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THE SOLUTION OF THE ILL-POSED CAUCHY PROBLEM FOR MATRIX FACTORIZATIONS OF THE HELMHOLTZ EQUATION IN A MULTIDIMENSIONAL BOUNDED DOMAIN

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Abstract In this paper, the problem of continuation of the solution of the ill-posed Cauchy problem for matrix factorizations of the Helmholtz equation in a multidimensional bounded domain is studied. It is assumed that the solution to the problem exists and is continuously differentiable in a closed domain with exactly given Cauchy data. For this case, an explicit formula for the continuation of the solution is established, as well as a regularization formula for the case when, under the indicated conditions, instead of the Cauchy data, their continuous approximations with a given error in the uniform metric are given. A stability estimate for the solution of the Cauchy problem in the classical sense is obtained.

1 Introduction

The theory of ill-posed problems is a direction of mathematics which has developed intensively in the last two decades and is connected with the most varied applied problems: interpretation of readings of many physical instruments and of geophysical, geological, and astronomical observations, optimization of control, management and planning, synthesis of automatic systems, etc. Development of the theory of ill-posed problems was occasioned by the advent of modern computing technology. Various areas of the theory of ill-posed problems can be included in traditional areas of mathematics such as function theory, functional analysis, differential equations, and linear algebra. The concept of a well-posed problem is connected with investigations by the famous French mathematician Hadamard of various boundary value problems for the equations of mathematical physics. Hadamard expressed the opinion that boundary value problems whose solutions do not satisfy certain continuity conditions are not physically meaningful, and he presented examples of such problems. It was subsequently found that Hadamard's opinion was erroneous. It turned out that many problems of mathematical physics which are ill-posed in the sense of Hadamard and, in particular, problems noted by Hadamard himself have real physical content. It also turned out that ill-posed problems arise in many other areas of mathematics which are connected with applications.

This problem concerns ill-posed problems, i.e., it is unstable. It is known that the Cauchy problem for elliptic equations is unstable relatively small change in the data, i.e., incorrect (example Hadamard, see, for instance [22], p. 39). There is a sizable literature on the subject (see, e.g. [24], [25], [4] [33], [34] and [23]). N.N. Tarkhanov [30] has published a criterion for the solvability of a larger class of boundary value problems for elliptic systems. In unstable problems, the image of the operator is not is closed, therefore, the solvability condition can not be is written in terms of continuous linear functionals. So, in the Cauchy problem for elliptic equations with data on part of the boundary of the domain the solution is usually unique, the problem is solvable for everywhere dense a set of data, but this set is not closed. Consequently, the theory of solvability of such problems is much more difficult and deeper than theory of solvability of Fredholm equations. The first results in this direction appeared only in the mid-1980s in the works of L.A. Aizenberg, A.M. Kytmanov, N.N. Tarkhanov (see, for instance [31]). Boundary value problems, as well as numerical solutions of some problems, are considered in [5], [6], [20]-[21], [38] and [39].

The uniqueness of the solution follows from Holmgren's general theorem (see [3]). The conditional stability of the problem follows from the work of A.N. Tikhonov (see [4]), if we restrict the class of possible solutions to a compactum.

In this paper we construct a family of vector-functions $U_{\sigma(\delta)}(x) = U(x, f_\delta)$ depending on a parameter σ, and prove that under certain conditions and a special choice of the parameter $\sigma = \sigma(\delta)$, at $\delta \to 0$, the family $U_{\sigma(\delta)}(x)$ converges in the usual sense to a solution $U(x)$ at a point $x \in G$.

Following A.N. Tikhonov [4], a family of vector-valued functions $U_{\sigma(\delta)}(x)$ is called a regularized solution of the problem. A regularized solution determines a stable method of approximate solution of the problem.

