{25n}\right).\end{aligned}\tag{3.6}$$

It follows from (3.5) and (3.6) that

$$\|R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}_n}(x)-R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}(x)\|\to 0,\quad \text{as } n\to\infty.$$

Thus, we have

$$R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}_n}(x)\to R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}}(x)\Leftrightarrow \tilde{M}_n \underset{\rightarrow}{G} \tilde{M}.$$

4 Conceptualization of the ordered inclusion problem and existence result

In what follows, we conceptualize the ordered inclusion problem associated to $\tilde{H}(\cdot,\cdot)$-compression (α,λ)-XOR-NODSM mapping. We establish an ordered fixed point problem for considered ordered inclusion problem and examine the existence of unique solution using XOR and XNOR operations.

Let $\mathcal{H}_p$ be a real ordered positive Hilbert space. Let $\tilde{U},\tilde{V},\tilde{g}:\mathcal{H}_p\to\mathcal{H}_p$ and $\tilde{H},\tilde{T}:\mathcal{H}_p\times\mathcal{H}_p\to\mathcal{H}_p$ be the single-valued mappings; let $\tilde{P},\tilde{Q},\tilde{R}:\mathcal{H}_p\to CB(\mathcal{H}_p)$ be the multi-valued mappings. Let $\tilde{M}:\mathcal{H}_p\times\mathcal{H}_p\to 2^{\mathcal{H}_p}$ be an $\tilde{H}(\cdot,\cdot)$-compression (α,λ)-ordered NODSM mapping with respect to $\tilde{U}$ and $\tilde{V}$. We propose the following ordered inclusion problem:

Find $\tilde{x}\in\mathcal{H}_p,\tilde{u}\in\tilde{P}(\tilde{x}),\tilde{v}\in\tilde{Q}(\tilde{x})$ and $\tilde{w}\in\tilde{R}(\tilde{x})$ such that

$$0\in\tilde{T}(\tilde{u},\tilde{v})\oplus\tilde{M}(\tilde{g}(\tilde{x}),\tilde{w}). \tag{4.1}$$

Note that the ordered inclusion problem (4.1) is more prevalent. For diverse selection of the mappings involved in the formulation, our problem include many problems existing in the literature as specialization; see, [6, 29, 32, 33].

Now, we establish the correspondence between ordered fixed point problem and ordered inclusion problem.

Lemma 4.1. *The ordered inclusion problem (4.1) has a solution $(\tilde{x},\tilde{u},\tilde{v},\tilde{w})$, if and only if $(\tilde{x},\tilde{u},\tilde{v},\tilde{w}),\tilde{x}\in\mathcal{H}_p,\tilde{u}\in\tilde{P}(\tilde{x}),\tilde{v}\in\tilde{Q}(\tilde{x})$ and $\tilde{w}\in\tilde{R}(\tilde{x})$ solves the following ordered fixed point problem*

$$\tilde{x}=R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}(\tilde{g}(\cdot),\tilde{w})}[\tilde{H}(\tilde{U}(\tilde{x}),\tilde{V}(\tilde{x}))\oplus\lambda\tilde{T}(\tilde{u},\tilde{v})].$$

where $\lambda>0$ is a constant.

Proof. By following the definition of resolvent operator, we have

$$\begin{aligned}&\tilde{x}=R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}(\tilde{g}(\cdot),\tilde{w})}[\tilde{H}(\tilde{U}(\tilde{x}),\tilde{V}(\tilde{x}))\oplus\lambda\tilde{T}(\tilde{u},\tilde{v})]\\ \Leftrightarrow&\tilde{H}(\tilde{U}(\tilde{x}),\tilde{V}(\tilde{x}))\oplus\lambda\tilde{M}(\tilde{g}(\tilde{x}),\tilde{w})=[\tilde{H}(\tilde{U}(\tilde{x}),\tilde{V}(\tilde{x}))\oplus\lambda\tilde{T}(\tilde{u},\tilde{v})]\\ \Leftrightarrow&0\in\tilde{T}(\tilde{u},\tilde{v})\oplus\tilde{M}(\tilde{g}(\tilde{x}),\tilde{w}).\end{aligned}$$

