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A variant of Birkhoff-Kakutani theorem on topological polygroups

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Abstract Results towards reaching Birkhoff-Kakutani theorem in topological polygroups are studied and metametric on first countable T_1 topological polygroup is obtained in a constructive way which induces stronger topology than the underlying one.

1 A brief history of Polygroups

Polygroups [1, 13, 15] appeared in the literature as a special subclass of Marty's hypergroups [3]. Only a few articles [2, 4, 6, 7, 8, 9, 10, 12, 14] have addressed the concept of topological hyperstructures so far. Heidari et al. [2] initiated the notion of topological polygroup as a generalization of topological group and later Shadkami et al. [9, 10], Singha et al. [8], Jamalzadeh [4] extended this field of study. In this setting, we define prenorm on a polygroup and study different properties associated with it. Later, we use this notion on a topological polygroup to obtain a metametric on it as in [11] and show that the topology induced by the metametric is stronger than the underlying topology on the polygroup.

Let's warm up with some basic definitions and results which will be treated as ready references in the sequel. Let P be a nonempty set and $\mathcal{P}^*(P)$ be the collection of all nonempty subsets of P. A function $\circ : P \times P \to \mathcal{P}^*(P)$ is called a *hyperoperation* on P and the ordered pair $(P, \circ)$ is called a *hypergroupoid*. The hyperoperation $\circ$ is extended to subsets of P in a usual way, i.e., for nonempty subsets H, K of P, $H \circ K = \bigcup\{h \circ k : h \in H,\ k \in K\}$. For $x \in P$, the notations $x \circ H$, $H \circ x$ are used for $\{x\} \circ H$ and $H \circ \{x\}$, respectively. The hypergroupoid $(P, \circ)$ is called (i) a *semihypergroup* if for all $x, y, z \in P$, $x \circ (y \circ z) = (x \circ y) \circ z$; (ii) a *quasihypergroup* if for all $x \in P$, $x \circ P = P \circ x = P$; (iii) a *hypergroup* if it is both semihypergroup and quasihypergroup. A hypergroup P is called a *polygroup* if the following conditions hold:

(1) for all $x \in P$, there exists an element e (called the *scalar identity*) in P such that $e \circ x = x \circ e = \{x\}$;

(2) for each $x \in P$, there exists a unique element $x^{-1} \in P$ (called the inverse of x) such that $e \in x \circ x^{-1} \cap x^{-1} \circ x$;

(3) $x \in y \circ z$ implies $y \in x \circ z^{-1}$ and $z \in y^{-1} \circ x$.

Denote the polygroup by $(P, \circ, e, ^{-1})$.
For a nonempty subset A of P, $A^{-1} := \{x^{-1} : x \in A\}$ and A is called *symmetric* if $A^{-1} = A$.
A nonempty subset K of a polygroup P is called a *subpolygroup* of P if (1) $a, b \in K$ implies $a \circ b \subseteq K$ and (2) $a \in K$ implies $a^{-1} \in K$.
A nonempty subset C of a hypergroup $(P, \circ)$ is said to be a *complete part* of P if for any nonzero natural number n and for all $x_1, x_2, \cdots, x_n$ of P, the following implication holds:

$$C \cap (x_1 \circ x_2 \circ x_3 \circ \cdots \circ x_n) \neq \phi \Rightarrow x_1 \circ x_2 \circ x_3 \circ \cdots \circ x_n \subseteq C.$$

Lemma 1.1. ([14]) *For a topological space (P, τ), the collection $\mathcal{B} = \{S_V : V \in \tau\}$, where $S_V = \{U \in \mathcal{P}^*(P) : U \subseteq V\}$ forms a base for some topology τ^* on $\mathcal{P}^*(P)$.*

Definition 1.2. ([2]) Let $(P, \circ, e, ^{-1})$ be a polygroup and (P, τ) be a topological space. Then, the system $(P, \circ, e, ^{-1}, \tau)$ is said to be a *topological polygroup* if the mappings $\circ : P \times P \to \mathcal{P}^*(P)$

and $^{-1} : P \to P$ are continuous while considering the product topology on $P \times P$ and the topology τ^* on $\mathcal{P}^*(P)$.