Formulas that allow finding a solution to an elliptic equation in the case when the Cauchy data are known only on a part of the boundary of the domain are called Carleman type formulas. In [37], Carleman established a formula giving a solution to the Cauchy - Riemann equations in a domain of a special form. Developing his idea, G.M. Goluzin and V.I. Krylov [17] derived a formula for determining the values of analytic functions from data known only on a portion of the boundary, already for arbitrary domains. A multidimensional analogue of Carleman's formula for analytic functions of several variables was constructed in (see [23]). A formula of the Carleman type, in which the fundamental solution of a differential operator with special properties (the Carleman function) is used, was obtained by M.M. Lavrent'ev (see, for instance [24]-[25]). Using this method, Sh. Yarmukhamedov (see, for instance [33]-[36]) constructed the Carleman functions for the Laplace and Helmholtz operators with $n(x, y) \equiv 1$ for spatial domains of a special form, when the part of the boundary of the domain where the data is unknown is a conical surface or a hyper surface $\{x_3 = 0\}$. In [31] an integral formula is proved for systems of equations of elliptic type of the first order, with constant coefficients in a bounded domain. Using the methodology of works [33]-[36], Ikehata [26] was considered the probe method and Carleman functions for the Laplace and Helmholtz equations in the three-dimensional domain. Using exponentially growing solutions, Ikehata [27] was obtained a formula for solving the Helmholtz equation with a variable coefficient for regions in space where the unknown data are located on a section of the hypersurface $\{x \cdot s = t\}$. Carleman type formulas for various elliptic equations and systems were also obtained in works [1]-[2], [16], [7]-[15], [17], [26]-[27], [33]-[36]. In work [16] it was considered the Cauchy problem for the Helmholtz equation in an arbitrary bounded plane domain with Cauchy data, known only on the region boundary. The solvability criterion the Cauchy problem for the Laplace equation in the space $\mathbb{R}^m$ it was considered by Shlapunov in work [1]. In work [18], was be continuation the problem for the Helmholtz equation is investigated and the results of numerical experiments are presented. The construction of the Carleman matrix for elliptic systems was carried out by: Sh. Yarmukhamedov, N.N. Tarkhanov, A.A. Shlapunov, I.E. Niyozov, D.A. Juraev and others (see, for instance [1]-[2], [7]-[15], [19], [28]-[31], [33]-[36]). The system considered in this paper was introduced by N.N. Tarkhanov. For this system, he studied correct boundary value problems and found an analogue of the Cauchy integral formula in a bounded domain (see, for instance [31]).

In many well-posed problems for systems of equations of elliptic type of the first order with constant coefficients that factorize the Helmholtz operator, it is not possible to calculate the values of the vector function on the entire boundary. Therefore, the problem of reconstructing the solution of systems of equations of first order elliptic type with constant coefficients, factorizing the Helmholtz operator (see, for instance [7], [8], [9], [10], [11], [12], [13], [14] and [15]), is one of the topical problems in the theory of differential equations.

For the last decades, interest in classical ill-posed problems of mathematical physics has remained. This direction in the study of the properties of solutions of the Cauchy problem for the Laplace equation was started in [24]-[25], [33]-[36] and subsequently developed in [16], [1]-[2], [7]-[15], [17], [26]-[27], [28]-[31] and [19].

In this paper, we present an explicit formula for the approximate solution of the Cauchy problem for the matrix factorizations of the Helmholtz equation in a multidimensional bounded domain. The odd dimensional case requires special consideration, in contrast to even dimensions in many mathematical problems. The even dimensional case will be further considered in other works of the author. Our formula for an approximate solution also includes the construction of a family of fundamental solutions for the Helmholtz operator a multidimensional bounded domain. This family is parametrized by some entire function $K(w)$, the choice of which depends on the dimension of the space. This motivates a separate study of regularization formulas in odd dimensional spatial domains.

Let $\mathbb{R}^m$, $(m = 2k+1,\, k \geq 1)$ be the $m-$dimensional real Euclidean space,

$$x = (x_1, \dots, x_m) \in \mathbb{R}^m, \quad y = (y_1, \dots, y_m) \in \mathbb{R}^m,$$
$$x' = (x_1, \dots, x_{m-1}) \in \mathbb{R}^{m-1}, \quad y' = (y_1, \dots, y_{m-1}) \in \mathbb{R}^{m-1}.$$

We introduce the following notation:

$$r = |y - x|, \quad \alpha = |y' - x'|, \quad w = i\sqrt{u^2 + \alpha^2} + y_m, \quad u \geq 0,$$

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m} \right)^T, \quad \frac{\partial}{\partial x} = \xi^T, \quad \xi^T = \begin{pmatrix} \xi_1 \\ ... \\ \xi_m \end{pmatrix} - \text{transposed vector } \xi,$$

$$U(x) = (U_1(x), \dots, U_n(x))^T, \quad u^0 = (1, \dots, 1) \in \mathbb{R}^n, \quad n = 2^m, \quad m \geq 3,$$

$$E(z) = \left\| \begin{matrix} z_1 \dots 0 \\ .... \\ 0 \dots z_n \end{matrix} \right\| - \text{diagonal matrix}, \quad z = (z_1, \dots, z_n) \in \mathbb{R}^n.$$

$G \subset \mathbb{R}^m -$be a bounded simply-connected domain with piecewise smooth boundary consisting of the plane T: $y_m = 0$ and of a smooth surface S, lying in the half-space $y_m > 0$, i.e., $\partial G = S \bigcup T$.