□

Theorem 4.2. *Let $\tilde{U},\tilde{V},\tilde{g}:\mathcal{H}_p\to\mathcal{H}_p$ and $\tilde{H},\tilde{T}:\mathcal{H}_p\times\mathcal{H}_p\to\mathcal{H}_p$ be the single-valued mappings such that $\tilde{H}$ is mixed strongly comparison mapping with respect to $\tilde{U}$ and $\tilde{V}$, β_1 and β_2-ordered compression mapping with respect to $\tilde{U}$ and $\tilde{V}$, respectively and $\tilde{T}$ is $\gamma'_{\tilde{T}}$ and $\gamma''_{\tilde{T}}$-ordered compression mapping in the first and second argument, respectively. Let $\tilde{P},\tilde{Q},\tilde{R}:\mathcal{H}_p\to CB(\mathcal{H}_p)$ be ordered $\mathcal{D}$-Lipschitz continuous mapping with constants $\zeta_{\tilde{P}},\zeta_{\tilde{Q}},\zeta_{\tilde{R}}$, respectively. Let $\tilde{M}:\mathcal{H}_p\times\mathcal{H}_p\to 2^{\mathcal{H}_p}$ be an $\tilde{H}(\cdot,\cdot)$-compression (α,λ) ordered NODSM mapping. Let $R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}(g(\cdot),\tilde{w}_1)}(\tilde{x}_1)\propto R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}(g(\cdot),\tilde{w}_2)}(\tilde{x}_2),\tilde{x}_1\propto\tilde{x}_2$ and the following condition is satisfied:*

$$\lambda_{\mathcal{C}}[\Omega(\beta_1+\beta_2)+|\lambda|(\gamma'_{\tilde{T}}\zeta_{\tilde{P}}+\gamma''_{\tilde{T}}\zeta_{\tilde{Q}})+\vartheta\zeta_{\tilde{R}}]<1, \tag{4.2}$$

where $\Omega=\dfrac{1}{\lambda\alpha-(\beta_1+\beta_2)}$, $\lambda\alpha>(\beta_1+\beta_2)$. In addition the following condition holds:

$$\begin{aligned}\|R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}(g(\cdot),\tilde{w}_1)}[\tilde{H}(\tilde{U}(\tilde{x}_2),\tilde{V}(\tilde{x}_2))\oplus\lambda\tilde{T}(\tilde{u}_2,\tilde{v}_2)]\quad&\oplus R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}(\tilde{g}(\cdot),\tilde{w}_2)}[\tilde{H}(\tilde{U}(\tilde{x}_2),\tilde{V}(\tilde{x}_2))\oplus\lambda\tilde{T}(\tilde{u}_2,\tilde{v}_2)]\|\\ &\le\vartheta\|\tilde{w}_1\oplus\tilde{w}_2\|.\end{aligned} \tag{4.3}$$

Then ordered inclusion problem (4.1) has a unique solution.

Proof. Define a single-valued mapping $\tilde{G}:\mathcal{H}_p\to\mathcal{H}_p$ by

$$\tilde{G}(\tilde{x})=R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}(g(\cdot),\tilde{w})}[\tilde{H}(\tilde{U}(\tilde{x}),\tilde{V}(\tilde{x}))\oplus\lambda\tilde{T}(\tilde{u},\tilde{v})].$$

It follows from Lemma 2.5 that

$$\begin{aligned} 0 &\le \tilde{G}(\tilde{x}_1)\oplus\tilde{G}(\tilde{x}_2) \\ &= R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}(\tilde{g}(\cdot),\tilde{w}_1)}[\tilde{H}(\tilde{U}(\tilde{x}_1),\tilde{V}(\tilde{x}_1))\oplus\lambda\tilde{T}(\tilde{u}_1,\tilde{v}_1)] \\ &\quad\oplus R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}(\tilde{g}(\cdot),\tilde{w}_2)}[\tilde{H}(\tilde{U}(\tilde{x}_2),\tilde{V}(\tilde{x}_2))\oplus\lambda\tilde{T}(\tilde{u}_2,\tilde{v}_2)] \\ &\le R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}(\tilde{g}(\cdot),\tilde{w}_1)}[\tilde{H}(\tilde{U}(\tilde{x}_1),\tilde{V}(\tilde{x}_1))\oplus\lambda\tilde{T}(\tilde{u}_1,\tilde{v}_1)] \\ &\quad\oplus R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}(\tilde{g}(\cdot),\tilde{w}_1)}[\tilde{H}(\tilde{U}(\tilde{x}_2),\tilde{V}(\tilde{x}_2))\oplus\lambda\tilde{T}(\tilde{u}_2,\tilde{v}_2)] \\ &\quad\oplus R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}(\tilde{g}(\cdot),\tilde{w}_1)}[\tilde{H}(\tilde{U}(\tilde{x}_2),\tilde{V}(\tilde{x}_2))\oplus\lambda\tilde{T}(\tilde{u}_2,\tilde{v}_2)] \\ &\quad\oplus R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}(\tilde{g}(\cdot),\tilde{w}_2)}[\tilde{H}(\tilde{U}(\tilde{x}_2),\tilde{V}(\tilde{x}_2))\oplus\lambda\tilde{T}(\tilde{u}_2,\tilde{v}_2)]. \end{aligned} \tag{4.4}$$

