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A new approach to fixed point result in non-Archimedean 2-Banach space and some of its applications

R. El Ghali and S. Kabbaj

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Abstract In this paper we extend the fixed point result of Brzdęk et al. [7] in non-Archimedean 2-Banach spaces. Moreover, we investigate the hyperstability of Cauchy-Jensen functional equation in the considered space by using the above result and we give some outcomes.

1 Introduction and preliminaries

A certain formula or equation is applicable to model a physical process of a small change of the formula or equation gives rise to a small change in the corresponding result. When this happens, we say that formula or equation is called *stable*. One of the unsolved problems was given by S. M. Ulam [21] tends to be the starting point for researching the stability problems of functional equations. Ulam asked the following question concerning the stability of group homomorphisms:
Given a group G, a metric group H with metric $d(.,.)$ and a positive number ε, does there exists a $\delta > 0$ such that if $f : G \to H$ satisfies : $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $\Phi : G \to H$ exists with $d(f(x), \Phi(x)) < \varepsilon$ for $x \in G$?
D. H. Hyers [15] gave the first partial answer to Ulam's problem for the Cauchy equation

$$f(x+y) = f(x) + f(y), \tag{1.1}$$

in Banach spaces with $\delta = \varepsilon$ and

$$\Phi(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}.$$

The most classical result concerning the Hyers-Ulam stability for the Cauchy equation (1.1) has been given by Th. M . Rassias [19].

Theorem 1.1. *[19] Let E_1 and E_1 be two normed spaces, $c \geq 0$ and $p \neq 1$ be fixed real numbers. Let $f : E_1 \to E_2$ be a mapping satisfying*

$$\|f(x+y) - f(x) - f(y)\| \leq c\left(\|x\|^p + \|y\|^p\right), \quad x, y \in E_1 \setminus \{0\}.$$

Then the following statements are valid

(1) If $p \geq 0$ and E_2 is complete, then there exists a unique additive function $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{c\|x\|^p}{|2^{p-1} - 1|}, \quad x \in E_1 \setminus \{0\}. \tag{1.2}$$

(2) $p < 0$, then f is additive.

This results is called the Hyers-Ulam - Rassias stability of Cauchy functional equation.

In 1994, P. Găvruţă [13] gave a generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings.

Theorem 1.2. *Let G be an abilean group and $(X, \|.\|)$ a Banach space. Let $\varphi : G \times G \to \mathbb{R}^+$ a mapping satisfying, for all $x, y \in G$, the condition:*

$$\tilde{\varphi}(x, y) = \sum_{k=0}^{\infty} 2^{-k} \varphi(2^k x, 2^k y) < +\infty$$

Let $f : G \to X$ be a mapping which fulfils, for each $x, y \in G$, the condition

$$\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y)$$

Then there exists a unique mapping $\mathcal{T} : G \to G$ such that

$$\mathcal{T}(x + y) = \mathcal{T}(x) + \mathcal{T}(y),$$

for all $x, y \in G$ and :

$$\|f(x) - \mathcal{T}(x)\| \leq \frac{1}{2}\tilde{\varphi}(x, y),$$

for all $x \in G$.

Since then, the problem of stability of several functional equations have been extensively studied by many mathematicians (see, for instance, [2, 6, 7, 8, 13, 15, 19]).

A functional equation is called *hyperstable* when any function f satisfying the equation approximately, in some sense, must be actually a solution to it. The term hyperstability was used for the first time probably in 2001 by Gy. Maksa and Zs. Páles [18], however, it seems that the first hyperstability result was published in [5] and concerned the ring homomorphisms. The hyperstability results for the Cauchy equation were investigated by J. Brzdęk in [9, 10]. E. Gselmann [14] studied the hyperstability of the parametric fundamental equation of information.

Note that the second statement of the Theorem 1.1, for $p < 0$ can be described as $\varphi-$ hyperstability of the additive equation with $\varphi(x, y) = c\left(\|x\|^p + \|y\|^p\right)$.

Throughout this paper, $\mathbb{N}$ stands for the set of all positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N}_{m_0}$ the set of all integers greater than or equals m_0 ($m_0 \in \mathbb{N}$), $\mathbb{R}_+ = [0, \infty)$ and we use the notation X_0 for the set $X \setminus \{0\}$.

Let us recall (see, for instance, [17]) some basic definitions and facts concerning non-Archimedean 2-normed spaces.

Definition 1.3. By a *non-Archimedean* field, we mean a field $\mathbb{K}$ equipped with a function (*valuation*) $|\cdot| : \mathbb{K} \to [0, \infty)$ such that for all $r, s \in \mathbb{K}$, the following conditions hold:

(1) $|r| = 0$ if and only if $r = 0$,

(2) $|rs| = |r||s|$,

(3) $|r + s| \leq \max\{|r|, |s|\}$.

The pair $(\mathbb{K}, |.|)$ is called a *valued field*.

Remark 1.4. In any non-Archimedean field, we have $|1| = |-1| = 1$ and $|n| \leq 1$ for $n \in \mathbb{N}$.

Example 1.5. In any field $\mathbb{K}$ the function $|\cdot| : \mathbb{K} \to \mathbb{R}_+$ given by

$$|x| := \begin{cases} 0, & x = 0, \\ 1, & x \neq 0 \end{cases}$$

is a valuation which is called *trivial valuation*, but the most important example of non-Archimedean fields are p-adic numbers which have gained the interest of physicists for their research in some problems coming from quantum physics, p-adic strings and super strings.

