A new approach to fixed point result in non-Archimedean 2-Banach space and some of its applications

R. El Ghali and S. Kabbaj

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Abstract

In this paper we extend the fixed point result of Brzdę̆k et al. [7] in non-Archimedean 2-Banach spaces. Moreover, we investigate the hyperstability of Cauchy-Jensen functional equation in the considered space by using the above result and we give some outcomes.

1 Introduction and preliminaries

A certain formula or equation is applicable to model a physical process of a small change of the formula or equation gives rise to a small change in the corresponding result. When this happens, we say that formula or equation is called stable. One of the unsolved problems was given by S. M. Ulam [21] tends to be the starting point for researching the stability problems of functional equations. Ulam asked the following question concerning the stability of group homomorphisms:

Given a group $G$, a metric group $H$ with metric $d(\cdot, \cdot)$ and a positive number $\varepsilon$, does there exists a $\delta > 0$ such that if $f : G \to H$ satisfies $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $\Phi : G \to H$ exists with $d(f(x), \Phi(x)) < \varepsilon$ for $x \in G$?

D. H. Hyers [15] gave the first partial answer to Ulam’s problem for the Cauchy equation (1.1) in Banach spaces with $\delta = \varepsilon$ and

$$\Phi(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n}.$$ 

The most classical result concerning the Hyers-Ulam stability for the Cauchy equation (1.1) has been given by Th. M. Rassias [19].

**Theorem 1.1.** [19] Let $E_1$ and $E_2$ be two normed spaces, $c \geq 0$ and $p \neq 1$ be fixed real numbers. Let $f : E_1 \to E_2$ be a mapping satisfying

$$\|f(x + y) - f(x) - f(y)\| \leq c \left(\|x\|^p + \|y\|^p\right), \quad x, y \in E_1 \setminus \{0\}.$$ 

Then the following statements are valid

1. If $p \geq 0$ and $E_2$ is complete, then there exists a unique additive function $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{c\|x\|^p}{2^{p-1} - 1}, \quad x \in E_1 \setminus \{0\}.$$  

2. If $p < 0$, then $f$ is additive.
This result is called the Hyers-Ulam-Rassias stability of Cauchy functional equation.


**Theorem 1.2.** Let $G$ be an abelian group and $(\mathcal{X}, \|\cdot\|)$ a Banach space. Let $\varphi: G \times G \to \mathbb{R}^+$ be a mapping satisfying, for all $x, y \in G$, the condition:

$$\varphi(x, y) = \sum_{k=0}^{\infty} 2^{-k} \varphi(2^k x, 2^k y) < +\infty$$

Let $f: G \to X$ be a mapping which fulfils, for each $x, y \in G$, the condition

$$\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y)$$

Then there exists a unique mapping $T: G \to G$ such that

$$T(x + y) = T(x) + T(y),$$

for all $x, y \in G$ and:

$$\|f(x) - T(x)\| \leq \frac{1}{2} \varphi(x, y),$$

for all $x \in G$.

Since then, the problem of stability of several functional equations have been extensively studied by many mathematicians (see, for instance, [2, 6, 7, 8, 13, 15, 19]).

A functional equation is called **hyperstable** when any function $f$ satisfying the equation approximately, in some sense, must be actually a solution to it. The term hyperstability was used for the first time probably in 2001 by Gy. Maksa and Zs. Páles [18], however, it seems that the first hyperstability result was published in [5] and concerned the ring homomorphisms. The hyperstability results for the Cauchy equation were investigated by J. Brzdęk in [9, 10]. E. Gselmann [14] studied the hyperstability of the parametric fundamental equation of information.

Note that the second statement of the Theorem 1.1, for $p < 0$ can be described as $\varphi$-hyperstability of the additive equation with $\varphi(x, y) = c \left(\|x\|^p + \|y\|^p\right)$.

Throughout this paper, $\mathbb{N}$ stands for the set of all positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N}_{m_0}$ the set of all integers greater than or equals $m_0 \ (m_0 \in \mathbb{N})$, $\mathbb{R}_+ = [0, \infty)$ and we use the notation $\mathbb{X}_0$ for the set $\mathbb{X} \setminus \{0\}$.

Let us recall (see, for instance, [17]) some basic definitions and facts concerning non-Archimedean 2-normed spaces.

**Definition 1.3.** By a non-Archimedean field, we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|: \mathbb{K} \to [0, \infty)$ such that for all $r, s \in \mathbb{K}$, the following conditions hold:

1. $|r| = 0$ if and only if $r = 0$,
2. $|rs| = |r||s|$, 
3. $|r + s| \leq \max\{|r|, |s|\}$.

The pair $(\mathbb{K}, |\cdot|)$ is called a valued field.

**Remark 1.4.** In any non-Archimedean field, we have $|1| = |-1| = 1$ and $|n| \leq 1$ for $n \in \mathbb{N}$.

**Example 1.5.** In any field $\mathbb{K}$ the function $|\cdot|: \mathbb{K} \to \mathbb{R}_+$ given by

$$|x| := \begin{cases} 0, & x = 0, \\ 1, & x \neq 0 \end{cases}$$

is a valuation which is called **trivial valuation**, but the most important example of non-Archimedean fields are $p$-adic numbers which have gained the interest of physicists for their research in some problems coming from quantum physics, $p$-adic strings and super strings.