Employing Lemma 2.3, we have

$$\begin{aligned} \|\tilde{G}(\tilde{x}_1)\oplus\tilde{G}(\tilde{x}_2)\| &\le \|R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}(\tilde{g}(\cdot),\tilde{w}_1)}[\tilde{H}(\tilde{U}(\tilde{x}_1),\tilde{V}(\tilde{x}_1))\oplus\lambda\tilde{T}(\tilde{u}_1,\tilde{v}_1)] \\ &\quad\oplus R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}(\tilde{g}(\cdot),\tilde{w}_1)}[\tilde{H}(\tilde{U}(\tilde{x}_2),\tilde{V}(\tilde{x}_2))\oplus\lambda\tilde{T}(\tilde{u}_2,\tilde{v}_2)] \\ &\quad\oplus R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}(\tilde{g}(\cdot),\tilde{w}_1)}[\tilde{H}(\tilde{U}(\tilde{x}_2),\tilde{V}(\tilde{x}_2))\oplus\lambda\tilde{T}(\tilde{u}_2,\tilde{v}_2)] \\ &\quad\oplus R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}(\tilde{g}(\cdot),\tilde{w}_2)}[\tilde{H}(\tilde{U}(\tilde{x}_2),\tilde{V}(\tilde{x}_2))\oplus\lambda\tilde{T}(\tilde{u}_2,\tilde{v}_2)]\| \\ &\le \lambda_{\mathcal{C}}\|R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}(\tilde{g}(\cdot),\tilde{w}_1)}[\tilde{H}(\tilde{U}(\tilde{x}_1),\tilde{V}(\tilde{x}_1))\oplus\lambda\tilde{T}(\tilde{u}_1,\tilde{v}_1)] \\ &\quad\oplus R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}(\tilde{g}(\cdot),\tilde{w}_1)}[\tilde{H}(\tilde{U}(\tilde{x}_2),\tilde{V}(\tilde{x}_2))\oplus\lambda\tilde{T}(\tilde{u}_2,\tilde{v}_2)]\| \\ &\quad+\lambda_{\mathcal{C}}\|R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}(\tilde{g}(\cdot),\tilde{w}_1)}[\tilde{H}(\tilde{U}(\tilde{x}_2),\tilde{V}(\tilde{x}_2))\oplus\lambda\tilde{T}(\tilde{u}_2,\tilde{v}_2)] \\ &\quad\oplus R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}(\tilde{g}(\cdot),\tilde{w}_2)}[\tilde{H}(\tilde{U}(\tilde{x}_2),\tilde{V}(\tilde{x}_2))\oplus\lambda\tilde{T}(\tilde{u}_2,\tilde{v}_2)]\|. \end{aligned} \tag{4.5}$$

Employing Lemma 2.17 and assumption (4.3), we get

$$\begin{aligned} \|\tilde{G}(\tilde{x}_1)\oplus\tilde{G}(\tilde{x}_2)\| &\le \lambda_{\mathcal{C}}\Omega\|[\tilde{H}(\tilde{U}(\tilde{x}_1),\tilde{V}(\tilde{x}_1))\oplus\lambda\tilde{T}(\tilde{u}_1,\tilde{v}_1)] \\ &\quad\oplus[\tilde{H}(\tilde{U}(\tilde{x}_2),\tilde{V}(\tilde{x}_2))\oplus\lambda\tilde{T}(\tilde{u}_2,\tilde{v}_2)]\|+\lambda_{\mathcal{C}}\vartheta\|\tilde{w}_1\oplus\tilde{w}_2\| \\ &\le \lambda_{\mathcal{C}}\Omega\|[\tilde{H}(\tilde{U}(\tilde{x}_1),\tilde{V}(\tilde{x}_1))\oplus\tilde{H}(\tilde{U}(\tilde{x}_2),\tilde{V}(\tilde{x}_2))] \\ &\quad-(\lambda\tilde{T}(\tilde{u}_1,\tilde{v}_1)\oplus\lambda\tilde{T}(\tilde{u}_2,\tilde{v}_2))\|+\lambda_{\mathcal{C}}\vartheta\|\tilde{w}_1\oplus\tilde{w}_2\| \\ &\le \lambda_{\mathcal{C}}\Omega\|\tilde{H}(\tilde{U}(\tilde{x}_1),\tilde{V}(\tilde{x}_1))\oplus\tilde{H}(\tilde{U}(\tilde{x}_2),\tilde{V}(\tilde{x}_2))\| \\ &\quad+|\lambda|\|\tilde{T}(\tilde{u}_1,\tilde{v}_1)\oplus\tilde{T}(\tilde{u}_2,\tilde{v}_2)\|+\lambda_{\mathcal{C}}\vartheta\|\tilde{w}_1\oplus\tilde{w}_2\|. \end{aligned} \tag{4.6}$$