Let p be a fixed prime number and x a rational number, there exists a unique integer $v_p(x) \in \mathbb{Z}$ such that $x = p^{v_p(x)}\ \frac{a}{b}$ where a and b are integer co-prime to p. The function defined in $\mathbb{Q}$ by $|x|_p = p^{v_p(x)}$ is called a p-adic, an ultrametric or simply a non-Archimedean absolute value on $\mathbb{Q}$. The completion, denoted by $\mathbb{Q}_p$ of $\mathbb{Q}$ with respect to the metric defined by the p-adic absolute is called p-adic numbers.

Definition 1.6. Let X be a vector space (with $dimX > 1$) over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $||.,.|| : X^2 \to \mathbb{R}_+$ is called a *non-Archimedean 2-norm (valuation)* if it satisfies the following conditions:

(1) $\|x,y\| = 0$ if and only if x and y are linearly independent, $x, y \in X$,

(2) $\|x,y\| = \|y,x\| \ \ x, y \in X$,

(3) $\|rx,y\| = |r| \, \|x,y\| \ \ (r \in \mathbb{K}, x, y \in X)$,

(4) $\|x, y+z\| \leq \max\left\{\|x,y\|, \|x,z\|\right\} \ \ x, y, z \in X$.

Then $(X, \|\cdot,\cdot\|)$ is called a non-Archimedean 2-normed space or an ultrametric 2-normed space.

Example 1.7. Let p be a fixed prime number. For $x = (x_1, x_2)$ and $y = (y_1, y_2)$ we define the non-Archimedean 2-norm in ${\mathbb{Q}_p}^2$ by $\|x,y\|_p = |x_1y_2 - x_2y_1|_p$.

Definition 1.8. Let $\{x_n\}$ be a sequence in a non-Archimedean 2-normed space X.

(1) A sequence$\{x_n\}_{n=1}^{\infty}$ is *a Cauchy sequence* if there are linearly independent $y, z \in X$ such that

$$\lim_{n\to\infty} \|x_{n+1} - x_n, y\| = 0 = \lim_{n\to\infty} \|x_{n+1} - x_n, z\|$$

(2) The sequence $\{x_n\}$ is said to be *convergent* if there exists $x \in X$ (called limit of this sequence and denoted by $\lim_{n\to\infty} x_n$) such that

$$\lim_{n\to\infty} \|x_n - x, y\| = 0 \quad y \in X$$

(3) If every Cauchy sequence in X converges, then the non-Archimedean 2-normed space X is called a non-Archimedean 2-Banach space or an ultrametric 2-Banach space.

Lemma 1.9. *[20]*

(1) Let X be a non-Archimedean 2-Banach space over a non-Archimedean field $\mathbb{K}$ and $x, y, z \in X$ such that y and z are linearly independent and $\|x,y\| = 0 = \|x,z\|$, then $x = 0$.

(2) $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence of element of X then :

$$\lim_{n\to\infty} \|x_n, y\| = \|\lim_{n\to\infty} x_n, y\| \quad y \in X.$$

In section 2, we introduce and prove a new version of fixed point theorem of Brzdęk [12] in non-Archimedean 2-Banach space. This theorem has been considered as an important tool for investigating the stability and hyperstability, in some way, of a several functional equations by many mathematicians (see for example [1, 4]). In section 3, we use our main results to investigate the hyperstability of the following Cauchy-Jensen functional equation

$$f(x+y) + f(x-y) = 2f(x), \tag{1.3}$$

in non-Archimedean 2-Banach space. We also give some outcomes as particular cases and we study the hyperstability of the inhomogeneous Cauchy-Jensen equation

$$f(x+y) + f(x-y) = 2f(x) + G(x,y).$$

2 Fixed point theorem

In 2018, J. Brzdęk and K. Ciepliński [12] presented and proved a new version of fixed point theorem in 2-Banach spaces with some applications in stability theory of functional equations. The following theorem is an analogous version of fixed theorem [12] in non-Archimedean 2-Banach spaces.

First, we need to present the following hypotheses.

(**H1**) X is a nonempty set, $(Y, \|.,.\|)$ is a non-Archimedean 2-Banach space over a non-Archimedean field, Y_0 is a subset of Y containing two linearly independent vectors, $f_1, ..., f_k : X \longrightarrow X$, $g_1, ..., g_k : Y_0 \longrightarrow Y_0$ and $L_1, ..., L_k : X \times Y_0 \longrightarrow \mathbb{R}_+$ are given.

(**H2**) $\mathcal{T} : Y^X \longrightarrow Y^X$ is an operator satisfying the inequality :

$$\left\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x), y\right\| \leq \max_{1\leq i\leq k} \left\{L_i(x,y)\left\|\xi\big(f_i(x)\big) - \mu\big(f_i(x)\big), g_i(y)\right\|\right\}, \quad \xi, \mu \in Y^X, x \in X, y \in Y_0.$$

(**H3**) $\Lambda : \mathbb{R}_+^{X\times Y_0} \longrightarrow \mathbb{R}_+^{X\times Y_0}$ is a non-decreasing linear operator defined by

$$\Lambda\delta(x,y) := \max_{1\leq i\leq k} \left\{L_i(x,y)\delta\big(f_i(x), g_i(y)\big)\right\}, \qquad \delta \in \mathbb{R}_+^{X\times Y_0}, \quad x \in X, y \in Y_0.$$

Theorem 2.1. *Let hypotheses (**H1**)-(**H3**) are valid and let $\varepsilon : X \times Y_0 \longrightarrow \mathbb{R}_+$ and $\varphi : X \longrightarrow Y$ be functions fulfilling the following two conditions*

$$\|\mathcal{T}\varphi(x) - \varphi(x), y\| \le \varepsilon(x, y), \qquad x \in X, y \in Y_0, \tag{2.1}$$

$$\lim_{n\to\infty} \Lambda^n \varepsilon(x, y) = 0, \qquad x \in X, y \in Y_0. \tag{2.2}$$

Then, for every $x \in X$, the limit

$$\psi(x) = \lim_{n\to\infty} \mathcal{T}^n \varphi(x)$$

exists and defines a fixed point ψ of $\mathcal{T}$ with

$$\|\varphi(x) - \psi(x), y\| \le \sup_{n\in\mathbb{N}_0} \Lambda^n \varepsilon(x, y) = \sigma(x, y), \qquad x \in X, y \in Y_0. \tag{2.3}$$

Moreover, if

$$(\Lambda\sigma)(x, y) \le \sup_{n\in\mathbb{N}_0} \Lambda^{n+1} \varepsilon(x, y), \qquad x \in X, x \in Y_0, \tag{2.4}$$

then ψ is a unique fixed point of $\mathcal{T}$ satisfying (2.3).