Lemma 1.3. ([2]) *In a topological polygroup $(P, \circ, e, ^{-1}, \tau)$, the following results hold:*

(i) the hyperoperation $\circ : P \times P \to \mathcal{P}^*(P)$ is continuous if and only if for $x, y \in P$ and $W \in \tau$ such that $x \circ y \subseteq W$, then there exist $U, V \in \tau$ such that $x \in U$, $y \in V$ and $U \circ V \subseteq W$;

(ii) if U is an open subset as well as a complete part of P, then for $a \in P$, $a \circ U$ and $U \circ a$ are open subsets of P;

(iii) if $W \in \tau$ and $x \in W$, then there exists $V \in \tau$ containing e such that $x \circ V \subseteq W$, $V \circ x \subseteq W$;

(iv) if $W \in \tau$ and contains e, then there exists $V \in \tau$ containing e such that $V \circ V \subseteq W$;

(v) if $W \in \tau$ and contains e, then there exists $V \in \tau$ containing e such that $V^{-1} \subseteq W$;

(vi) for every neighborhood U of e, there exists a symmetric neighborhood V of e such that $V \subseteq U$.

Metametric, introduced by Väisälä [5] is one of the generalizations of the well known concept of metric in literature.
A *metametric* on a nonempty set P is a function $m : P \times P \to [0, \infty)$ satisfying the following properties:

(1) for $x, y \in P$, $m(x, y) = m(y, x)$;

(2) for $x, y, z \in P$, $m(x, y) \leq m(x, z) + m(z, y)$;

(2) if $m(x, y) = 0$ for $x, y \in P$, then $x = y$.

The ordered pair (P, m) is called a metametric space. For $a \in P$ and $r > 0$ write $B_m(a, r) = \{x \in P : m(x, a) < r\}$. The metametric m on P induces a Hausdorff topology τ_m in the following way:

A subset U of P is open in P if for each $a \in U$ there exists $r > 0$ such that $B_m(a, r) \subseteq U$. Throughout this paper, all neighborhoods are assumed to be open.

2 Prenorms on a polygroup

Here, we obtain a metametric by the help of prenorm on a first countable T_1 topological polygroup and show that the topology induced by the metametric on the polygroup is stronger than the underlying topology. So, let's define prenorm on a polygroup first.

Definition 2.1. A *Prenorm* N on a polygroup P is a real-valued function with the following properties

(PN1) $N(e) = 0$,

(PN2) for $x \in P$, $N(x) = N(x^{-1})$,

(PN3) for $x, y \in P$, the set $N(x \circ y)$ is bounded above in $\mathbb{R}$ and $\sup N(x \circ y) \leq N(x) + N(y)$.

The following results are immediate after the above definition.

Proposition 2.2. *The following results hold for a prenorm N on a polygroup P:*

(i) *for $x \in P$, $N(x) \geq 0$,*

(ii) *for $x, y \in P$, $|N(x) - N(y)| \leq \sup N(x^{-1} \circ y)$,*

Proof. (i) For $x \in P$, $e \in x \circ x^{-1}$. Then, $0 = N(e) \leq \sup N(x \circ x^{-1}) \leq N(x) + N(x^{-1}) = 2N(x)$. i.e., $N(x) \geq 0$ for $x \in P$.

