Palestine Journal of Mathematics

Vol. 11(3)(2022) , 57–62

Roman and k-rainbow Domination of Degree Splitting Graph

Shailaja S. Shirkol and Pavitra P. Kumbargoudra

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 05C69, 05C38.

Keywords and phrases: Roman Domination number, k-rainbow domination number, Degree Splitting Graph.

Abstract Consider a graph $G(V,\ E)$ with vertex partition $V = S_1 \cup S_2 \cup \ldots, \cup S_t \cup T$ where each S_i is a set with minimum two vertices having the same degree and $T = V \setminus \cup S_i$. The degree splitting graph $DS(G)$ is obtained from G by adding vertices $w_1, w_2, \ldots, w_t$ and joining w_i to each vertex of $S_i (1 \leq i \leq t)$. In this research article we characterize roman domination number of degree splitting graph $\gamma_R(DS(G))$ and we obtain roman domination number and k-rainbow domination number of degree splitting graphs. Also we establish many bounds on $\gamma_R(DS(G))$ and $\gamma_{rk}(DS(G))$ in terms of elements of G.

1 Introduction

In this paper we consider finite, simple and undirected graph $G = (V,\ E)$ with vertex set $V = V(G)$ and edge set $E = E(G)$. The number of edges incident on the vertex v is called degree of a vertex $d(v)$. The minimum and maximum degree of G is denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$ respectively. If the degree of each vertex is r then the graph is called r-regular graph i.e. if $\forall v \in V(G), d(v) = r$. For any vertex $v \in V$, the open neighborhood $N(v) = \{u \in V(G) \setminus uv \in E(G)\}$ and the closed neighborhood $N[v] = N(v) \cup v$. A connected acyclic graph is called tree. We denote K_n for complete graph with n vertices, C_n for a cycle of length n, P_n for a path of length n. For notation and graph theory terminology we refer [1].

The concept of Roman domination number was introduced by Cockaynea and et.al, and has been well studied by many authors [3, 5, 6]. A Roman dominating function (RDF) on graph $G = (V, E)$ is a function $f : V(G) \to \{0, 1, 2\}$ such that whenever $f(v) = 0$ there exist a neighboring vertex u of v such that $f(u) = 2$. The weight of f is $w(f) = \sum_{v \in V(G)} f(v)$. The minimum weight of RDF of G is called the roman domination number $\gamma_R(G)$. A roman dominating function f can be represented by the ordered partition (V_0, V_1, V_2) of V, where $V_i = \{v \in V | f(v) = i\}$. Clearly the weight $w(f) = |V_1| + 2|V_2|$.

Proposition 1.1. *([5]) For any graph G,* $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$.

Theorem 1.2. *([4]) If G is a connected graph with* n *vertices, then* $\gamma_R(G) \leq 4n/5$.

In this paper we consider another variation of domination that is k-rainbow domination number. In the year 2003, M. A. Henning introduced and studied [8, 9] the application of k-rainbow domination number. Assume that there are k different type of guards and for each vertex we assign an arbitrary subset of these guards. If any vertex is not assigned by any type of guards(an empty set) then it should have all type of guards in its neighboring locations and this assignment is known as k-rainbow dominating function (kRDF). On the other hand, a k-rainbow dominating function of a graph G is a function $f : V(G) \to P(\{1, \ldots, k\})$ such that for every vertex $v \in V(G)$ with $f(v) = \emptyset$ then we have

$$\bigcup_{u \in N(v)} f(u) = S.$$

The weight $w(f)$ is the sum of carnality of $f(v)$. Mathematically $w(f) = \sum_{v \in V(G)} \mid f(v) \mid$. The minimum weight of a kRDF is called the k-rainbow domination number of G and is denoted by $\gamma_{rk}(G)$.