Proof. We show by induction that, for any $n \in \mathbb{N}_0$

$$\|\mathcal{T}^n\varphi(x) - \mathcal{T}^{n+1}\varphi(x), y\| \le \Lambda^n \varepsilon(x, y), \ \ x \in X, \ y \in Y_0. \tag{2.5}$$

Indeed, it's easy to see that if $n = 0$, then the inequality (2.5) is exactly (2.1). Now, we fix $n \in \mathbb{N}$ and suppose that (2.5) hold for n, then by using the non-decreasing property of the operator Λ and (**H2**), for any $x \in X, y \in Y_0$, we get

$$\begin{aligned}\Big\|\mathcal{T}^{n+1}\varphi(x) - \mathcal{T}^{n+2}\varphi(x), y\Big\| &\le \max_{1\le i\le k} \Big\{L_i(x, y)\Big\|\mathcal{T}^n\varphi\big(f_i(x)\big) - \mathcal{T}^{n+1}\varphi\big(f_i(x)\big), g_i(y)\Big\|\Big\} \\ &\le \max_{1\le i\le k} \big\{L_i(x, y)\Lambda^n\varepsilon\big(f_i(x), g_i(y)\big)\big\} \\ &= \Lambda^{n+1}\varepsilon(x, y),\end{aligned} \tag{2.6}$$

then (2.5) holds for any $n \in \mathbb{N}$. Moreover, by using (2.2) and (2.5), for any $k \in \mathbb{N}$, $n \in \mathbb{N}_0$ and $x \in X$ and $y \in Y_0$, we have

$$\begin{aligned}\Big\|\mathcal{T}^n\varphi(x) - \mathcal{T}^{n+k}\varphi(x), y\Big\| &\le \max_{0\le i\le k-1} \Big\{\Big\|\mathcal{T}^{n+i}\varphi(x) - \mathcal{T}^{n+i+1}\varphi(x), y\Big\|\Big\} \\ &\le \max_{0\le i\le k-1} \big\{\Lambda^{n+i}\varepsilon(x, y)\big\},\end{aligned} \tag{2.7}$$

The sequence $\big(\mathcal{T}^n\varphi(x)\big)_{n\in\mathbb{N}}$, for each $x \in X$, is a Cauchy sequence. Because Y is a complete space, so this sequence is convergent and the limit $\psi(x) = \lim_{n\to\infty} \mathcal{T}^n\varphi(x)$ exists. Letting $k \to \infty$ in (2.7), we obtain, for any $n \in \mathbb{N}$, $x \in X$ and $y \in Y_0$, that :

$$\begin{aligned}\|\mathcal{T}^n\varphi(x) - \psi(x), y\| &\le \sup_{i\ge n}(\Lambda^i\varepsilon(x, y)) \\ &= \sigma_n(x, y).\end{aligned} \tag{2.8}$$

For $n = 0$, it's easy to show that (2.8) gives (2.3). Moreover, by using (2.8) and (**H2**), we find

$$\begin{aligned}\Big\|\mathcal{T}^{n+1}\varphi(x) - \mathcal{T}\psi(x), y\Big\| &\le \max_{0\le i\le k} \Big\{L_i(x, y)\Big\|\mathcal{T}^n\varphi\big(f_i(x)\big) - \psi\big(f_i(x)\big), g_i(y)\|\Big\} \\ &\le \Lambda\Big(\Big\|\mathcal{T}^n\varphi(x) - \psi(x), y\Big\|\Big) \\ &\le \Lambda\Big(\sup_{i\ge n}(\Lambda^i\varepsilon(x, y))\Big) \\ &\le \Lambda(\sigma_n(x, y)),\end{aligned} \tag{2.9}$$

for all $n \in \mathbb{N}$, $x \in X$ and $y \in Y_0$. Letting $n \to \infty$ in (2.9) and using (2.2), we get

$$\mathcal{T}\psi(x) = \lim_{n\to\infty} \mathcal{T}^{n+1}\varphi(x) = \psi(x)$$

for all $x \in X$ which means that ψ is a fixed point of the operator $\mathcal{T}$.
Next, we will prove the uniqueness of a fixed point. To do it, we suppose that (2.4) holds and there exists an other fixed point $\chi \in Y^X$ of $\mathcal{T}$ satisfying

$$\|\varphi(x) - \chi(x), y\| \le \sup_{n\in\mathbb{N}_0} \Lambda^n \varepsilon(x, y) = \sigma(x, y), \quad x \in X, y \in Y_0.$$

Then, for each $x \in X$ and $y \in Y_0$, we have

$$\|\psi(x) - \chi(x), y\| \le \max \{\|\psi(x) - \varphi(x), y\|, \|\varphi(x) - \chi(x), y\|\}.$$

By a similar proof of (2.7), we have ,for any $k \in \mathbb{N}$,

$$\begin{aligned}\|\psi(x) - \chi(x), y\| =& \|\mathcal{T}^k\psi(x) - \mathcal{T}^k\chi(x), y\| \\ &\le \Lambda^k \left(\|\psi(x) - \chi(x), y\|\right) \\ &\le \Lambda^k(\sigma(x, y)) \\ &\le \sup_{n\in\mathbb{N}_0} \Lambda^{n+k}(\varepsilon(x, y)).\end{aligned}$$