Let $p$ be a fixed prime number and $x$ a rational number, there exists a unique integer $v_p(x) \in \mathbb{Z}$ such that $x = p^{v_p(x)} \frac{a}{b}$ where $a$ and $b$ are integer co-prime to $p$. The function defined in $\mathbb{Q}$ by $|x|_p = p^{v_p(x)}$ is called a $p$-adic, an ultrametric or simply a non-Archimedean absolute value on $\mathbb{Q}$. The completion, denoted by $\mathbb{Q}_p$ of $\mathbb{Q}$ with respect to the metric defined by the $p$-adic absolute is called $p$-adic numbers.
Definition 1.6. Let $X$ be a vector space (with $\dim X > 1$) over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $\cdot |. A function $\|,\| : X^2 \to \mathbb{R}_+$ is called a non-Archimedean 2-norm (valuation) if it satisfies the following conditions:

1. $\|x, y\| = 0$ if and only if $x$ and $y$ are linearly independent, $x, y \in X$.
2. $\|x, y\| = \|y, x\|$ for $x, y \in X$.
3. $\|rx, y\| = |r| \|x, y\|$ for $r \in \mathbb{K}, x, y \in X$.
4. $\|x, y + z\| \leq \max \{\|x, y\|, \|x, z\|\}$ for $x, y, z \in X$.

Then $(X, \|,\|)$ is called a non-Archimedean 2-normed space or an ultrametric 2-normed space.

Example 1.7. Let $p$ be a fixed prime number. For $x = (x_1, x_2)$ and $y = (y_1, y_2)$ we define the non-Archimedean 2-norm in $\mathbb{Q}_p^2$ by $\|x, y\|_p = |x_1 y_2 - x_2 y_1|_p$.

Definition 1.8. Let $\{x_n\}$ be a sequence in a non-Archimedean 2-normed space $X$.

1. A sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence if there are linearly independent $y, z \in X$ such that
   \[
   \lim_{n \to \infty} \|x_{n+1} - x_n, y\| = \lim_{n \to \infty} \|x_{n+1} - x_n, z\| = 0
   \]
2. The sequence $\{x_n\}$ is said to be convergent if there exists $x \in X$ (called limit of this sequence and denoted by $\lim_{n \to \infty} x_n$) such that
   \[
   \lim_{n \to \infty} \|x_n - x, y\| = 0 \quad y \in X
   \]
3. If every Cauchy sequence in $X$ converges, then the non-Archimedean 2-normed space $X$ is called a non-Archimedean 2-Banach space or an ultrametric 2-Banach space.

Lemma 1.9. [20]

1. Let $X$ be a non-Archimedean 2-Banach space over a non-Archimedean field $\mathbb{K}$ and $x, y, z \in X$ such that $y$ and $z$ are linearly independent and $\|x, y\| = 0 = \|x, z\|$, then $x = 0$.
2. $(x_n)_{n \in \mathbb{N}}$ is a convergent sequence of element of $X$ then:
   \[
   \lim_{n \to \infty} \|x_n, y\| = \| \lim_{n \to \infty} x_n, y \| \quad y \in X.
   \]

In section 2, we introduce and prove a new version of fixed point theorem of Brzdek [12] in non-Archimedean 2-Banach space. This theorem has been considered as an important tool for investigating the stability and hyperstability, in some way, of several functional equations by many mathematicians (see for example [1, 4]). In section 3, we use our main results to investigate the hyperstability of the following Cauchy-Jensen functional equation

\[
\|f(x + y) + f(x - y) - 2f(x)\| = \text{0}, \quad (1.3)
\]

in non-Archimedean 2-Banach space. We also give some outcomes as particular cases and we study the hyperstability of the inhomogeneous Cauchy-Jensen equation

\[
\|f(x + y) + f(x - y) - 2f(x) + G(x, y)\| = \text{0}.
\]

2 Fixed point theorem


First, we need to present the following hypotheses.

\textbf{(H1)} $X$ is a nonempty set, $(Y, \|,\|)$ is a non-Archimedean 2-Banach space over a non-Archimedean field, $Y_0$ is a subset of $Y$ containing two linearly independent vectors, $f_1, \ldots, f_k : X \to X, g_1, \ldots, g_k : Y_0 \to Y_0$ and $L_1, \ldots, L_k : X \times Y_0 \to \mathbb{R}_+$ are given.

\textbf{(H2)} $T : Y^X \to Y^X$ is an operator satisfying the inequality:

\[
\|T\xi(x) - T\mu(x), y\| \leq \max_{1 \leq i \leq k} \left\{ L_i(x, y) \|\xi(f_i(x)) - \mu(f_i(x)), g_i(y)\| \right\}, \quad \xi, \mu \in Y^X, x \in X, y \in Y_0.
\]

\textbf{(H3)} $\Lambda : \mathbb{R}^+ \times Y_0 \to \mathbb{R}^+ \times Y_0$ is a non-decreasing linear operator defined by

\[
\Lambda \delta(x, y) := \max_{1 \leq i \leq k} \left\{ L_i(x, y) \delta(f_i(x), g_i(y)) \right\}, \quad \delta \in \mathbb{R}^+ \times Y_0, \quad x \in X, y \in Y_0.
\]
Theorem 2.1. Let hypotheses (H1)-(H3) are valid and let \( \varepsilon : X \times Y_0 \to \mathbb{R}_+ \) and \( \varphi : X \to Y \) be functions fulfilling the following two conditions

\[
\|T\varphi(x) - \varphi(x), y\| \leq \varepsilon(x, y), \quad x \in X, y \in Y_0, \tag{2.1}
\]

\[
\lim_{n \to \infty} \Lambda^n \varepsilon(x, y) = 0, \quad x \in X, y \in Y_0. \tag{2.2}
\]