Since $\tilde{H}$ is β_1 and β_2-ordered compression mapping with respect to $\tilde{U}$ and $\tilde{V}$, respectively, then

$$\begin{aligned} \|\tilde{H}(\tilde{U}(\tilde{x}_1),\tilde{V}(\tilde{x}_1)) &\oplus\tilde{H}(\tilde{U}(\tilde{x}_2),\tilde{V}(\tilde{x}_2))\| \\ &\le \|\tilde{H}(\tilde{U}(\tilde{x}_1),\tilde{V}(\tilde{x}_1))\oplus\tilde{H}(\tilde{U}(\tilde{x}_2),\tilde{V}(\tilde{x}_1)) \\ &\quad-[\tilde{H}(\tilde{U}(\tilde{x}_2),\tilde{V}(\tilde{x}_1))\oplus\tilde{H}(\tilde{U}(\tilde{x}_2),\tilde{V}(\tilde{x}_2))]\| \\ &\le \|\tilde{H}(\tilde{U}(\tilde{x}_1),\tilde{V}(\tilde{x}_1))\oplus\tilde{H}(\tilde{U}(\tilde{x}_2),\tilde{V}(\tilde{x}_1))\| \\ &\quad+\|\tilde{H}(\tilde{U}(\tilde{x}_2),\tilde{V}(\tilde{x}_1))\oplus\tilde{H}(\tilde{U}(\tilde{x}_2),\tilde{V}(\tilde{x}_2))\| \\ &\le (\beta_1+\beta_2)\|\tilde{x}_1\oplus\tilde{x}_2\|. \end{aligned} \tag{4.7}$$

Since $\tilde{T}$ is $\gamma'_{\tilde{T}}$ and $\gamma''_{\tilde{T}}$-ordered compression mapping in the first and second argument, respectively, $\tilde{P}$ is ordered $\mathcal{D}$-Lipschitz continuous mapping with constant $\zeta_{\tilde{P}}$ and $\tilde{Q}$ is ordered $\mathcal{D}$-Lipschitz continuous mapping with constant $\zeta_{\tilde{Q}}$, therefore

$$\begin{aligned} \|\tilde{T}(\tilde{u}_1,\tilde{v}_1)\oplus\tilde{T}(\tilde{u}_2,\tilde{v}_2)\| &\le \|\tilde{T}(\tilde{u}_1,\tilde{v}_1)\oplus\tilde{T}(\tilde{u}_2,\tilde{v}_1)\oplus\tilde{T}(\tilde{u}_2,\tilde{v}_1)\oplus\tilde{T}(\tilde{u}_2,\tilde{v}_2)\| \\ &\le \|(\tilde{T}(\tilde{u}_1,\tilde{v}_1)\oplus\tilde{T}(\tilde{u}_2,\tilde{v}_1))-(\tilde{T}(\tilde{u}_2,\tilde{v}_1)\oplus\tilde{T}(\tilde{u}_2,\tilde{v}_2))\| \\ &\le \|\tilde{T}(\tilde{u}_1,\tilde{v}_1)\oplus\tilde{T}(\tilde{u}_2,\tilde{v}_1)\|+\|\tilde{T}(\tilde{u}_2,\tilde{v}_1)\oplus\tilde{T}(\tilde{u}_2,\tilde{v}_2)\| \\ &\le \gamma'_{\tilde{T}}\|\tilde{u}_1\oplus\tilde{u}_2\|+\gamma''_{\tilde{T}}\|\tilde{v}_1\oplus\tilde{v}_2\| \\ &\le \gamma'_{\tilde{T}}\mathcal{D}(\tilde{P}(\tilde{x}_1),\tilde{P}(\tilde{x}_2))+\gamma''_{\tilde{T}}\mathcal{D}(\tilde{Q}(\tilde{x}_1),\tilde{Q}(\tilde{x}_2)) \\ &\le \gamma'_{\tilde{T}}\zeta_{\tilde{P}}\|\tilde{x}_1\oplus\tilde{x}_2\|+\gamma''_{\tilde{T}}\zeta_{\tilde{Q}}\|\tilde{x}_1\oplus\tilde{x}_2\| \\ &= (\gamma'_{\tilde{T}}\zeta_{\tilde{P}}+\gamma''_{\tilde{T}}\zeta_{\tilde{Q}})\|\tilde{x}_1\oplus\tilde{x}_2\|. \end{aligned} \tag{4.8}$$

Utilizing the assumption that $\tilde{R}$ is ordered $\mathcal{D}$-Lipschitz continuous with constant $\zeta_{\tilde{R}}$, we get

$$\|\tilde{w}_1 \oplus \tilde{w}_2\| \leq \mathcal{D}(\tilde{R}(\tilde{x}_1), \tilde{R}(\tilde{x}_2)) \leq \zeta_{\tilde{R}} \|\tilde{x}_1 \oplus \tilde{x}_2\|. \tag{4.9}$$