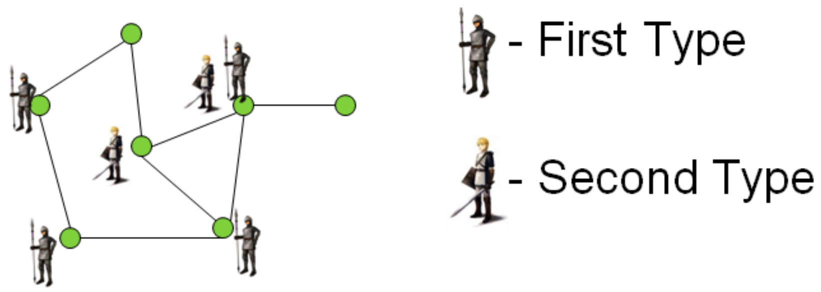

Figure 1. Example for 2-rainbow dominating function. Here $\gamma_{2r}(G) = 4$.

In 2004, R. Ponraj and S Somasundaram defined degree splitting graph [2]. Let $G(V,\ E)$ be a graph with vertex partition $V = S_1 \cup S_2 \cup \ldots, \cup S_t \cup T$ where each $S_i(1 \leq i \leq t)$ is a set of minimum two vertices having the same degree and $T = V \setminus \cup S_i$. The degree splitting graph of G is denoted by $DS(G)$ is obtained from G by adding vertices $w_1, w_2, \ldots, w_t$ and joining w_i to each vertex of $S_i(1 \leq i \leq t)$.

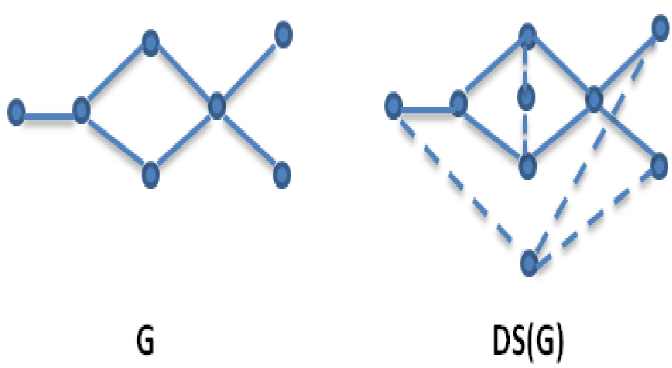

Figure 2. Example of Degree Splitting Graph.

Later B. Basavangouda, P. V. Patil and S. M. Hosamani [7] worked on Domination in Degree Spliting graph. They studied variation in domination from the graph G to the degree splitting graph $DS(G)$. They established $\gamma(DS(G)) \leq \lceil \frac{p}{2} \rceil$, $\gamma(DS(G)) \leq |W_i \bigcup T|$. Also they worked on Domatic number of $DS(G)$ found some bounds.

In this article, initially we determine numerical value of roman domination and k-rainbow domination number for some graphs. We also obtain some bounds for $\gamma_R(DS(G))$ and $\gamma_{rk}(DS(G))$.

2 Roman Domination Number of Degree Splitting Graph

Theorem 2.1. *If $G = P_n$ be a path on n vertices, then*

$$\gamma_R(DS(P_n)) = 4$$

Proof. Let $G = P_n$ be a path with $V(G) = \{v_i : 1 \leq i \leq n\}$. Let S_1 and S_2 are the two partition of $V(G)$ such that $S_1 = \{v_1, v_n\}$ and $S_2 = \{v_i : 2 \leq i \leq n-1\}$. Clearly $DS(G)$ is obtained by adding two vertices w_1 and w_2 to $V(G)$ and connecting two vertices v_1 and v_n to w_1 and all vertices of $\{v_2,\ v_3, \ldots, v_{n-1}\}$ to w_2. Then $|\ V(DS(G))\ | = n+2$ and $|\ E(DS(G))\ | = 2n+1$. Let us define a roman dominating function $g : V(DS(P_n)) \to \{0,1,2\}$ with minimum weight such that $f(v_i) = 0$, $f(w_1) = f(w_2) = 2$, Hence $\gamma_R(DS(P_n)) = 4$.