Letting $n \to \infty$ in the previous inequality and using (2.2), we obtain that $\psi = \chi$. □

3 Hyperstability results in non-Archimedean 2-Banach space

Taking $Y_0 = Y$ and $g_i : Y \to Y$ as identities mapping for all $i \in \{1, 2, ..., k\}$. In the following theorem, we use the fixed point Theorem 2.1 as a basic tool to investigate the hyperstability of the Cauchy-Jensen functional equation (1.3) in a non-Archimedean 2-Banach space.
In the remaining part of the paper, we use X as a non empty set, $(Y, \|., .\|)$ a non-Archimedean 2-Banach space, and X' a non empty subset of X.

Theorem 3.1. *Let $h_1, h_2 : X' \times Y \to \mathbb{R}_+$ be two functions such that*

$$\mathcal{U} := \Big\{n \in \mathbb{N} : \alpha_n = \max\{\lambda_1(n + 1)\lambda_2(n + 1)\ ,\ \lambda_1(2n + 1)\lambda_2(2n + 1)\} < 1\Big\},$$

where

$$\lambda_i(n) = \inf\{t \in \mathbb{R}_+ : h_i(nx, z) \le t h_i(x, z), \ \ x \in X', z \in Y\}$$

for all $n \in \mathbb{N}$, where $i = 1, 2$ such that

$$\lim_{n\to\infty} \lambda_1(n)\lambda_2(n) = 0.$$

Suppose that $f : X' \to Y$ satisfies the inequality

$$\|f(x + y) + f(x - y) - 2f(x), z\| \le h_1(x, z)h_2(y, z), \tag{3.1}$$

for all $x, y \in X'$ and $z \in Y$ such that $x + y, x - y \in X'$. Then f is a Cauchy-Jensen on X'.

Proof. Replacing x by $(m + 1)x$ and y by mx where $x, y \in X'$ and $m \in \mathbb{N}$ in the inequality (3.1), we get

$$\|2f((m+1)x) - f((2m+1)x) - f(x), z\| \le h_1((m+1)x, z)h_2(mx, z), \quad x \in X', z \in Y. \tag{3.2}$$

For each $m \in \mathbb{N}$, we define the operator $\mathcal{T}_m : Y^{X'} \to Y^{X'}$ and the function $\varepsilon_m : X' \times Y \to \mathbb{R}_+$ by

$$\mathcal{T}_m\xi(x) := 2\xi((m + 1)x) - \xi((2m + 1)x), \ \ \xi \in Y^{X'}, \ x \in X', \ z \in Y, \ m \in \mathbb{N},$$

$$\varepsilon_m(x,z) := h_1((m+1)x,z)h_2(mx,z), \ \ x\in X',\ z\in Y,\ m\in\mathbb{N}.$$

For every $x \in X'$, $z\in Y$ and $m\in\mathbb{N}$, the inequality (3.2) becomes

$$\|\mathcal{T}_m f(x)-f(x),z\| \le \varepsilon_m(x,z) \quad x\in X',\ z\in Y.$$

Furthermore, for every $\xi,\mu\in Y^{X'}$, $x\in X'$, $z\in Y$ and $m\in\mathbb{N}$, we have

$$\begin{aligned}\|\mathcal{T}_m\xi(x)-\mathcal{T}_m\mu(x),z\| &= \|2\xi((m+1)x)-\xi((2m+1)x)-2\mu((m+1)x)+\mu((2m+1)x),z\|\\ &\le \max\left\{2\|\xi((m+1)x)-\mu((m+1)x),z\|,\ \|\xi((2m+1)x)-\mu((2m+1)x),z\|\right\}\\ &\le \max\left\{\|\xi((m+1)x)-\mu((m+1)x),z\|,\ \|\xi((2m+1)x)-\mu((2m+1)x),z\|\right\}.\end{aligned}$$

It brings us to define the operator $\Lambda_m : \mathbb{R}_+^{X'\times Y}\to\mathbb{R}_+^{X'\times Y}$ by

$$\Lambda_m\delta(x,z) := \max\left\{\delta((m+1)x,z),\delta((2m+1)x,z)\right\},\ \ \delta\in\mathbb{R}_+^{X'\times Y},\ x\in X',z\in Y.$$

Therefore, for each $m\in\mathbb{N}$, the operator $\Lambda := \Lambda_m$ has the form described in (**H3**) with $k=2$, $f_1(x)=(m+1)x$, $f_2(x)=(2m+1)x$, $L_1(x,z)=L_2(x,z)=1$, $g_i = Id_Y, i=1,2$ for all $x\in X'$ and $z\in Y$. Observe that

$$\varepsilon_m(x,z)\le \lambda_1(m+1)\lambda_2(m)h_1(x,z)h_2(x,z), \tag{3.3}$$

for all $x\in X'$ and $z\in Y$. By induction, we will show that for each $n\in\mathbb{N}_0$, we have

$$\Lambda_m^n\varepsilon_m(x,z)\le\lambda_1(m+1)\lambda_2(m)\alpha_m^n h_1(x,z)h_2(x,z),\ \ x\in X',z\in Y. \tag{3.4}$$

for all $m\in\mathcal{U}$. For $n=0$, it's obvious to see that (3.4) is exactly (3.3). We fix $k\in\mathbb{N}$ and assume that (3.4) holds for $n=k$. Then, using the non-decreasing of Λ_m, we have