Then, for every \( x \in X \), the limit

\[
\psi(x) = \lim_{n \to \infty} T^n \varphi(x)
\]

exists and defines a fixed point \( \psi \) of \( T \) with

\[
\|\varphi(x) - \psi(x), y\| \leq \sup_{n \in \mathbb{N}_0} \Lambda^n \varepsilon(x, y) = \sigma(x, y), \quad x \in X, y \in Y_0. \tag{2.3}
\]

Moreover, if

\[
(\Lambda \sigma)(x, y) \leq \sup_{n \in \mathbb{N}_0} \Lambda^{n+1} \varepsilon(x, y), \quad x \in X, x \in Y_0, \tag{2.4}
\]

then \( \psi \) is a unique fixed point of \( T \) satisfying (2.3).

Proof. We show by induction that, for any \( n \in \mathbb{N}_0 \)

\[
\|T^n \varphi(x) - T^{n+1} \varphi(x), y\| \leq \Lambda^n \varepsilon(x, y), \quad x \in X, y \in Y_0. \tag{2.5}
\]

Indeed, it’s easy to see that if \( n = 0 \), then the inequality (2.5) is exactly (2.1). Now, we fix \( n \in \mathbb{N} \) and suppose that (2.5) hold for \( n \), then by using the non-decreasing property of the operator \( \Lambda \) and (H2), for any \( x \in X, y \in Y_0 \), we get

\[
\|T^{n+1} \varphi(x) - T^{n+2} \varphi(x), y\| \leq \max_{1 \leq i \leq k} \left\{ L_i(x, y) \left\| T^n \varphi(f_i(x)) - T^{n+1} \varphi(f_i(x)), g_i(y) \right\| \right\}
\leq \max_{1 \leq i \leq k} \left\{ L_i(x, y) \Lambda^n \varepsilon(f_i(x), g_i(y)) \right\}
= \Lambda^{n+1} \varepsilon(x, y),
\]

then (2.5) holds for any \( n \in \mathbb{N} \). Moreover, by using (2.2) and (2.5), for any \( k \in \mathbb{N}, n \in \mathbb{N}_0 \) and \( x \in X \) and \( y \in Y_0 \), we have

\[
\|T^n \varphi(x) - T^{n+k} \varphi(x), y\| \leq \max_{0 \leq i \leq k-1} \left\{ \left\| T^{n+i} \varphi(x) - T^{n+i+1} \varphi(x), y\right\| \right\}
\leq \max_{0 \leq i \leq k-1} \left\{ \Lambda^{n+i+1} \varepsilon(x, y) \right\}, \tag{2.7}
\]

The sequence \( (T^n \varphi(x))_{n \in \mathbb{N}} \) for each \( x \in X \), is a Cauchy sequence. Because \( Y \) is a complete space, so this sequence is convergent and the limit \( \psi(x) = \lim_{n \to \infty} T^n \varphi(x) \) exists. Letting \( k \to \infty \) in (2.7), we obtain, for any \( n \in \mathbb{N}, x \in X \) and \( y \in Y_0 \), that:

\[
\|T^n \varphi(x) - \psi(x), y\| \leq \sup_{i \geq n} \|\Lambda^i \varepsilon(x, y)\) \]

\[
= \sigma_n(x, y), \tag{2.8}
\]

For \( n = 0 \), it’s easy to show that (2.8) gives (2.3). Moreover, by using (2.8) and (H2), we find

\[
\|T^{n+1} \varphi(x) - T \psi(x), y\| \leq \max_{0 \leq i \leq k} \left\{ L_i(x, y) \left\| T^n \varphi(f_i(x)) - \psi(f_i(x)), g_i(y) \right\| \right\}
\leq \Lambda \left( \|T^n \varphi(x) - \psi(x), y\| \right)
\leq \Lambda \left( \sup_{i \geq n} (\Lambda^i \varepsilon(x, y)) \right)
\leq \Lambda (\sigma_n(x, y)), \tag{2.9}
\]
for all \( n \in \mathbb{N} \), \( x \in X \) and \( y \in Y_0 \). Letting \( n \to \infty \) in (2.9) and using (2.2), we get
\[
\mathcal{T}\psi(x) = \lim_{n \to \infty} \mathcal{T}^{n+1}\varphi(x) = \psi(x)
\]
for all \( x \in X \) which means that \( \psi \) is a fixed point of the operator \( \mathcal{T} \).