(ii) For $x, y \in P$, $y \in x \circ (x^{-1} \circ y)$. Then, $y \in x \circ t$ for some $t \in x^{-1} \circ y$ and $N(y) \leq \sup N(x \circ t) \leq N(x) + N(t) \leq N(x) + \sup N(x^{-1} \circ y)$. i.e., $N(y) - N(x) \leq \sup N(x^{-1} \circ y)$. Again, $x^{-1} \in (x^{-1} \circ y) \circ y^{-1}$, then $x^{-1} \in s \circ y^{-1}$ for some $s \in x^{-1} \circ y$. So, $N(x) = N(x^{-1}) \leq \sup N(s \circ y^{-1}) \leq N(s) + N(y^{-1}) \leq \sup N(x^{-1} \circ y) + N(y)$. i.e., $N(x) - N(y) \leq \sup N(x^{-1} \circ y)$. □

Proposition 2.3. *For a prenorm N on a polygroup P, the set $\mathcal{K} = \{x \in P : N(x) = 0\}$ is a subpolygroup of P.*

Proof. For $a, b \in \mathcal{K}$, take $t \in a \circ b$. Then, $0 \leq N(t) \leq \sup N(a \circ b) \leq N(a) + N(b) = 0$. Thus, $a \circ b \subseteq \mathcal{K}$. Also, for $a \in \mathcal{K}$, $N(a^{-1}) = N(a) = 0$ implies $a^{-1} \in \mathcal{K}$. □

The following lemma illustrates the construction of a prenorm from a bounded real-valued function on a polygroup.

Lemma 2.4. *Consider a bounded real-valued function f on a polygroup P. Then, the function N_f defined on P by*

$$N_f(x) = \sup\{|f(t) - f(y)| : t \in y \circ x \text{ and } y \in P\}$$

for $x \in P$, is a prenorm on P.

Proof. $N_f(e) = 0$. For $x \in P$,

$$\begin{aligned} N_f(x^{-1}) &= \sup\{|f(t) - f(y)| : t \in y \circ x^{-1}, y \in P\} \\ &= \sup\{|f(y) - f(t)| : y \in t \circ x, t \in P\} = N_f(x). \end{aligned}$$

For $x, y \in P$, $N_f(x \circ y)$ is bounded. For $t \in x \circ y$,

$$\begin{aligned} N_f(t) &= \sup\{|f(r) - f(s)| : r \in s \circ t, s \in P\} \\ &\leq \sup\{|f(r) - f(p)| : r \in s \circ t, p \in s \circ x, s \in P\} \\ &\quad + \sup\{|f(p) - f(s)| : r \in s \circ t, p \in s \circ x, s \in P\} \\ &\leq \sup\{|f(r) - f(p)| : r \in p \circ y, p \in s \circ x, s \in P\} \\ &\quad + \sup\{|f(p) - f(s)| : r \in p \circ y, p \in s \circ x, s \in P\} \\ &\leq \sup\{|f(r) - f(p)| : r \in p \circ y, p \in P\} \\ &\quad + \sup\{|f(s) - f(p)| : s \in p \circ x^{-1}, p \in P\} \\ &= N_f(y) + N_f(x^{-1}) = N_f(y) + N_f(x). \end{aligned}$$

Hence, $\sup N_f(x \circ y) \leq N_f(x) + N_f(y)$. □

Proposition 2.5. *A prenorm N on a topological polygroup P, where the open subsets are complete parts is continuous if and only if for every $\epsilon > 0$ there exists a neighborhood U of e such that $N(x) < \epsilon$, for each $x \in U$.*

Proof. If N is continuous on P, then the condition holds. For the converse take $x \in P$ and $\epsilon > 0$. Consider a neighborhood U of the identity e satisfying the condition as stated in Proposition 2.5. Then, $x \circ U$ is a neighborhood of x. Take $y \in x \circ U$. Then, $y \in x \circ u$ for some $u \in U$ and $u \in x^{-1} \circ y$, which implies $x^{-1} \circ y \subseteq U$. Then, (ii) of Proposition 2.2 implies $|N(x) - N(y)| \leq \sup N(x^{-1} \circ y) \leq \epsilon$. Thus, N is continuous at x. □

Let's define different type of balls for a polygroup.