$$\begin{aligned}&\Lambda_m^{k+1}\varepsilon_m(x,z) = \Lambda_m(\Lambda_m^k\varepsilon_m(x,z))\\ &=\max\{\Lambda_m^k\varepsilon_m((m+1)x,z),\Lambda_m^k\varepsilon_m((2m+1)x,z)\}\\ &=\lambda_1(m+1)\lambda_2(m)\alpha_m^k\max\{h_1((m+1)x,z)h_2((m+1)x,z),h_1((2m+1)x,z)h_2((2m+1)x,z)\}\\ &\le\lambda_1(m+1)\lambda_2(m)\alpha_m^k h_1(x,z)h_2(x,z)\max\{\lambda_1(m+1)\lambda_2(m+1),\lambda_1(2m+1)\lambda_2(2m+1)\}\\ &=\lambda_1(m+1)\lambda_2(m)\alpha_m^{k+1}h_1(x,z)h_2(x,z),\end{aligned}$$

for all $x\in X'$ and $z\in Y$. Letting $n\to\infty$ in (3.4), we get

$$\lim_{n\to\infty}\Lambda_m^n\varepsilon_m(x,z)=0$$

for all $x\in X'$,$z\in Y$ and all $m\in\mathcal{U}$. Then, according to Theorem 2.1, there exists , for each $m\in\mathcal{U}$, a fixed point J_m of $\mathcal{T}_m$ such that

$$\|f(x)-J_m(x),z\|\le\sup_{n\in\mathbb{N}_0}\Lambda_m^n\varepsilon_m(x,z), \tag{3.5}$$

for all $x\in X'$ and all $z\in Y$ and

$$\lim_{n\to\infty}\mathcal{T}_m^n f(x)=J_m(x),\ \ x\in X'. \tag{3.6}$$

Next, we will show, by induction, that for each $n\in\mathbb{N}_0$

$$\|\mathcal{T}_m^n f(x+y)+\mathcal{T}_m^n f(x-y)-2\mathcal{T}_m^n f(x),z\|\le\alpha_m^n h_1(x,z)h_2(y,z), \tag{3.7}$$

for all $x,y,x-y,x+y\in X'$, $z\in Y$ and all $m\in\mathcal{U}$.
Since the case $n=0$ is just (3.1), we fix $k\in\mathbb{N}$ and suppose that (3.7) holds for $n=k$. Then, for

all $x, y \in X'$ such that $x-y, x+y \in X'$ and $z \in Y$ we have

$$\begin{aligned}
&\|\mathcal{T}_m^{k+1} f(x+y) + \mathcal{T}_m^{k+1} f(x-y) - 2\mathcal{T}_m^{k+1} f(x), z\| \\
&= \|\mathcal{T}_m\left(\mathcal{T}_m^k f(x+y)\right) + \mathcal{T}_m\left(\mathcal{T}_m^k f(x-y)\right) - 2\mathcal{T}_m\left(\mathcal{T}_m^k f(x)\right), z\| \\
&= \|2\mathcal{T}_m^k f((m+1)(x+y)) - \mathcal{T}_m^k f((2m+1)(x+y)) + 2\mathcal{T}_m^k f((m+1)(x-y)) \\
&- \mathcal{T}_m^k f((2m+1)(x-y)) - 4\mathcal{T}_m^k f((m+1)x) + 2\mathcal{T}_m^k f((2m+1)x), z\| \\
&\leq \max\{2\|\mathcal{T}_m^k f((m+1)(x+y)) + \mathcal{T}_m^k f((m+1)(x-y)) - 2\mathcal{T}_m^k f((m+1)x), z\|; \\
&\|\mathcal{T}_m^k f((2m+1)(x+y)) + \mathcal{T}_m^k f((2m+1)(x-y)) - 2\mathcal{T}_m^k f((2m+1)x), z\|\} \\
&\leq \max\{\|\mathcal{T}_m^k f((m+1)(x+y)) + \mathcal{T}_m^k f((m+1)(x-y)) - 2\mathcal{T}_m^k f((m+1)x), z\|; \\
&\|\mathcal{T}_m^k f((2m+1)(x+y)) + \mathcal{T}_m^k f((2m+1)(x-y)) - 2\mathcal{T}_m^k f((2m+1)x), z\|\} \\
&\leq \alpha_m^k \max\{h_1((m+1)x, z)h_2((m+1)y, z); h_1((2m+1)x, z)h_2((2m+1)y, z)\} \\
&\leq \alpha_m^k h_1(x, z)h_2(y, z) \max\{\lambda_1(m+1)\lambda_2(m+1); \lambda_1(2m+1)\lambda_2(2m+1)\} \\
&= \alpha_m^{k+1} h_1(x, z)h_2(y, z).
\end{aligned}$$

Thus, we have shown that (3.7) holds for every $n \in \mathbb{N}_0$. Letting $n \to \infty$ in (3.7), we obtain, for each $m \in \mathcal{U}$, that

$$J_m(x+y) + J_m(x-y) = 2J_m(x), \quad x, x+y, x-y \in X'.$$

In this way, we find a sequence $\{J_m\}_{m\in\mathcal{U}}$ of a Cauchy-Jensen functions on X' such that

$$\|f(x) - J_m(x), z\| \leq \sup_{n\in\mathbb{N}}\{\lambda_1(m+1)\lambda_2(m)\alpha_m^n h_1(x, z)h_2(x, z)\}, \quad x \in X', z \in Y$$

It follows, with $m \to \infty$, that f is Cauchy-Jensen on X'. □

By similar method, we prove the following theorem.