□

Theorem 2.2. *If G be any r regular graph then $\gamma_R(DS(G)) = 2$.*

Proof. Given G be any r-regular graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. Then $DS(G) = G + K_1$ and $\triangle(DS(G)) = p - 1$. Let us define roman dominating function $f : V(DS(G)) \to \{0, 1, 2\}$ with minimum weight such that $f(v_i) = 0, \forall v \in V(G)$ and $f(w_1) = 2$. Hence $\gamma_{rk}(DS(G)) = 2$. □

Theorem 2.3. *For any graph G, $\gamma_R(DS(G)) \leq 2 \mid w_i \bigcup T \mid$.*

Proof. Let G be any graph with p vertices. By the definition of degree splitting of graph $DS(G)$, $V(DS(G)) = \{S_1, S_2, \ldots, S_t, T\}$ where each $S_i, 1 \leq i \leq t$ and T is as defined in the definition.
Case 1: $T = \emptyset$ Since each $w_i, 1 \leq i \leq t$ is independent in $DS(G)$ and clearly the set containing each w_i will be the maximal independent set in $DS(G)$. Hence $\gamma_R(DS(G)) \leq 2 \mid w_i \mid$
Case 2: $T \neq \emptyset$. Clearly there exist at least one vertex v_i in G of degree r and no other vertex of same degree i.e. $v_i \notin S_i; 1 \leq i \leq t$. Since G is induced subgraph of $DS(G)$, to define roman dominating function $f(v_i \neq 0)$. Hence

$$\gamma_R(DS(G)) \leq 2 \mid w_i \bigcup T \mid .$$

□

Theorem 2.4. *For any graph G with p vertices, $\gamma_R(DS(G)) \leq 2\lceil \frac{p}{2} \rceil$.*

Proof. To prove the results we have the following cases.
Case 1: If $T = \emptyset$, then G has atmost $S_i \leq \dfrac{p}{2}$. Hence $w_i \leq \dfrac{p}{2}$. Therefore by previous Theorem , we get,

$$\gamma_R(DS(G)) \leq 2. \mid w_i \mid \leq 2\tfrac{p}{2} \leq 2\lceil \frac{p}{2} \rceil.$$

Case 2: If $T \neq \emptyset$. Then G has atmost $S_i \leq \dfrac{p}{2} - T$. Hence $w_i \leq \dfrac{p}{2} - T$. We have,

$$\gamma_R(DS(G)) \leq 2. \mid w_i + T \mid \leq 2 \mid \frac{p}{2} - T + T \mid 2 \leq \tfrac{p}{2}\, 2 \leq \lceil \frac{p}{2} \rceil.$$

□

Theorem 2.5. *Let G be any graph then $\gamma_R(G).\gamma_R(DS(G)) \leq 2 \mid w_i \mid$.*

Proof. Let G be any graph with p vertices. By the definition of degree splitting of graph $DS(G)$, $V(DS(G)) = \{S_1, S_2, \ldots, S_t, T\}$ where each $S_i, 1 \leq i \leq t$ and T is as defined in the definition.

Since each $w_i, 1 \leq i \leq t$ is independent in $DS(G)$ and clearly the set containg each w_i will be the maximal independent set in $DS(G)$. Hence $\gamma_R(DS(G)) \leq 2 \mid w_i \mid$

$$\gamma_R(G).\gamma_R(DS(G)) \leq \gamma_R(G).2 \mid w_i \mid .$$

□

Theorem 2.6. *For any graph G, $\gamma_R(G).\gamma_R(DS(G)) \leq \dfrac{8.p.(p+1)}{5}$*