Making use of (4.7)-(4.9), (4.6) becomes

$$\begin{aligned} \|\tilde{G}(\tilde{x}_1) \oplus \tilde{G}(\tilde{x}_2)\| &\leq [\lambda_{\mathcal{C}}\Omega(\beta_1 + \beta_2) + |\lambda|\lambda_{\mathcal{C}}(\gamma'_{\tilde{T}}\zeta_{\tilde{P}} + \gamma''_{\tilde{T}}\zeta_{\tilde{Q}}) + \lambda_{\mathcal{C}}\vartheta\zeta_R]\|\tilde{x}_1 \oplus \tilde{x}_2\| \\ &= \Theta\|\tilde{x}_1 \oplus \tilde{x}_2\|, \end{aligned} \tag{4.10}$$

where $\Theta = \lambda_{\mathcal{C}}[\Omega(\beta_1+\beta_2)+|\lambda|(\gamma'_{\tilde{T}}\zeta_{\tilde{P}}+\gamma''_{\tilde{T}}\zeta_{\tilde{Q}})+\vartheta\zeta_{\tilde{R}}]$ and $\Omega = \dfrac{1}{\lambda\alpha - (\beta_1 + \beta_2)}$. It follows from the assumption (4.2) that $0 < \Theta < 1$. Hence, (4.10) and Banach contraction principle guarantees that $\tilde{G}$ is a contraction mapping. Then there exists a unique $\tilde{x} \in \mathcal{H}_p$ such that

$$\tilde{G}(\tilde{x}) = R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}(\tilde{g}(\cdot),\tilde{w})}[\tilde{H}(\tilde{U}(\tilde{x}), \tilde{V}(\tilde{x})) \oplus \lambda\tilde{T}(\tilde{u}, \tilde{v})].$$

Thus, by Lemma 4.1, we can deduce that $(\tilde{x}, \tilde{u}, \tilde{v}, \tilde{w}), \tilde{x} \in \mathcal{H}_p, \tilde{u} \in \tilde{P}(\tilde{x}), \tilde{v} \in \tilde{Q}(\tilde{x})$ and $\tilde{w} \in \tilde{R}(\tilde{x})$ is a unique solution of the ordered inclusion problem (4.1). □

5 Iterative algorithm and convergence analysis

This section begins with the construction of iterative algorithm. Finally, the convergence analysis of the sequences generated by the proposed iterative algorithm to the unique solution of ordered inclusion problem (4.1) is discussed.

Algorithm 5.1. Let $\tilde{U}, \tilde{V}, \tilde{g} : \mathcal{H}_p \to \mathcal{H}_p$ and $\tilde{H}, \tilde{T} : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings; let $\tilde{P}, \tilde{Q}, \tilde{R} : \mathcal{H}_p \to CB(\mathcal{H}_p)$ be the multi-valued mappings. Let $\tilde{M} : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a set-valued $\tilde{H}(\cdot,\cdot)$-compression (α, λ)-ordered NODSM mapping with respect to $\tilde{U}$ and $\tilde{V}$.

For any $\tilde{x}_0 \in \mathcal{H}_p$, $\tilde{u}_0 \in \tilde{P}(\tilde{x}_0)$, $\tilde{v} \in \tilde{Q}(\tilde{x}_0)$ and $\tilde{w} \in \tilde{R}(\tilde{x}_0)$, compute the sequences $\{\tilde{x}_n\}$, $\{\tilde{u}_n\}$, $\{\tilde{v}_n\}$ and $\{\tilde{w}_n\}$ by the following iteration process:

$$\begin{aligned} &\tilde{x}_{n+1} = (1-a)\tilde{x}_n + a[R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}_n(\tilde{g}(\cdot),\tilde{w}_n)}[\tilde{H}(\tilde{U}(\tilde{x}_n), \tilde{V}(\tilde{x}_n)) \oplus \lambda\tilde{T}(\tilde{u}_n, \tilde{v}_n)]], \\ &\tilde{u}_{n+1} \in \tilde{P}(\tilde{x}_{n+1}) : \|\tilde{u}_{n+1} \oplus \tilde{u}_n\| \leq \mathcal{D}(\tilde{P}(\tilde{x}_{n+1}), \tilde{P}(\tilde{x}_n)), \\ &\tilde{v}_{n+1} \in \tilde{Q}(\tilde{x}_{n+1}) : \|\tilde{v}_{n+1} \oplus \tilde{v}_n\| \leq \mathcal{D}(\tilde{Q}(\tilde{x}_{n+1}), \tilde{Q}(\tilde{x}_n)), \\ &\tilde{w}_{n+1} \in \tilde{R}(\tilde{x}_{n+1}) : \|\tilde{w}_{n+1} \oplus \tilde{w}_n\| \leq \mathcal{D}(\tilde{R}(\tilde{x}_{n+1}), \tilde{R}(\tilde{x}_n)), \end{aligned}$$

where $a \in [0, 1]$, $\lambda > 0$ and $n = 0, 1, 2, \cdots$.