Theorem 3.2. *Let $h : X' \times Y \to \mathbb{R}_+$ be a function such that*

$$\mathcal{U} := \left\{n \in \mathbb{N} : \alpha_n = \max\left\{\lambda(n+1)\ ,\ \lambda(n)\right\} < 1\right\} \neq \phi,$$

where

$$\lambda(n) := \inf\{t \in \mathbb{R}_+ : h(nx, z) \leq th(x, z), \quad x \in X', \ z \in Y\}$$

for all $n \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \lambda(n) = 0.$$

Suppose that $f : X' \to Y$ satisfies the inequality

$$\|f(x+y) + f(x-y) - 2f(x), z\| \leq h(x, z) + h(y, z), \tag{3.8}$$

for all $x, y \in X'$ such that $x+y, x-y \in X'$ and $z \in Y$. Then f is a Cauchy-Jensen on X'.

Proof. Replacing x by $(m+1)x$ and y by mx for $m \in \mathbb{N}$ in (3.8), we get

$$\|2f((m+1)x) - f((2m+1)x) - f(x), z\| \leq h((m+1)x, z) + h(mx, z), \tag{3.9}$$

for all $x \in X'$ and $z \in Y$. For each $m \in \mathcal{U}$, we define the operator $\mathcal{T}_m : Y^{X'} \to Y^{X'}$ by

$$\mathcal{T}_m\xi(x) := 2\xi((m+1)x) - \xi((2m+1)x), \quad \xi \in Y^{X'}, \ x \in X'.$$

Further, putting

$$\varepsilon_m(x, z) = h((m+1)x, z) + h((m+1)x, z) \leq \big(\lambda(m+1) + \lambda(m)\big)h(x, z), \tag{3.10}$$

for all $x \in X'$ and $z \in Y$, then the inequality (3.9) takes the form

$$\|\mathcal{T}_m f(x) - f(x), z\| \le \varepsilon_m(x,z), \quad x \in X',\ z \in Y.$$

For each $m \in \mathcal{U}$, the operator $\Lambda_m : \mathbb{R}_+^{X'\times Y} \to \mathbb{R}_+^{X'\times Y}$ which is defined by

$$\Lambda_m \delta(x) = \max\{\delta((m+1)x,z)\ ,\ \delta(mx,z)\},\ \ \delta \in \mathbb{R}_+^{X'\times Y},\ x \in X',\ z \in Y$$

has the form described in (**H3**) with $k = 2$ and

$$f_1(x) = (m+1)x,\ \ f_2(x) = mx,\ \ L_1(x,z) = L_2(x,z) = 1$$

for all $x \in X'$. Moreover, for every $\xi, \mu \in Y^{X'}$, $x \in X'$ and $z \in Y$, we have

$$\begin{aligned}
&\|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x), z\|\\
&= \|2\xi((m+1)x) - \xi((2m+1)x) - 2\mu((m+1)x) + \mu((2m+1)x), z\|\\
&\le \max\left\{2\|\xi((m+1)x,z) - \mu((m+1)x,z)\|\ ,\ \|\xi((2m+1)x,z) - \mu((2m+1)x,z)\|\right\}\\
&\le \max\left\{\|\xi((m+1)x,z) - \mu((m+1)x,z)\|\ ,\ \|\xi((2m+1)x,z) - \mu((2m+1)x,z)\|\right\}.
\end{aligned}$$

So, (**H2**) is valid. Also, by using mathematical induction on $n \in \mathbb{N}_0$, we will show , for each $x \in X'$ and $z \in Y$, that

$$\Lambda_m^n \varepsilon_m(x,z) \le \Big(\lambda(m+1) + \lambda(m)\Big)\alpha_m^n h(x,z), \tag{3.11}$$

where $\alpha_m := \max\{\lambda(m+1)\ ,\ \lambda(m)\}$ for all $m \in \mathcal{U}$. From (3.10), we obtain that the inequality (3.11) holds for $n = 0$. Next, we will assume that (3.11) holds for $n = k$, where $k \in \mathbb{N}$. Then we have

$$\begin{aligned}
\Lambda_m^{k+1}\varepsilon_m(x,z) =&\Lambda_m\left(\Lambda_m^k\varepsilon_m(x,z)\right) = \max\left\{\Lambda_m^k\varepsilon_m\big((m+1)x,z\big)\ ,\ \Lambda_m^k\varepsilon_m(mx,z)\right\}\\
&\le \big(\lambda(m+1) + \lambda(m)\big)\alpha_m^k\ \max\left\{\varphi\big((m+1)x,z\big)\ ,\ \varphi(mx,z)\right\}\\
&\le \Big(\lambda(m+1) + \lambda(m)\Big)\alpha_m^{k+1}\varphi(x,z),\ \ x \in X_0',\ z \in Y.
\end{aligned}$$

This shows that (3.11) holds for $n = k+1$. Now we can conclude that the inequality (3.11) holds for all $n \in \mathbb{N}_0$. From (3.11), we obtain

$$\lim_{n\to\infty} \Lambda^n \varepsilon_m(x,z) = 0,$$

for all $x \in X'$, $z \in Y$ and all $m \in \mathcal{U}$. Hence, according to Theorem 2.1, there exists, for each $m \in \mathcal{U}$, a unique solution $J_m : X' \to Y$ of the equation

$$J_m(x) = J_m((m+1)x) - 2J_m((2m+1)x), \quad x \in X', \tag{3.12}$$

such that

$$\|f(x) - J_m(x), z\| \le \sup_{n\in\mathbb{N}_0}\left\{\Big(\lambda(m+1) + \lambda(m)\Big)\alpha_m^n h(x,z)\right\},\ \ x \in X',\ z \in Y. \tag{3.13}$$