Definition 2.6. For a prenorm N on a topological polygroup P, define *unit ball* to be the set $B_N = \{x \in P : N(x) < 1\}$ and N-*ball* of radius r to be the set $B_N(r) = \{x \in P : N(x) < r\}$.
If N is continuous, then the above balls are open in P.

To use later in the sequel, let's develop few results.

Lemma 2.7. *Let $\{U_n : n \in \omega = \mathbb{N} \cup \{0\}\}$ be a sequence of symmetric neighborhoods of the identity e in a topological polygroup P such that $U_{n+1} \circ U_{n+1} \subseteq U_n$, for each $n \in \omega$. Then, there exists a prenorm N on P satisfying the following condition:*

(PN4) $\{x \in P : N(x) < 1/2^n\} \subseteq U_n \subseteq \{x \in P : N(x) \leq 2/2^n\}$

Proof. Take $V(1) = U_0$, fixing $n \in \omega$, assume that $V(m/2^n)$ are neighborhoods of e for $m = 1, 2, \cdots, 2^n$. Then, put $V(1/2^{n+1}) = U_{n+1}$, $V(2m/2^{n+1}) = V(m/2^n)$ for $m = 1, 2, \cdots, 2^n$ and $V((2m+1)/2^{n+1}) = V(m/2^n) \circ U_{n+1} = V(m/2^n) \circ V(1/2^{n+1})$ for $m = 1, 2, \cdots, 2^n - 1$. This defines neighborhoods $V(r)$ of the identity e for every positive dyadic rational number $r \leq 1$. Put $V(m/2^n) = P$ for $m > 2^n$. As in general case [11] the following condition holds:

(‡) $V(m/2^n) \circ V(1/2^n) \subseteq V((m+1)/2^n)$, for all integers $m > 0$ and $n \geq 0$.

Define a real-valued function f on P as follows:
For $x \in P$,

$$f(x) = \inf\{r > 0 : x \in V(r)\}.$$

Then f is well-defined, as $x \in V(2) = P$, for $x \in P$. (‡) implies if $0 < r < s$ for positive dyadic rational numbers r and s, then $V(r) \subseteq V(s)$. Here, r and s are positive dyadic rational numbers. Thus the following fact arises:

$$\text{If } f(x) < r, \text{ then } x \in V(r).$$

f is non-negative and bounded above by 2. Then, Lemma 2.4 ensures the function N defined on P by

$$N(x) = \sup_{t \in y \circ x,\, y \in P} |f(t) - f(y)|, \text{ for all } x \in P$$

is a prenorm on P.
To prove the condition (PN4), take $x \in P$ such that $N(x) < 1/2^n$. Then $f(x) = |f(e \circ x) - f(e)| \leq N(x) < 1/2^n$, as $f(e) = 0$. This implies $x \in V(1/2^n) = U_n$. To prove the remaining part take $x \in V(1/2^n)$. For any $y \in P$ there exists a positive integer k such that $(k-1)/2^n \leq f(y) < k/2^n$. Then $y \in V(k/2^n)$. $V(1/2^n)$ is symmetric, so $x^{-1} \in V(1/2^n)$ and $y \circ x$ and $y \circ x^{-1}$ both are contained in $V(k/2^n) \circ V(1/2^n) \subseteq V((k+1)/2^n)$. Therefore,

$$f(t) \leq (k+1)/2^n, \text{ for all } t \in y \circ x$$

and

$$f(t) \leq (k+1)/2^n, \text{ for all } t \in y \circ x^{-1}.$$

So we obtain

$$f(t) - f(y) \leq (k+1)/2^n - (k-1)/2^n \leq 2/2^n, \text{ for all } t \in y \circ x$$

and

$$f(t) - f(y) \leq (k+1)/2^n - (k-1)/2^n \leq 2/2^n, \text{ for all } t \in y \circ x^{-1}.$$