Theorem 5.2. *Suppose that the mappings $\tilde{U}, \tilde{V}, \tilde{g}, \tilde{H}, \tilde{T}, \tilde{P}, \tilde{Q}, \tilde{R}$ and $\tilde{M}$ are same as in Theorem 4.1. Let $\tilde{M}_n : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be an $\tilde{H}(\cdot,\cdot)$-compression (α, λ)-ordered NODSM mapping with respect to $\tilde{U}$ and $\tilde{V}$ such that $\tilde{M}_n \underset{\rightarrow}{G} \tilde{M}$ and $x_n \propto x$. If the following condition holds:*

$$0 < \lambda_{\mathcal{C}}[(1-a) + a\Omega(\beta_1 + \beta_2) + a\Omega|\lambda|(\gamma'_{\tilde{T}}\zeta_{\tilde{P}} + \gamma''_{\tilde{T}}\zeta_{\tilde{Q}})] < 1. \tag{5.1}$$

Then the iterative sequences $\{\tilde{x}_n\}, \{\tilde{u}_n\}, \{\tilde{v}_n\}$ and $\{\tilde{w}_n\}$ generated by Algorithm 5.1 converge strongly to the unique solution $(\tilde{x}, \tilde{u}, \tilde{v}, \tilde{w}), \tilde{x} \in \mathcal{H}_p, \tilde{u} \in \tilde{P}(\tilde{x}), \tilde{v} \in \tilde{Q}(\tilde{x})$ and $\tilde{w} \in \tilde{R}(\tilde{x})$ of ordered inclusion problem (4.1).

Proof. It follows from Algorithm 5.1 and Lemma 2.5 that

$$\begin{aligned} 0 &\leq \tilde{x}_{n+1} \oplus \tilde{x} \\ &= \Big[(1-a)\tilde{x}_n + a\Big(R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}_n(\tilde{g}(\cdot),\tilde{w}_n)}[\tilde{H}(\tilde{U}(\tilde{x}_n), \tilde{V}(\tilde{x}_n)) \oplus \lambda\tilde{T}(\tilde{u}_n, \tilde{v}_n)]\Big)\Big] \\ &\quad \oplus \Big[(1-a)\tilde{x} + a\Big(R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}(\tilde{g}(\cdot),\tilde{w})}[\tilde{H}(\tilde{U}(\tilde{x}), \tilde{V}(\tilde{x})) \oplus \lambda\tilde{T}(\tilde{u}, \tilde{v})]\Big)\Big] \\ &\leq (1-a)(\tilde{x}_n \oplus \tilde{x}) + a\Big[R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}_n(\tilde{g}(\cdot),\tilde{w}_n)}[\tilde{H}(\tilde{U}(\tilde{x}_n), \tilde{V}(\tilde{x}_n)) \oplus \lambda\tilde{T}(\tilde{u}_n, \tilde{v}_n)] \\ &\quad \oplus R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}(\tilde{g}(\cdot),\tilde{w})}[\tilde{H}(\tilde{U}(\tilde{x}), \tilde{V}(\tilde{x})) \oplus \lambda\tilde{T}(\tilde{u}, \tilde{v})]\Big]. \end{aligned} \tag{5.2}$$