Next, we will prove the uniqueness of a fixed point. To do it, we suppose that (2.4) holds and there exists an other fixed point \( \chi \in Y^X \) of \( \mathcal{T} \) satisfying
\[
\|\varphi(x) - \chi(x), y\| \leq \sup_{n \in \mathbb{N}_0} \Lambda^n \varepsilon(x, y) = \sigma(x, y), \quad x \in X, \ y \in Y_0.
\]
Then, for each \( x \in X \) and \( y \in Y_0 \), we have
\[
\|\psi(x) - \chi(x), y\| \leq \max \{\|\psi(x) - \varphi(x), y\|, \|\varphi(x) - \chi(x), y\|\}.
\]
By a similar proof of (2.7), we have, for any \( k \in \mathbb{N} \),
\[
\|\psi(x) - \chi(x), y\| = \|\mathcal{T}^k\psi(x) - \mathcal{T}^k\chi(x), y\|
\]
\[
\leq \Lambda^k(\|\psi(x) - \chi(x), y\|)
\]
\[
\leq \Lambda^k(\sigma(x, y))
\]
\[
\leq \sup_{n \in \mathbb{N}_0} \Lambda^{n+k}(\varepsilon(x, y)).
\]
Letting \( n \to \infty \) in the previous inequality and using (2.2), we obtain that \( \psi = \chi \). \( \square \)

3 Hyperstability results in non-Archimedean 2-Banach space

Taking \( Y_0 = Y \) and \( q_i : Y \to Y \) as identities mapping for all \( i \in \{1, 2, \ldots, k\} \). In the following theorem, we use the fixed point Theorem 2.1 as a basic tool to investigate the hyperstability of the Cauchy-Jensen functional equation (1.3) in a non-Archimedean 2-Banach space.

In the remaining part of the paper, we use \( X \) as a non empty set, \( (Y, \|\cdot\|) \) a non-Archimedean 2-Banach space, and \( X' \) a non empty subset of \( X \).

Theorem 3.1. Let \( h_1, h_2 : X' \times Y \to \mathbb{R}_+ \) be two functions such that
\[
\mathcal{U} := \left\{ n \in \mathbb{N} : \alpha_n = \max(\lambda_1(n+1)\lambda_2(n+1), \lambda_1(2n+1)\lambda_2(2n+1)) < 1 \right\},
\]
where
\[
\lambda_i(n) = \inf\{t \in \mathbb{R}_+ : h_i(nx, z) \leq th_i(nz, z), \quad x \in X', \ z \in Y \}
\]
for all \( n \in \mathbb{N} \), where \( i = 1, 2 \) such that
\[
\lim_{n \to \infty} \lambda_1(n)\lambda_2(n) = 0.
\]
Suppose that \( f : X' \to Y \) satisfies the inequality
\[
\|f(x + y) + f(x - y) - 2f(x), z\| \leq h_1(x, z)h_2(y, z),
\]
for all \( x, y \in X' \) and \( z \in Y \) such that \( x + y, x - y \in X' \). Then \( f \) is a Cauchy-Jensen on \( X' \).

Proof. Replacing \( x \) by \((m+1)x\) and \( y \) by \( mx \) where \( x, y \in X' \) and \( m \in \mathbb{N} \) in the inequality (3.1), we get
\[
\|2f((m+1)x) - f((2m+1)x) - f(x), z\| \leq h_1((m+1)x, z)h_2(mx, z), \quad x \in X', z \in Y. \quad (3.2)
\]
For each \( m \in \mathbb{N} \), we define the operator \( \mathcal{T}_m : Y^{X'} \to Y^{X'} \) and the function \( \varepsilon_m : X' \times Y \to \mathbb{R}_+ \) by
\[
\mathcal{T}_m(x) := 2\xi((m+1)x) - \xi((2m+1)x), \quad \xi \in Y^{X'}, \ x \in X', \ z \in Y, \ m \in \mathbb{N},
\]
that (3.4) holds for all $x \in X'$, $z \in Y$ and $m \in \mathbb{N}$, the inequality (3.2) becomes

$$\|T_m f(x) - f(x), z\| \leq \varepsilon_m(x, z) \quad x \in X', \ z \in Y.$$ 

Furthermore, for every $\xi, \mu \in Y^{X'}$, $x \in X'$, $z \in Y$ and $m \in \mathbb{N}$, we have

$$\|T_m \xi(x) - T_m \mu(x), z\| = \|2\xi((m + 1)x) - \xi((2m + 1)x) - 2\mu((m + 1)x) + \mu((2m + 1)x), z\|$$

$$\leq \max \left\{ 2\|\xi((m + 1)x) - \mu((m + 1)x), z\|, \|\xi((2m + 1)x) - \mu((2m + 1)x), z\| \right\}$$

It brings us to define the operator $\Lambda_m : \mathbb{R}_{+}^{X' \times Y} \to \mathbb{R}_{+}^{X' \times Y}$ by

$$\Lambda_m \delta(x, z) := \max \left\{ \delta((m + 1)x), z), \delta((2m + 1)x), z) \right\}, \quad \delta \in \mathbb{R}_{+}^{X' \times Y}, \ x \in X', \ z \in Y.$$ 

Therefore, for each $m \in \mathbb{N}$, the operator $\Lambda := \Lambda_m$ has the form described in (H3) with $k = 2$, $f_1(x) = (m + 1)x$, $f_2(x) = (2m + 1)x$, $L_1(x, z) = L_2(x, z) = 1$, $g_i = \text{Id}_Y$, $i = 1, 2$ for all $x \in X'$ and $z \in Y$.