Proof. E. J. Cockayne et al. [5] showed that $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$, for any connected graph G with p vertices. Clearly by [2] we obtain,

$$\begin{aligned} \gamma_R(G).\gamma_R(DS(G)) &\leq 2.\gamma(G).2.\gamma(DS(G)) \\ &\leq 4.\gamma(G).\gamma(DS(G)) \\ &\leq 4.\gamma(G).\lceil \frac{p}{2} \rceil \\ &\leq 4.\frac{4p}{5}.\lceil \frac{p}{2} \rceil \\ &\leq 16.\frac{p}{5}.\frac{p+1}{2} \\ &\leq \frac{8.p.(p+1)}{5} \end{aligned}$$

□

Theorem 2.7. *Let G be any graph then* $\gamma_R(G) \leq \gamma_R(DS(G))$.

Theorem 2.8. *Let G be graph of order n,* $\gamma_R(G) = \gamma_R(DS(G))$ *iff* $G = K_n$

3 *k*-rainbow Domination Number of Degree Splitting Graph

Lemma 3.1. *If* K_n *be a complete graph with n vertices then* $\gamma_{rk}(DS(K_n)) = min\{k,\ n+1\}$.

Proof: Since complete graph K_n is $(n-1)$-regular graph hence $DS(K_n)$ is obtained by adding one vertex w_1 and connecting each vertex of K_n to w_1. Therefore $DS(K_n) = K_{n+1}$. $\gamma_{rk}(K_n) = min\{k,\ n\}$, so $\gamma_{rk}(DS(K_n)) = min\{k,\ n+1\}$.

Lemma 3.2. *If* C_n *be a cycle of length n then* $\gamma_{rk}(C_n) = min\{k,\ n+1\}$.

Proof: Clearly $DS(C_n) = W_{n+1}$. Therefore $\gamma_{rk}(C_n) = min\{k,\ n+1\}$.

Lemma 3.3. *If* $K_{m,\ n}$ *be a complete bipartite graph* $(m \neq n)$ *and* $\gamma_{rk}(K_{m,\ n}) = \gamma_{rk}$ *then* $\gamma_{rk}(DS(K_{m,n})) \leq \gamma_{rk} + 2$.

Proof: The complete bipartite graph $K_{m,\ n}$ is $(n,\ m)-$ regular graph. The degree splitting graph $DS(K_{m,\ n})$ contains $V(DS(K_{m,\ n})) = V(K_{m,\ n}) \cup w_1 \cup w_2$ and each vertex $v_i, 1 \leq i \leq m$, joins w_1 and each vertex $u_i, 1 \leq i \leq n$, joins w_2. Hence $DS(K_{m,\ n}) = K_{m+1,\ n+1} - w_1w_2$. Let f be a k-rainbow dominating function with minimum weight γ_{rk} of $K_{m,\ n}$. Let us define the k-rainbow dominating function $g : V(Ds(K_{m,\ n})) \to P(\{1,\ldots,k\})$ such that $g(v_i) = f(v_i), \forall v_i \in V(K_{m,\ n})$ and $|g(w_1)| \leq |g(w_2)| \leq 1$. Hence $\gamma_{rk}(DS(K_{m,\ n})) \leq \gamma_{rk} + 2$.

Theorem 3.4. *If* $G = P_n$ *be a path on n vertices, then*

$$\gamma_{rk}(DS(P_n)) = \begin{cases} \gamma_{rk} + 2 & if\ n = k \\ \gamma_{rk} + 1 & if\ n < k \\ k + 3 & if\ n > k. \end{cases}$$