Utilizing Lemma 2.3, we have

$$\begin{aligned}\|\tilde{x}_{n+1}\oplus\tilde{x}\| &\le \lambda_{\mathcal{C}}[\|(1-a)(\tilde{x}_n\oplus\tilde{x})+a[R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}_n(\tilde{g}(\cdot),\tilde{w}_n)}[\tilde{H}(\tilde{U}(\tilde{x}_n),\tilde{V}(\tilde{x}_n))\oplus\lambda\tilde{T}(\tilde{u}_n,\tilde{v}_n)]\\
&\quad\oplus R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}(\tilde{g}(\cdot),\tilde{w})}[\tilde{H}(\tilde{U}(\tilde{x}),\tilde{V}(\tilde{x}))\oplus\lambda\tilde{T}(\tilde{u},\tilde{v})]]\|]\\
&\le \lambda_{\mathcal{C}}(1-a)\|\tilde{x}_n\oplus\tilde{x}\|+\lambda_{\mathcal{C}}a\|R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}_n(\tilde{g}(\cdot),\tilde{w}_n)}[\tilde{H}(\tilde{U}(\tilde{x}_n),B(\tilde{x}_n))\oplus\lambda\tilde{T}(\tilde{u}_n,\tilde{v}_n)]\\
&\quad\oplus R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}(\tilde{g}(\cdot),\tilde{w})}[\tilde{H}(\tilde{U}(\tilde{x}),\tilde{V}(\tilde{x}))\oplus\lambda\tilde{T}(\tilde{u},\tilde{v})]\|\\
&\le \lambda_{\mathcal{C}}(1-a)\|\tilde{x}_n\oplus\tilde{x}\|+\lambda_{\mathcal{C}}a\|R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}_n(\tilde{g}(\cdot),\tilde{w}_n)}[\tilde{H}(\tilde{U}(\tilde{x}_n),\tilde{V}(\tilde{x}_n))\oplus\lambda\tilde{T}(\tilde{u}_n,\tilde{v}_n)]\\
&\quad\oplus R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}_n(\tilde{g}(\cdot),\tilde{w}_n)}[\tilde{H}(\tilde{U}(\tilde{x}),\tilde{V}(\tilde{x}))\oplus\lambda\tilde{T}(\tilde{u},\tilde{v})]\\
&\quad\oplus R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}_n(\tilde{g}(\cdot),\tilde{w}_n)}[\tilde{H}(\tilde{U}(\tilde{x}),\tilde{V}(\tilde{x}))\oplus\lambda\tilde{T}(\tilde{u},\tilde{v})]\\
&\quad\oplus R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}(\tilde{g}(\cdot),\tilde{w})}[\tilde{H}(\tilde{U}(\tilde{x}),\tilde{V}(\tilde{x}))\oplus\lambda\tilde{T}(\tilde{u},\tilde{v})]\|\\
&\le \lambda_{\mathcal{C}}(1-a)\|\tilde{x}_n\oplus\tilde{x}\|+\lambda_{\mathcal{C}}a\|R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}_n(\tilde{g}(\cdot),\tilde{w}_n)}[\tilde{H}(\tilde{U}(\tilde{x}_n),\tilde{V}(\tilde{x}_n))\oplus\lambda\tilde{T}(\tilde{u}_n,\tilde{v}_n)]\\
&\quad\oplus R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}_n(\tilde{g}(\cdot),\tilde{w}_n)}[\tilde{H}(\tilde{U}(\tilde{x}),\tilde{V}(\tilde{x}))\oplus\lambda\tilde{T}(\tilde{u},\tilde{v})]\|\\
&\quad+\lambda_{\mathcal{C}}a\|R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}_n(\tilde{g}(\cdot),\tilde{w}_n)}[\tilde{H}(\tilde{U}(\tilde{x}),\tilde{V}(\tilde{x}))\oplus\lambda\tilde{T}(\tilde{u},\tilde{v})]\\
&\quad\oplus R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}(\tilde{g}(\cdot),\tilde{w})}[\tilde{H}(\tilde{U}(\tilde{x}),\tilde{V}(\tilde{x}))\oplus\lambda\tilde{T}(\tilde{u},\tilde{v})]\|.\end{aligned}\tag{5.3}$$

It follows from Lemma 2.17 that

$$\begin{aligned}\|\tilde{x}_{n+1}\oplus\tilde{x}\| &\le \lambda_{\mathcal{C}}(1-a)\|\tilde{x}_n\oplus\tilde{x}\|+\lambda_{\mathcal{C}}a\boldsymbol{\Omega}\|[\tilde{H}(\tilde{U}(\tilde{x}_n),\tilde{V}(\tilde{x}_n))\oplus\lambda\tilde{T}(\tilde{u}_n,\tilde{v}_n)]\\
&\quad\oplus[\tilde{H}(\tilde{U}(\tilde{x}),\tilde{V}(\tilde{x}))\oplus\lambda\tilde{T}(\tilde{u},\tilde{v}))]\|+\lambda_{\mathcal{C}}a\kappa_n\\
&\le \lambda_{\mathcal{C}}(1-a)\|\tilde{x}_n\oplus\tilde{x}\|+\lambda_{\mathcal{C}}a\boldsymbol{\Omega}\|[\tilde{H}(\tilde{U}(\tilde{x}_n),\tilde{V}(\tilde{x}_n))\oplus\lambda\tilde{T}(\tilde{u}_n,\tilde{v}_n)]\\
&\quad-[\tilde{H}(\tilde{U}(\tilde{x}),\tilde{V}(\tilde{x}))\oplus\lambda\tilde{T}(\tilde{u},\tilde{v}))]\|+\lambda_{\mathcal{C}}a\kappa_n\\
&\le \lambda_{\mathcal{C}}(1-a)\|\tilde{x}_n\oplus\tilde{x}\|+\lambda_{\mathcal{C}}a\boldsymbol{\Omega}\|\tilde{H}(\tilde{U}(\tilde{x}_n),\tilde{V}(\tilde{x}_n))\oplus\tilde{H}(\tilde{U}(\tilde{x}),\tilde{V}(\tilde{x}))\|\\
&\quad+|\lambda|\lambda_{\mathcal{C}}a\boldsymbol{\Omega}\|\tilde{T}(\tilde{u}_n,\tilde{v}_n)\oplus\tilde{T}(\tilde{u},\tilde{v})\|+\lambda_{\mathcal{C}}a\kappa_n,\end{aligned}\tag{5.4}$$

where

$$\begin{aligned}\kappa_n &= \|R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}_n(\tilde{g}(\cdot),\tilde{w}_n)}[\tilde{H}(\tilde{U}(\tilde{x}),\tilde{V}(\tilde{x}))\oplus\lambda\tilde{T}(\tilde{u},\tilde{v})]\\
&\quad\oplus R^{\tilde{H}(\cdot,\cdot)}_{\lambda,\tilde{M}(\tilde{g}(\cdot),\tilde{w})}[\tilde{H}(\tilde{U}(\tilde{x}),\tilde{V}(\tilde{x}))\oplus\lambda\tilde{T}(\tilde{u},\tilde{v})]\|.\end{aligned}\tag{5.5}$$