Moreover,

$$J_m(x) := \lim_{n\to\infty}\left(\mathcal{T}_m^n f\right)(x)$$

for all $x \in X'$. Now, we show that

$$\|\mathcal{T}_m^n f(x+y) + \mathcal{T}_m^n f(x-y) - 2\mathcal{T}_m^n f(x), z\| \le \alpha_m^n\ \big(h(x,z) + h(y,z)\big) \tag{3.14}$$

for every $z \in Y$, $x, y \in X'$ such that $x+y$, $x-y \in X'$ and $n \in \mathbb{N}_0$. Since the case $n = 0$ is just (3.8), take $k \in \mathbb{N}$ and assume that (3.14) holds for $n = k$, where $k \in \mathbb{N}$ and every $x, y \in X'$

such that $x+y, x-y \in X'$. Then

$$
\begin{aligned}
&\left\|\mathcal{T}_m^{k+1} f(x+y)+\mathcal{T}_m^{k+1} f(x-y)-2\mathcal{T}_m^{k+1} f(x), z\right\| \\
&=\left\|\mathcal{T}_m\Big(\mathcal{T}_m^k f(x+y)\Big)+\mathcal{T}_m\Big(\mathcal{T}_m^k f(x-y)\Big)-2\mathcal{T}_m\Big(\mathcal{T}_m^k f(x)\Big), z\right\| \\
&=\|2\mathcal{T}_m^k f\big((m+1)(x+y)\big)-\mathcal{T}_m^k f\big((2m+1)(x+y)\big)+2\mathcal{T}_m^k f\big((m+1)(x-y)\big) \\
&-\mathcal{T}_m^k f\big((2m+1)(x-y)\big)-4\mathcal{T}_m^k f\big((m+1)x\big)+2\mathcal{T}_m^k f\big((2m+1)x\big), z \\
&\leq \max\{2\left\|\mathcal{T}_m^k f\big((m+1)(x+y)\big)+\mathcal{T}_m^k f\big((m+1)(x-y)\big)-2\mathcal{T}_m^k f\big((m+1)x\big), z\right\|, \\
&\left\|\mathcal{T}_m^k f\big((2m+1)(x+y)\big)+\mathcal{T}_m^k f\big((2m+1)(x-y)\big)-2\mathcal{T}_m^k f\big((2m+1)x\big), z\right\| \\
&\leq \max\{\left\|\mathcal{T}_m^k f\big((m+1)(x+y)\big)+\mathcal{T}_m^k f\big((m+1)(x-y)\big)-2\mathcal{T}_m^k f\big((m+1)x\big), z\right\|, \\
&\left\|\mathcal{T}_m^k f\big((2m+1)(x+y)\big)+\mathcal{T}_m^k f\big((2m+1)(x-y)\big)-2\mathcal{T}_m^k f\big((2m+1)x\big), z\right\| \\
&\leq \max\Big\{\alpha_m^k\Big(h\big((m+1)x, z\big)+h\big((m+1)y, z\big)\Big), \alpha_m^k\Big(h\big((2m+1)x, z\big)+h\big((2m+1)y, z\big)\Big)\Big\} \\
&\leq \alpha_m^k \max\Big\{\lambda(m+1)\,,\ \lambda(2m+1)\Big\}\Big(h(x, z)+h(y, z)\Big) \\
&=\alpha_m^{k+1}\Big(h(x, z)+h(y, z)\Big).
\end{aligned}
$$

Thus, by induction we have shown that (3.14) holds for every $n \in \mathbb{N}_0$. Letting $n \to \infty$ in (3.14), we obtain that

$$
J_m(x+y)+J_m(x-y)=2J_m(x),
$$

for all $x, y \in X'$ such that $x+y, x-y \in X'$. In this way, we obtain a sequence $\{J_m\}_{m\in\mathcal{U}}$ of Cauchy-Jensen functions on X' such that

$$
\|f(x)-J_m(x), z\| \leq \sup_{n\in\mathbb{N}_0}\Big\{\Big(\lambda(m+1)+\lambda(2m+1)\Big)\alpha_m^n h(x, z)\Big\}, \quad x \in X',\ z \in Y.
$$

It implies that

$$
\|f(x)-J_m(x), z\| \leq \Big(\lambda(m+1)+\lambda(2m+1)\Big)\alpha_m^n h(x, z), \quad x \in X',\ z \in Y.
$$

It follows, with $m \to \infty$, that f is a Cauchy-Jensen on X'.

4 Consequences

In this section, we assume that $X'=X_0=X\setminus\{0\}$. According the Theorem 3.1 and Theorem 3.2, we derive the following two corollaries.

Corollary 4.1. *Let $(X, \|.\|_X)$ be a normed space and $(Y, \|.,.\|)$ be a non-Archimedean 2-Banach space, s be a fixed element in Y and let $c \geq 0$, $r \geq 0$,$p, q \in \mathbb{R}$ such that $p+q<0$. Suppose that $f : X' \to Y$ is a function satisfying the inequality*

$$
\|f(x+y)+f(x-y)-2f(x), z\| \leq c\,\|x\|_X^p\ \|y\|_X^q\ \|z, s\|^r, \tag{4.1}
$$

for all $x, y \in X'$ such that $x+y, x-y \in X'$ and $z \in Y$. Then f is a Cauchy-Jensen on X'.