Proof. Let $G = P_n$ be a path with $V(G) = \{v_i : 1 \leq i \leq n\}$. Let S_1 and S_2 are subset of $V(G)$ such that $S_1 = \{v_1, v_n\}$ and $S_2 = \{v_i : 2 \leq i \leq n-1\}$. Clearly $DS(G)$ is obtained by adding two vertices w_1 and w_2 to $V(G)$ and connecting two vertices v_1 and v_n to w_1 and all vertices of $\{v_2,\ v_3, \ldots, v_{n-1}\}$ to w_2. Then $\mid V(DS(G)) \mid = n+2$ and $\mid E(DS(G)) \mid = 2n+1$. Now $f : V(P_n) \to P(\{1,\ldots,k\})$ be a k-rainbow dominating function with minimum weight γ_{rk}. Let us define a k-rainbow dominating function $g : V(DS(P_n)) \to P(\{1,\ldots,k\})$ with minimum weight the following cases arises,
Case 1: $n = k$.
Since $|f(v_i)| = 1$, $g(v_i) = f(v_i) \forall v_i \in V(P_n)$ and $|g(w_1|)| = |g(w_2)| = 1$. Hence

$$\gamma_{rk}(DS(P_n)) = \gamma_{rk} + 2. \tag{3.1}$$

Case 2: $n > k$
Clearly $d(w_2) = n-2$. Hence $|g(w_1)| = |g(v_1)| = |g(v_n)| = 1$ and $g(w_2) = \{1,\ldots,k\}$ $g(v_i) = \emptyset\ \forall\ 2 \leq v_i \leq n$. Hence

$$\gamma_{rk}(DS(P_n)) = k + 3. \tag{3.2}$$

case 3: $n < k$
Here $g(v_i) = f(v_i) \forall v_i \in V(P_n)$ $|g(w_1|)| = 1$ and $g(w_2) = \emptyset$ Hence

$$\gamma_{rk}(DS(P_n)) = \gamma_{rk} + 1. \tag{3.3}$$

From equations (2), (3) and (4),

$$\gamma_{rk}(DS(P_n)) = \begin{cases} \gamma_{rk} + 2 & if\ n = k \\ \gamma_{rk} + 1 & if\ n < k \\ k + 3 & if\ n > k. \end{cases}$$

□

Theorem 3.5. *If G be any r regular graph then $\gamma_{rk}(DS(G)) = k$.*

Proof. Given G be any r-regular graph with ertex set $\{v_1, v_2, \ldots, v_n\}$. Then $DS(G) = G + K_1$ and $\triangle(DS(G)) = p - 1$. Let us define k-rainbow dominating function $f : V(DS(G)) \rightarrow P(1, 2, \ldots, k)$ with minimum weight such that $f(v_i) = \emptyset, \forall v \in V(G)$ and $f(w_1) = \{1, 2, \ldots, k\}$. Hence $\gamma_{rk}(DS(G)) = k$. □

Theorem 3.6. *For any graph G, $\gamma_{rk}(DS(G)) \leq k \mid w_i \bigcup T \mid$.*

Proof. Let G be any graph with p vertices. By the definition of degree splitting of graph $DS(G)$, $V(DS(G)) = \{S_1, S_2, \ldots, S_t, T\}$ where each $S_i, 1 \leq i \leq t$ and T is as defined in the defination.
Case 1: $T = \emptyset$ Since each $w_i, 1 \leq i \leq t$ is independent in $DS(G)$ and clearly the set containg each w_i will be the maximal independent set in $DS(G)$. Hence $\gamma_{rk}(DS(G)) \leq k \mid w_i \mid$
Case 2: $T \neq \emptyset$. Clearly there exist at least one vertex v_i in G of degree r and no other vertex of same degree i.e. $v_i \notin S_i; 1 \leq i \leq t$. Since G is induced subgraph of $DS(G)$, to define k-rainbow dominating function $f(v_i \neq \emptyset)$. Hence

$$\gamma_{rk}(DS(G)) \leq k \mid w_i \bigcup T \mid . \tag{3.4}$$

□

Theorem 3.7. *For any graph G with p vertices, $\gamma_{rk}(DS(G)) \leq k\lceil \frac{p}{2} \rceil$.*

Proof. To prove the results we have the following cases.
Case 1: If $T = \emptyset$, then G has atmost $S_i \leq \dfrac{p}{2}$. Hence $w_i \leq \dfrac{p}{2}$. Therefore by Theorem 3.7, we get,

$$\gamma_{rk}(DS(G)) \leq k. \mid w_i \mid \leq k\tfrac{p}{2} \leq k\lceil \frac{p}{2} \rceil.$$