Observe that

$$\varepsilon_m(x, z) \leq \lambda_1(m + 1)\lambda_2(m)\alpha_m h_1(x, z)h_2(x, z),$$

(3.3)

for all $x \in X'$ and $z \in Y$. By induction, we will show that for each $n \in \mathbb{N}_0$, we have

$$\Lambda_m^{n+1} \varepsilon_m(x, z) \leq \lambda_1(m + 1)\lambda_2(m)\alpha_m h_1(x, z)h_2(x, z), \quad x \in X', \ z \in Y.$$ 

(3.4)

for all $m \in U$. For $n = 0$, it’s obvious to see that (3.4) is exactly (3.3). We fix $k \in \mathbb{N}$ and assume that (3.4) holds for $n = k$. Then, using the non-decreasing of $\Lambda_m$, we have

$$\Lambda_m^{k+1} \varepsilon_m(x, z) = \Lambda_m(\Lambda_m^k \varepsilon_m(x, z))$$

$$= \max \{ \Lambda_m^k \varepsilon_m((m + 1)x, z), \Lambda_m^k \varepsilon_m((2m + 1)x, z) \}$$

$$= \lambda_1(m + 1)\lambda_2(m)\alpha_m^k \max \{ h_1((m + 1)x, z)h_2((m + 1)x, z), h_1((2m + 1)x, z)h_2((2m + 1)x, z) \}$$

$$\leq \lambda_1(m + 1)\lambda_2(m)\alpha_m^k \max \{ \lambda_1(m + 1)\lambda_2(m + 1), \lambda_1(2m + 1)\lambda_2(2m + 1) \}$$

$$= \lambda_1(m + 1)\lambda_2(m)\alpha_m^{k+1} h_1(x, z)h_2(x, z),$$

for all $x \in X'$ and $z \in Y$. Letting $n \to \infty$ in (3.4), we get

$$\lim_{n \to \infty} \Lambda_m^n \varepsilon_m(x, z) = 0$$

for all $x \in X', z \in Y$ and all $m \in U$. Then, according to Theorem 2.1, there exists , for each $m \in U$, a fixed point $J_m$ of $T_m$ such that

$$\|f(x) - J_m(x), z\| \leq \sup_{n \in \mathbb{N}_0} \Lambda_m^n \varepsilon_m(x, z),$$

(3.5)

for all $x \in X'$ and all $z \in Y$ and

$$\lim_{n \to \infty} T_m^n f(x) = J_m(x), \quad x \in X'.$$ 

(3.6)

Next, we will show, by induction, that for each $n \in \mathbb{N}_0$

$$\|T_m^n f(x + y) + T_m^n f(x - y) - 2T_m^n f(x), z\| \leq \alpha_m^n h_1(x, z)h_2(y, z),$$

(3.7)

for all $x, y, x - y, x + y \in X'$, $z \in Y$ and all $m \in U$. Since the case $n = 0$ is just (3.1), we fix $k \in \mathbb{N}$ and suppose that (3.7) holds for $n = k$. Then, for
all $x, y \in X'$ such that $x - y, x + y \in X'$ and $z \in Y$ we have

$$
\|T^{k+1}_m f(x + y) + T^{k+1}_m f(x - y) - 2T^{k+1}_m f(x), z\| \\
= \|T_m (T^k_m f(x + y)) + T_m (T^k_m f(x - y)) - 2T_m (T^k_m f(x)), z\| \\
= \|2T^k_m f((m + 1)(x + y)) - T^k_m f((2m + 1)(x + y)) + 2T^k_m f((m + 1)(x - y)) \\
- T^k_m f((2m + 1)(x - y)) - 4T^k_m f((m + 1)x) + 2T^k_m f((2m + 1)x), z\| \\
\leq \max\{2\|T^k_m f((m + 1)(x + y)) + T^k_m f((m + 1)(x - y)) - 2T^k_m f((m + 1)x), z\|; \\
\|T^k_m f((2m + 1)(x + y)) + T^k_m f((2m + 1)(x - y)) - 2T^k_m f((m + 1)x), z\|; \\
\|T^k_m f((m + 1)(x + y)) + T^k_m f((m + 1)(x - y)) - 2T^k_m f((m + 1)x), z\|; \\
\|T^k_m f((2m + 1)(x + y)) + T^k_m f((2m + 1)(x - y)) - 2T^k_m f((2m + 1)x), z\|\} \\
\leq \alpha_m^k \max\{h_1((m + 1)x, z); h_2((m + 1)y, z); h_1((2m + 1)x, z)h_2((2m + 1)y, z)\} \\
\leq \alpha_m^k h_1(x, z)h_2(y, z) \max\{\lambda_1(m + 1)\lambda_2(m + 1); \lambda_1(2m + 1)\lambda_2(2m + 1)\} \\
= \alpha_m^k h_1(x, z)h_2(y, z).
$$

Thus, we have shown that (3.7) holds for every $n \in \mathbb{N}_0$. Letting $n \to \infty$ in (3.7), we obtain, for each $m \in \mathcal{U}$, that

$$
J_m(x+y) + J_m(x-y) = 2J_m(x), \quad x, x+y, x-y \in X'.
$$

In this way, we find a sequence $\{J_m\}_{m \in \mathcal{U}}$ of a Cauchy-Jensen functions on $X'$ such that

$$
\|f(x) - J_m(x), z\| \leq \sup\{\lambda_1(m + 1)\lambda_2(m)\alpha_m^kh_1(x, z)h_2(x, z)\}, \quad x \in X', \ z \in Y
$$

It follows, with $m \to \infty$, that $f$ is Cauchy-Jensen on $X'$. □

By similar method, we prove the following theorem.