Proof. The proof follows from Theorem 3.1 by taking $h_1, h_2 : X' \times Y \to \mathbb{R}_+$ as follows:

$$
h_1(x, z)=c_1\|x\|_X^p\|z, s\|^{r_1}
$$

and

$$
h_2(y, z)=c_2\|y\|_X^q\|z, s\|^{r_2}
$$

for all $x, y \in X'$ and all $z \in Y$, where $c_1, c_2 \in \mathbb{R}_+$, $r_1, r_2 \in \mathbb{R}$ and $p, q \in \mathbb{R}$ such that , $r_1 + r_2 \geq 0$ and $p + q < 0$.
For each $m \in \mathbb{N}$, we define $\lambda_1(m)$ as in Theorem 3.1

$$\begin{aligned}\lambda_1(m) &= \inf\{t \in \mathbb{R}_+ : h_1(mx, z) \leq t\, h_1(x, z)\}, \quad x \in X',\ z, s \in Y\\ &= \inf\{t \in \mathbb{R}_+ : c_1 m^p \|x\|_X^p \|z, s\|^{r_1} \leq t c_1 \|x\|_X^p \|z, s\|^{r_1}\}, \quad x \in X',\ z, s \in Y\\ &= m^p.\end{aligned}$$

Also, for $m \in \mathbb{N}$, we have $\lambda_2(m) = m^q$. Therefore,

$$\lim_{m\to\infty} \lambda_1(m+1)\lambda_2(m) = \lim_{m\to\infty} (m+1)^p (m)^q = \lim_{m\to\infty} (m+1)^{p+q} = 0.$$

Furthermore, we get

$$\begin{aligned}\alpha_m &= \max\left\{\lambda_1(m+1)\lambda_2(m+1)\ ,\ \lambda_1(2m+1)\lambda_2(2m+1)\right\}\\ &= \max\left\{(m+1)^{p+q}\ ,\ (2m+1)^{p+q}\right\}\\ &= (m+1)^{p+q}.\end{aligned}$$

Then $\mathcal{U}$ is a non empty set. According to Theorem 3.1, f is a Cauchy-Jensen on X'. □

By a similar method, we can prove the following corollary as a particular case of Theorem 3.2 where $h(x, z) = c\, \|x\|_X^p \|z, s\|^r$ with $c \geq 0$, $p < 0$, and $r \geq 0$.

Corollary 4.2. *Let $(X, \|.\|_X)$ be a normed space and $(Y, \|., .\|)$ be a non-Archimedeen 2-Banach space, s be a fixed element in Y and let $c \geq 0$, $p < 0$, $r \geq 0$ and $f : X' \to Y$ satisfy*

$$\|f(x+y) + f(x-y) - 2f(x), z\| \leq c\left(\|x\|_X^p + \|y\|_X^p\right) \|z, s\|^r,$$

for all $x, y \in X'$ such that $x + y, x - y \in X'$ and $z \in Y$. Then f is a Cauchy-Jensen on X'.

In the following two corollaries, we discuss the hyperstability of the inhomogeneous Cauchy-Jensen functional equation.

Corollary 4.3. *Let $(X, \|.\|_X)$ be a normed space and $(Y, \|., .\|)$ be a non-Archimedean 2-Banach space, s be a fixed element in Y and let $G : X'^2 \to Y$ such that $G(x_0, y_0) \neq 0$ for some $x_0, y_0 \in X'$ and*

$$\left\|G(x, y), z\right\| \leq c\, \|x\|_X^p\ \|y\|_X^q\ \|z, s\|^r, \tag{4.2}$$

for all $x, y \in X'$ such that $x+y, x-y \in X'$ and $z \in Y$ where $c \geq 0$, $p, q \in \mathbb{R}$ such that $p+q < 0$. Then the functional equation

$$g(x+y) + g(x-y) = 2g(x) + G(x, y), \tag{4.3}$$

for all $x, y \in X'$ with $x + y, x - y \in X'$, has no solution in the class of functions $g : X \to Y$.

Proof. Suppose that $f : X \to Y$ is a solution to (4.3). Then

$$\begin{aligned}\left\|f(x+y) + f(x-y) - 2f(x), z\right\| &= \left\|2f(x) + G(x, y) - 2f(x), z\right\|\\ &= \left\|G(x, y), z\right\|\\ &\leq c\, \|x\|_X^p\ \|y\|_X^q\ \|z, s\|^r, \quad x, y, \in X', \quad z \in Y.\end{aligned}$$

Consequently, by Theorem 3.1, f is a Cauchy-Jensen on X', whence

$$G(x_0, y_0) = f(x_0 + y_0) + f(x_0 - y_0) - 2f(x_0) = 0,$$

which is a contradiction. □

Corollary 4.4. *Let $(X, \|.\|_X)$ be a normed space and $(Y, \|.,.\|)$ be a non-Archimedean 2-Banach space, s be a fixed element in Y and $p, q \in \mathbb{R}$ such that $p+q<0$. Assume that $G : X'^2 \to Y$ and $f : X' \to Y$ satisfy the inequality*

$$\left\| f(x+y) + f(x-y) - 2f(x) - G(x,y), z \right\| \leq c\, \|x\|_X^p \, \|y\|_X^q \, \|z,s\|^r, \tag{4.4}$$

for all $x, y \in X'$ with $x+y, x-y \in X'$ and $z \in Y$. If the functional equation

$$f(x+y) + f(x-y) = 2f(x) + G(x,y), \quad x, y \in X', \tag{4.5}$$

has a solution $f_0 : X' \to Y$, then f is a solution of functional equation 4.5 on X'.

Proof. From (4.4), we get that the function $K : X' \to Y$ defined by $K := f - f_0$ satisfies (4.1). Consequently, Corollary 4.1 implies that K is a Cauchy-Jensen on X'. Therefore,

$$\begin{aligned} f(x+y) + f(x-y) - 2f(x) - G(x,y) &= K(x+y) + f_0(x+y) + K(x-y) + f_0(x-y) \\ &\quad - 2K(x) - 2f_0(x) - G(x,y) \\ &= 0. \end{aligned}$$

for all $x, y \in X'$ with $x+y, x-y \in X'$. Which means f is a solution of functional equation 4.5 on X'. □

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Author information

R. El Ghali and S. Kabbaj, Department of Mathematics, Faculty of Sciences, University of Ibn Tofail, BP 133 Kenitra, Morocco.
E-mail: rachid2810@gmail.com; samkabbaj@yahoo.fr

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