After replacing y, t in the last inequality, it becomes

$$f(t) - f(y) \geq -2/2^n, \text{ for all } t \in y \circ x.$$

Combining we get $|f(t) - f(y)| \leq 2/2^n$, for all $t \in y \circ x$.
Hence, $N(x) = \sup_{t \in y \circ x,\, y \in P} |f(t) - f(y)| \leq 2/2^n$. □

Theorem 2.8. *In a topological polygroup P, for each neighborhood U of e there exists a prenorm N on P such that the unit ball B_N is contained in U. Moreover, if the open subsets of P are complete parts, then N is continuous.*

Proof. P being a topological polygroup, we can construct a sequence of symmetric neighborhoods $\{U_n : n \in \omega\}$ of e in P satisfying all the conditions of Lemma 2.7 such that $U_0 = U$. Then, by Lemma 2.7 there exists a prenorm N on P such that the unit ball B_N of N is contained in $U_0 = U$.

The second assertion follows from Proposition 2.5. □

Let's prove the main result of the paper.

Theorem 2.9. *Every first countable topological polygroup satisfying T_1-axiom of separability possesses a metametric. Moreover, the topology induced by the metametric is stronger than the underlying topology on the polygroup.*

Proof. Let P be a first countable topological polygroup which satisfies T_1 separation axiom. Suppose $\{W_n : n \in \omega\}$ be a countable base of P at e. Then, a sequence of symmetric neighborhoods $\{U_n : n \in \omega\}$ of e can be constructed such that $U_n \subseteq W_n$ and $U_{n+1} \circ U_{n+1} \subseteq U_n$ for $n \in \omega$. So, by Lemma 2.7, there exists a prenorm N on P such that $B_N(1/2^n) \subseteq U_n$, for each $n \in \omega$. For $x, y \in P$, set $m(x, y) = \sup N(x \circ y^{-1})$. Then,

(1) for $x \in P$, $m(x, x) = \sup N(x \circ x^{-1}) \geq 0$ as $e \in x \circ x^{-1}$.

(2) For $x, y \in P$,

$$\begin{aligned} m(x,y) &= \sup\{N(t) : t \in x \circ y^{-1}\} \\ &= \sup\{N(t^{-1}) : t^{-1} \in y \circ x^{-1}\} \\ &= \sup\{N(s) : s \in y \circ x^{-1}\} = m(y,x). \end{aligned}$$

(3) For $x, y, z \in P$, $x \circ y^{-1} \subseteq (x \circ z^{-1}) \circ (z \circ y^{-1})$. Take $t \in x \circ y^{-1}$, then $t \in p \circ q$ for some $p \in x \circ z^{-1}$ and $q \in z \circ y^{-1}$. So, $N(t) \leq \sup N(p \circ q) \leq N(p) + N(q) \leq \sup N(x \circ z^{-1}) + \sup N(z \circ y^{-1}) = m(x, z) + m(z, y)$. This implies $m(x, y) \leq m(x, z) + m(z, y)$.

(4) If $m(x, y) = 0$ for some $x, y \in P$, i.e., $\sup N(x \circ y^{-1}) = 0$, then $x \circ y^{-1} \subseteq B_N(1/2^n) \subseteq U_n$ for each $n \in \omega \Rightarrow x \circ y^{-1} \subseteq \bigcap_{n\in\omega} U_n = \{e\}$, as P is a T_1 space. This implies $x = y$.

To prove the second assertion, let τ, τ_m be the underlying topology and the topology induced by the metametric m on P, respectively. Take a member W of τ and $x \in W$. Then, there exists U_k, for some $k \in \omega$ such that $U_k \circ x \subseteq W$. So, by Lemma 2.7, $B_N(1/2^k) \circ x \subseteq U_k \circ x \subseteq W$. Claim that $B_m(x, 1/2^k) \subseteq B_N(1/2^k) \circ x$. For, take $p \in B_m(x, 1/2^k)$, then $m(x, p) < 1/2^k$, i.e., $\sup N(p \circ x^{-1}) < 1/2^k$, which implies $N(t) < 1/2^k$ for all $t \in p \circ x^{-1}$. Consequently, $p \in t \circ x \subseteq B_N(1/2^k) \circ x$, which implies $W \in \tau_m$. □

Let's conclude the section with following remark.