**Theorem 3.2.** Let $h : X' \times Y \to \mathbb{R}_+$ be a function such that

$$
\mathcal{U} := \{n \in \mathbb{N} : \alpha_n = \max\{\lambda(n + 1), \lambda(n)\} < 1\} \neq \emptyset,
$$

where

$$
\lambda(n) := \inf\{t \in \mathbb{R}_+ : h(nx, z) \leq th(x, z), \quad x \in X', \ z \in Y\}
$$

for all $n \in \mathbb{N}$ such that $\lim_{n \to \infty} \lambda(n) = 0$.

Suppose that $f : X' \to Y$ satisfies the inequality

$$
\|f(x+y) + f(x-y) - 2f(x), z\| \leq h(x, z) + h(y, z), \tag{3.8}
$$

for all $x, y \in X'$ such that $x + y, x - y \in X'$ and $z \in Y$. Then $f$ is a Cauchy-Jensen on $X'$.

**Proof.** Replacing $x$ by $(m+1)x$ and $y$ by $mx$ for $m \in \mathbb{N}$ in (3.8), we get

$$
\|2f((m+1)x) - f((2m+1)x) - f(x), z\| \leq h((m+1)x, z) + h(mx, z), \tag{3.9}
$$

for all $x \in X'$ and $z \in Y$. For each $m \in \mathcal{U}$, we define the operator $T_m : YX' \to YX'$ by

$$
T_m \xi(x) := 2\xi((m+1)x) - \xi((2m+1)x), \quad \xi \in YX', \ x \in X'.
$$

Further, putting

$$
\varepsilon_m(x, z) = h((m+1)x, z) + h((m+1)x, z) \leq \lambda(m+1) + \lambda(m)h(x, z), \tag{3.10}
$$
for all $x \in X'$ and $z \in Y$, then the inequality (3.9) takes the form

$$\|T_m f(x) - f(x), z\| \leq \varepsilon_m(x, z), \ x \in X', \ z \in Y.$$  

For each $m \in U$, the operator $A_m : \mathbb{R}^{X' \times Y} \rightarrow \mathbb{R}^{X' \times Y}$ which is defined by

$$A_m \delta(x) = \max \{\delta((m+1)x, z), \delta(mx, z)\}, \ \delta \in \mathbb{R}^{X' \times Y}, \ x \in X', \ z \in Y$$

has the form described in (H3) with $k = 2$ and

$$f_1(x) = (m+1)x, \ f_2(x) = mx, \ L_1(x, z) = L_2(x, z) = 1$$

for all $x \in X'$. Moreover, for every $\xi, \mu \in Y^X$, $x \in X'$ and $z \in Y$, we have

$$\|T_m \xi(x) - T_m \mu(x), z\|$$

$$= \|2\xi((m+1)x) - \xi((m+1)x) - 2\mu((m+1)x) + \mu((m+1)x), z\|$$

$$\leq \max \{2\|\xi((m+1)x, z) - \mu((m+1)x, z)\|, \|\xi((m+1)x, z) - \mu((m+1)x, z)\|\}$$

$$\leq \max \{\|\xi((m+1)x, z) - \mu((m+1)x, z)\|, \|\xi((m+1)x, z) - \mu((m+1)x, z)\|\}.$$  

So, (H2) is valid. Also, by using mathematical induction on $n \in \mathbb{N}_0$, we will show, for each $x \in X'$ and $z \in Y$, that

$$A_m^n \varepsilon_m(x, z) \leq (\lambda(m+1) + \lambda(m)) \alpha_m^n h(x, z), \tag{3.11}$$

where $\alpha_m := \max \{\lambda((m+1), \lambda(m))\}$ for all $m \in U$. From (3.10), we obtain that the inequality (3.11) holds for $n = 0$. Next, we will assume that (3.11) holds for $n = k$, where $k \in \mathbb{N}$. Then we have

$$A_m^{k+1} \varepsilon_m(x, z) = A_m \left(A_m^k \varepsilon_m(x, z)\right) = \max \left\{A_m^k \varepsilon_m((m+1)x, z), A_m^k \varepsilon_m(mx, z)\right\}$$

$$\leq (\lambda(m+1) + \lambda(m)) \alpha_m^k \max \left\{\varphi((m+1)x, z), \varphi(mx, z)\right\}$$

$$\leq (\lambda(m+1) + \lambda(m)) \alpha_m^{k+1} \varphi(x, z), \ x \in X', \ z \in Y.$$  

This shows that (3.11) holds for $n = k + 1$. Now we can conclude that the inequality (3.11) holds for all $n \in \mathbb{N}_0$. From (3.11), we obtain

$$\lim_{n \to \infty} A_m^n \varepsilon_m(x, z) = 0,$$

for all $x \in X'$, $z \in Y$ and all $m \in U$. Hence, according to Theorem 2.1, there exists, for each $m \in U$, a unique solution $J_m : X' \rightarrow Y$ of the equation

$$J_m(x) = J_m((m+1)x) - 2J_m((m+1)x), \ x \in X', \tag{3.12}$$

such that

$$\|f(x) - J_m(x), z\| \leq \sup_{n \in \mathbb{N}_0} \left\{(\lambda(m+1) + \lambda(m)) \alpha_m^n h(x, z)\right\}, \ x \in X', \ z \in Y. \tag{3.13}$$