By using similar arguments as from (4.6)-(4.8) and Lemma 2.3, (5.4) yields

$$\begin{aligned}\|\tilde{x}_{n+1}\oplus\tilde{x}\| &\le [\lambda_{\mathcal{C}}(1-a)+\lambda_{\mathcal{C}}a\boldsymbol{\Omega}(\beta_1+\beta_2)+|\lambda|\lambda_{\mathcal{C}}a\boldsymbol{\Omega}(\gamma'_{\tilde{T}}\zeta_{\tilde{P}}+\gamma''_{\tilde{T}}\zeta_{\tilde{Q}})]\|\tilde{x}_n\oplus\tilde{x}\|+\lambda_{\mathcal{C}}a\kappa_n\\
&\le [\lambda_{\mathcal{C}}(1-a)+\lambda_{\mathcal{C}}a\boldsymbol{\Omega}(\beta_1+\beta_2)+\lambda|\lambda_{\mathcal{C}}a\boldsymbol{\Omega}|(\gamma'_{\tilde{T}}\zeta_{\tilde{P}}+\gamma''_{\tilde{T}}\zeta_{\tilde{Q}})]\|\tilde{x}_n-\tilde{x}\|+\lambda_{\mathcal{C}}a\kappa_n.\end{aligned}\tag{5.6}$$

It follows from (5.5) and Theorem 3.2 that

$$\kappa_n\to 0 \ \text{ as } \ n\to\infty.\tag{5.7}$$

Thus, we have

$$\|\tilde{x}_{n+1}\oplus\tilde{x}\|\le\boldsymbol{\Theta}_2\|\tilde{x}_n-\tilde{x}\|,\tag{5.8}$$

where $\boldsymbol{\Theta}_2=\lambda_{\mathcal{C}}[(1-a)+a\boldsymbol{\Omega}(\beta_1+\beta_2)+a\boldsymbol{\Omega}|\lambda|(\gamma'_{\tilde{T}}\zeta_{\tilde{P}}+\gamma''_{\tilde{T}}\zeta_{\tilde{Q}})]$. It follows from the condition (5.1) that $0<\boldsymbol{\Theta}_2<1$. Therefore $\{\tilde{x}_n\}$ is a Cauchy sequence in $\mathcal{H}_p$ and $\mathcal{H}_p$ is complete therefore there exists a point $\tilde{x}\in\mathcal{H}_p$ such that $\tilde{x}_n\to\tilde{x}$ as $n\to\infty$. It follows from the Algorithm 5.1 that

$$\|\tilde{u}_{n+1}\oplus\tilde{u}\|\le\mathcal{D}(\tilde{P}(\tilde{x}_{n+1}),\tilde{P}(\tilde{x}))\le\zeta_{\tilde{P}}\|\tilde{x}_{n+1}-\tilde{x}\|.\tag{5.9}$$

$$\|\tilde{v}_{n+1}\oplus\tilde{v}\|\le\mathcal{D}(\tilde{Q}(\tilde{x}_{n+1}),\tilde{Q}(\tilde{x}))\le\zeta_{\tilde{Q}}\|\tilde{x}_{n+1}-\tilde{x}\|.\tag{5.10}$$

and

$$\|\tilde{w}_{n+1}\oplus\tilde{w}\|\le\mathcal{D}(\tilde{R}(\tilde{x}_{n+1}),\tilde{R}(\tilde{x}))\le\zeta_{\tilde{R}}\|\tilde{x}_{n+1}-\tilde{x}\|.\tag{5.11}$$

From (5.9), (5.10) and (5.11), one can see that $\{\tilde{u}_n\}$, $\{\tilde{v}_n\}$ and $\{\tilde{w}_n\}$ are also Cauchy sequences in $\mathcal{H}_p$. Therefore there exist $\tilde{u},\tilde{v},\tilde{w}\in\mathcal{H}_p$ such that $\tilde{u}_n\to\tilde{u}$, $\tilde{v}_n\to\tilde{v}$ and $\tilde{w}_n\to\tilde{w}$ as $n\to\infty$. Thus $\{(\tilde{x}_n,\tilde{u}_n,\tilde{v}_n,\tilde{w}_n)\}$ converges strongly to the unique solution $(\tilde{x},\tilde{u},\tilde{v},\tilde{w})$ of ordered inclusion problem (4.1). This completes the proof. □

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