Remark 2.10. The reverse inequality in Theorem 2.9, i.e., $\tau_m \subseteq \tau$ may not be true in general. For, consider $([0, 1], \tau)$ as a subspace of $\mathbb{R}$ with standard topology. For $x, y \in [0, 1]$, let $\circ$ be the hyperoperation defined as follows

$$x \circ y = \begin{cases} \{\max\{x, y\}\}, & \text{if} \quad x \neq y; \\ [0, x], & \text{if} \quad x = y. \end{cases}$$

Then, $([0, 1], \circ, \tau)$ is a topological polygroup which is first countable and satisfies T_1-axiom of separability. So, there exists a prenorm which induces a metametric m on $[0, 1]$. A basis for the topology τ_m on $[0, 1]$ generated by the metametric m is the collection $\mathfrak{B} = \{B_m(a, r) : a \in [0, 1] \text{ with } m(a, a) = 0, r > 0\} \bigcup \{\{b\} : b \in [0, 1] \text{ with } m(b, b) > 0\}$. Take $x \in [0, 1]$ such that $m(x, x) > 0$. Then, there exists no element containing such x in τ which is contained in $\{x\}$.

References

[1] B. Davvaz, *Polygroup Theory and Related Systems*, World Scientic Publishing Co. Pte. Ltd., Hackensack, NJ., 2013.

[2] D. Heidari, B. Davvaz and S. M. S. Modarres, *Topological polygroups*, Bull. Malays. Math. Sci. Soc., **39** (2016), 707-721.

[3] F. Marty, *Sur une généralization de la notion de groupe*, 8th Congress Math. Scandenaves, Stockholm, Sweden, (1934), 45-49.

[4] J. Jamalzadeh, *Paratopological Polygroups versus Topological Polygroups*, Filomat, **32(8)** (2018), 2755-2761.

[5] J. Väisälä, *Gromov hyperbolic spaces*, Expo. Math., **23(3)** (2005), 187-231.

[6] M. A. Tahan, S. Hoskova-Mayerova and B. Davvaz, *An overview of topological hypergroupoids*, Journal of Intelligent and Fuzzy Systems, **34** (2018), 1907-1916.

[7] M. Singha, K. Das and B. Davvaz, *On topological complete hypergroups*, Filomat, **31(16)** (2017), 5045-5056.

[8] M. Singha, K. Das and B. Davvaz, *Uniformities on OCP-polygroups*, J. Hyperstructures, **7(2)** (2018), 104-123.

[9] M. S. Shadkami, M. R. Ahmadi Zand and B. Davvaz, *Left Big Subsets of Topological Polygroups*, Filomat, **30(12)** (2016), 3139-3147.

[10] M. S. Shadkami, M. R. Ahmadi Zand and B. Davvaz, *The role of complete parts in topological polygroups*, Int. J. Anal. Appl., **11(1)** (2016), 54-60.

[11] M. Tkačenko, *Introduction to topological groups*, Topology and its Applications, **86** (1998), 179-231.

[12] S. Borhani-Nejad and B. Davvaz, *On proximity spaces and topological hyper nearrings*, Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat., **69(2)** (2020), 1418-1427.

[13] S. D. Comer, *Polygroups derived from cogroups*, J. Algebra, **89** (1984), 397-405.

[14] S. Hoskova-Mayerova, *Topological hypergroupoids*, Comput. Math. Appl., **64(9)** (2012), 2845-2849.

[15] S. Ioulidis, *Polygroups et certain de leurs properties*, Bull. Greek Math. Soc., **22** (1981), 95-104.

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