Moreover,

$$J_m(x) := \lim_{n \to \infty} (T_m^n f)(x)$$

for all $x \in X'$. Now, we show that

$$\|T_m^n f(x + y) + T_m^n f(x - y) - 2T_m^n f(x), z\| \leq \alpha_m^n (h(x, z) + h(y, z)) \tag{3.14}$$

for every $z \in Y$, $x, y \in X'$ such that $x + y, x - y \in X'$ and $n \in \mathbb{N}_0$. Since the case $n = 0$ is just (3.8), take $k \in \mathbb{N}$ and assume that (3.14) holds for $n = k$, where $k \in \mathbb{N}$ and every $x, y \in X'$.
Thus, by induction we have shown that (3.14) holds for every $n \in \mathbb{N}_0$. Letting $n \to \infty$ in (3.14), we obtain that

\[ J_{m}(x + y) + J_{m}(x - y) = 2J_{m}(x), \]

for all $x, y \in X'$ such that $x + y, x - y \in X'$. In this way, we obtain a sequence $\{J_{m}\}_{m \in \mathbb{U}}$ of Cauchy-Jensen functions on $X'$ such that

\[ \|f(x) - J_{m}(x), z\| \leq \sup_{n \in \mathbb{N}_0} \left\{ \left( \lambda(m + 1) + \lambda(2m + 1) \right) \alpha_{m}^{\alpha} h(x, z) \right\}, \quad x \in X', \quad z \in Y. \]

It implies that

\[ \|f(x) - J_{m}(x), z\| \leq \left( \lambda(m + 1) + \lambda(2m + 1) \right) \alpha_{m}^{\alpha} h(x, z), \quad x \in X', \quad z \in Y. \]

It follows, with $m \to \infty$, that $f$ is a Cauchy-Jensen on $X'$.

## 4 Consequences

In this section, we assume that $X' = X_0 = X \setminus \{0\}$. According the Theorem 3.1 and Theorem 3.2, we derive the following two corollaries.

**Corollary 4.1.** Let $(X, \| \cdot \|_X)$ be a normed space and $(Y, \| \cdot \|, || \cdot ||)$ be a non-Archimedean 2-Banach space, $s$ be a fixed element in $Y$ and let $c \geq 0, r \geq 0, p, q \in \mathbb{R}$ such that $p + q < 0$. Suppose that $f : X' \to Y$ is a function satisfying the inequality

\[ ||f(x + y) + f(x - y) - 2f(x), z|| \leq c \|x\|_X^p \|y\|_X^q \|z, s\|_r, \quad (4.1) \]

for all $x, y \in X'$ such that $x + y, x - y \in X'$ and $z \in Y$. Then $f$ is a Cauchy-Jensen on $X'$.

**Proof.** The proof follows from Theorem 3.1 by taking $h_1, h_2 : X' \times Y \to \mathbb{R}_+$ as follows:

\[ h_1(x, z) = c_1 \|x\|_X^p \|z, s\|_r \]

and

\[ h_2(y, z) = c_2 \|y\|_X^q \|z, s\|_r \]

such that $x + y, x - y \in X'$. Then

\[ \|T_{m+1}^k f(x + y) + T_{m+1}^k f(x - y) - 2T_{m+1}^k f(x, z)\| \]

\[ = \|T_m \left( T_{m}^k f(x + y) \right) + T_m \left( T_{m}^k f(x - y) \right) - 2T_m \left( T_{m}^k f(x) \right), z\| \]

\[ = \|2T_{m}^k f((m+1)(x+y)) - T_{m}^k f((m+1)(x+y)) + 2T_{m}^k f((m+1)(x-y)) - T_{m}^k f((m+1)(x-y)) - 4T_{m}^k f((m+1)x) + 2T_{m}^k f((m+1)x), z \|
\]

\[ \leq \max \left\{ 2 \|T_{m}^k f((m+1)(x+y)) + T_{m}^k f((m+1)(x-y)) - 2T_{m}^k f((m+1)x), z \|, \right\} \]

\[ \|T_{m}^k f((m+1)(x+y)) + T_{m}^k f((m+1)(x-y)) - 2T_{m}^k f((m+1)x), z\| \]

\[ \leq \max \left\{ \alpha_{m}^k \left( h((m+1)x, z) + h((m+1)y, z) \right), \alpha_{m}^k \left( h((2m+1)x, z) + h((2m+1)y, z) \right) \right\} \]

\[ \leq \alpha_{m}^k \max \left\{ \lambda(m+1), \lambda(2m+1) \right\} \left( h(x, z) + h(y, z) \right) \]

\[ = \alpha_{m}^{k+1} \left( h(x, z) + h(y, z) \right). \]
for all \( x, y \in X' \) and all \( z \in Y \), where \( c_1, c_2 \in \mathbb{R}_+, r_1, r_2 \in \mathbb{R} \) and \( p, q \in \mathbb{R} \) such that \( r_1 + r_2 \geq 0 \) and \( p + q < 0 \).