Case 2: If $T \neq \emptyset$. Then G has atmost $S_i \leq \dfrac{p}{2} - T$. Hence $w_i \leq \dfrac{p}{2} - T$. We have,

$$\gamma_{rk}(DS(G)) \leq k. \mid w_i + T \mid \leq k \mid \frac{p}{2} - T + T \mid k \leq \tfrac{p}{2}\, k \leq \lceil \frac{p}{2} \rceil.$$

□

Theorem 3.8. *Let G be any graph then $\gamma_{rk}(G).\gamma_{rk}(DS(G)) \leq k \mid w_i \mid$.*

Proof. Let G be any graph with p vertices. By the definition of degree splitting of graph $DS(G)$, $V(DS(G)) = \{S_1, S_2, \ldots, S_t, T\}$ where each $S_i, 1 \leq i \leq t$ and T is as defined in the defination.

Since each $w_i, 1 \leq i \leq t$ is independent in $DS(G)$ and clearly the set containg each w_i will be the maximal independent set in $DS(G)$. Hence $\gamma_{rk}(DS(G)) \leq k \mid w_i \mid$

$$\gamma_{rk}(G).\gamma_{rk}(DS(G)) \leq \gamma_{rk}(G).k \mid w_i \mid . \tag{3.5}$$

□

4 Concluding Remarks

In this article we have mainly focused on finding the roman and k rainbow domination number for degree splitting graph. We obtained some bounds for $\gamma_R(DS(G))$ and $\gamma_R(DS(G))$. The derived results in this paper can be extended to study the roman domination number for $DS(P(n, 2))$, $(DS(P(n, 3)))$, $DS(C_n \Box C_m))$, $DS(C_n \Box P_m)$ and $DS(P_n \Box P_m)$.

References

[1] F. Harary, *Graph Theory*, Narosa Publishing House, New Delhi, (1988).

[2] R. Ponraj and S. Somasundaram, On the degree splitting graph of a graph, *Natlacadscilett* , **27**, (No. 7 and 8), 275–278 (2004).

[3] F. Xueliang, Y. Yuansheng and J. Baoqi, Roman domination in regular graphs, *Discrete Mathematics* , **309**, 1528–1537 (2009).

[4] O. Favaron, H. Karami, R. Khoeilar and S. M. Sheikholeslami, On the Roman domination number of a graph, *Discrete Mathematics* , **309**, 3447–3451 (2009).

[5] E. J. Cockaynea, P. A. Dreyer and S. M. Hedetniemic, Roman domination in graphs, *Discrete Mathematics* , **278**, 11–22 (2004).

[6] C. H. Liu and G. J. Changa, Upper bounds on Roman domination numbers of graphs, *Discrete Mathematics* , **312**, 1386–1391 (2012).

[7] B. Basavanagoud, P. V. Patil and S. M. Hosamani, Domination in Degree Splitting Graphs, *Journal of Analysis and Computation* , **8**, (No. 1) 1–8 (2012).

[8] B. Bresar, M. A. Henning and D. F. Rall, Rainbow Domination in Graphs,*Taiwanese Journal of Mathematics* , **12**, (No. 1) 213–225 (2008).

[9] M. A. Henning, Defending the Roman Empire from multiple attacks, *Discrete Applied Mathematics* , **271**, 101–115 (2003).

Author information

Shailaja S. Shirkol and Pavitra P. Kumbargoudra,
Department of Mathematics,
S. D. M. College of Engineering and Technology,
Dharwad, Karnataka

Department of Mathematics,
The Oxford Engineering College,
Bangalore, Karnataka, India.
E-mail: shailajashirkol@gmail.com and pavitra2504@gmail.com

Received: March 8, 2021
Accepted: June 13, 2021