For each \( m \in \mathbb{N} \), we define \( \lambda_1(m) \) as in Theorem 3.1

\[
\lambda_1(m) = \inf \{ t \in \mathbb{R}_+ : h_1(mx, z) \leq t h_1(x, z) \}, \quad x \in X', \ z, s \in Y
\]

\[
= \inf \{ t \in \mathbb{R}_+ : c_1m^p ||x||_X^p ||z, s||^r_t \leq tc_1 ||x||_X^p ||z, s||^r_t \}, \quad x \in X', \ z, s \in Y
\]

\[
= m^p.
\]

Also, for \( m \in \mathbb{N} \), we have \( \lambda_2(m) = m^q \). Therefore,

\[
\lim_{m \to \infty} \lambda_1(m + 1)\lambda_2(m) = \lim_{m \to \infty} (m + 1)^p(m)^q = \lim_{m \to \infty} (m + 1)^{p+q} = 0.
\]

Furthermore, we get

\[
\alpha_m = \max \{ \lambda_1(m + 1)\lambda_2(m + 1) \}, \quad \lambda_1(2m + 1)\lambda_2(2m + 1) \}
\]

\[
= \max \{ (m + 1)^{p+q}, (2m + 1)^{p+q} \}
\]

\[
= (m + 1)^{p+q}.
\]

Then \( \mathcal{U} \) is a non empty set. According to Theorem 3.1, \( f \) is a Cauchy-Jensen on \( X' \). \( \square \)

By a similar method, we can prove the following corollary as a particular case of Theorem 3.2 where \( h(x, z) = c ||x||_X^p ||z, s||^r \) with \( c \geq 0 \), \( p < 0 \), and \( r \geq 0 \).

**Corollary 4.2.** Let \( (X, ||||_X) \) be a normed space and \( (Y, ||||_Y) \) be a non-Archimedean 2-Banach space, \( s \) be a fixed element in \( Y \) and let \( c \geq 0 \), \( p < 0 \), \( r \geq 0 \) and \( f : X' \to Y \) satisfy

\[
||f(x + y) + f(x - y) - 2f(x), z|| \leq c (||x||_X^p + ||y||_X^p) ||z, s||^r,
\]

for all \( x, y \in X' \) such that \( x + y, x - y \in X' \) and \( z \in Y \). Then \( f \) is a Cauchy-Jensen on \( X' \).

In the following two corollaries, we discuss the hyperstability of the inhomogeneous Cauchy-Jensen functional equation.

**Corollary 4.3.** Let \( (X, ||||_X) \) be a normed space and \( (Y, ||||_Y) \) be a non-Archimedean 2-Banach space, \( s \) be a fixed element in \( Y \) and let \( G : X' \to Y \) such that \( G(x_0, y_0) \neq 0 \) for some \( x_0, y_0 \in X' \) and

\[
||G(x, y), z|| \leq c ||x||_X^p ||y||_X^p ||z, s||^r,
\]

(4.2)

for all \( x, y \in X' \) such that \( x + y, x - y \in X' \) and \( z \in Y \) where \( c \geq 0 \), \( p, q \in \mathbb{R} \) such that \( p + q < 0 \). Then the functional equation

\[
g(x + y) + g(x - y) = 2g(x) + G(x, y),
\]

(4.3)

for all \( x, y \in X' \) with \( x + y, x - y \in X' \), has no solution in the class of functions \( g : X \to Y \).

**Proof.** Suppose that \( f : X \to Y \) is a solution to (4.3). Then

\[
||f(x + y) + f(x - y) - 2f(x), z|| = ||2f(x) + G(x, y) - 2f(x), z||
\]

\[
= ||G(x, y), z||
\]

\[
\leq c ||x||_X^p ||y||_X^p ||z, s||^r, \quad x, y, z \in X', \ y, z \in Y.
\]

Consequently, by Theorem 3.1, \( f \) is a Cauchy-Jensen on \( X' \), whence

\[
G(x_0, y_0) = f(x_0 + y_0) + f(x_0 - y_0) - 2f(x_0) = 0,
\]

which is a contradiction. \( \square \)
Corollary 4.4. Let \((X, \|\cdot\|_X)\) be a normed space and \((Y, \|\cdot\|)\) be a non-Archimedean 2-Banach space, \(s\) be a fixed element in \(Y\) and \(p, q \in \mathbb{R}\) such that \(p + q < 0\). Assume that \(G : X^2 \to Y\) and \(f : X' \to Y\) satisfy the inequality

\[
\| f(x + y) + f(x - y) - 2f(x) - G(x, y), z \| \leq c \| x \|_X^p \| y \|_X^q \| z, s \|, \tag{4.4}
\]

for all \(x, y, z \in X'\) with \(x + y, x - y \in X'\) and \(z \in Y\). If the functional equation

\[
f(x + y) + f(x - y) = 2f(x) + G(x, y), \quad x, y \in X', \tag{4.5}
\]

has a solution \(f_0 : X' \to Y\), then \(f\) is a solution of functional equation 4.5 on \(X'\).

Proof. From (4.4), we get that the function \(K : X' \to Y\) defined by \(K := f - f_0\) satisfies (4.1). Consequently, Corollary 4.1 implies that \(K\) is a Cauchy-Jensen on \(X'\). Therefore,

\[
f(x + y) + f(x - y) - 2f(x) - G(x, y) = K(x + y) + f_0(x + y) + K(x - y) + f_0(x - y) - 2K(x) - 2f_0(x) - G(x, y)
\]

\[= 0.
\]

for all \(x, y \in X'\) with \(x + y, x - y \in X'\). Which means \(f\) is a solution of functional equation 4.5 on \(X'\).

\[\square\]

References


**Author information**

R. El Ghali and S. Kabbaj, Department of Mathematics, Faculty of Sciences, University of Ibn Tofail, BP 133 Kenitra, Morocco.
E-mail: rachid2810@gmail.com; samkabbaj@yahoo.fr

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