Let $D(\xi^T)$, $(n \times n)-$dimensional matrix with elements consisting of a set of linear functions with constant coefficients of the complex plane for which the following condition is satisfied:

$$D^*(\xi^T) D(\xi^T) = E((|\xi|^2 + \lambda^2) u^0),$$

where $D^*(\xi^T)$ is the Hermitian conjugate matrix $D(\xi^T)$, $\lambda -$ is a real number.

We consider in the region G a system of differential equations

$$D\left(\frac{\partial}{\partial x}\right) U(x) = 0, \tag{1.1}$$

where $D\left(\frac{\partial}{\partial x}\right)$ is the matrix of first-order differential operators.

We denote by $A(G)-$the class of vector functions in the domain G continuous on $\overline{G} = G \bigcup \partial G$ and satisfying system (1.1).

2 Construction of the Carleman matrix and the Cauchy problem

Formulation of the problem. Suppose $U(y) \in A(G)$ and

$$U(y)|_S = f(y), \quad y \in S. \tag{2.1}$$

Here, $f(y)$ a given continuous vector-function on S. It is required to restore the vector function $U(y)$ in the domain G, based on it's values $f(y)$ on S.

If $U(y) \in A(G)$, then the following integral formula of Cauchy type is valid

$$U(x) = \int_{\partial G} N(y, x; \lambda) U(y) ds_y, \quad x \in G, \tag{2.2}$$

where

$$N(y, x; \lambda) = \left(E \left(\varphi_m(\lambda r) u^0 \right) D^* \left(\frac{\partial}{\partial x} \right) \right) D(t^T).$$

Here $t = (t_1, \ldots, t_m)-$is the unit exterior normal, drawn at a point y, the surface ∂G, $\varphi_m(\lambda r)-$ is the fundamental solution of the Helmholtz equation in $\mathbb{R}^m$, $(m = 2k+1,\ k \geq 1)$, where $\varphi_m(\lambda r)$ defined by the following formula (see [32]):

$$\varphi_m(\lambda r) = P_m \lambda^{(m-2)/2} \frac{H^{(1)}_{(m-2)/2}(\lambda r)}{r^{(m-2)/2}},$$
$$P_m = \frac{1}{2i(2\pi)^{(m-2)/2}}, \quad m = 2k+1, \quad k \geq 1. \tag{2.3}$$

We denote by $K(w)$ is an entire function taking real values for real w, $(w = u+iv,\ \ u, v-$real numbers) and satisfying the following conditions:

$$K(u) \neq 0, \quad \sup_{v \geq 1} \left| v^p K^{(p)}(w) \right| = B(u, p) < \infty,$$
$$-\infty < u < \infty, \quad p = 0, 1, \ldots, m. \tag{2.4}$$

We define the function $\Phi(y, x; \lambda)$ at $y \neq x$ by the following equality

$$\Phi(y, x; \lambda) = \frac{1}{c_m K(x_m)} \frac{\partial^{k-1}}{\partial s^{k-1}} \int_0^\infty \mathrm{Im} \left[\frac{K(w)}{w - x_m} \right] \frac{\cos(\lambda u)}{\sqrt{u^2 + \alpha^2}} du,$$
$$m = 2k+1, \quad k \geq 1, \tag{2.5}$$

where $c_m = (-1)^k 2^{-k} (2k-1)!(m-2)\pi\omega_m$, $k \geq 1$; ω_m- area of a unit sphere in space $\mathbb{R}^m$.

In the formula (2.5), choosing

$$K(w) = \exp(\sigma w), \quad K(x_m) = \exp(\sigma x_m), \quad \sigma > 0, \tag{2.6}$$

we get

$$\Phi_\sigma(y, x; \lambda) = \frac{e^{-\sigma x_m}}{c_m} \frac{\partial^{k-1}}{\partial s^{k-1}} \int_0^\infty \mathrm{Im} \left[\frac{\exp(\sigma w)}{w - x_m} \right] \frac{\cos(\lambda u)}{\sqrt{u^2 + \alpha^2}} du. \tag{2.7}$$

The formula (2.2) is true if instead $\varphi_m(\lambda r)$ of substituting the function

$$\Phi_\sigma(y, x; \lambda) = \varphi_m(\lambda r) + g_\sigma(y, x; \lambda), \tag{2.8}$$

where $g_\sigma(y, x)-$ is the regular solution of the Helmholtz equation with respect to the variable y, including the point $y = x$.

Then the integral formula has the form:

$$U(x) = \int_{\partial G} N_\sigma(y, x; \lambda) U(y) ds_y, \quad x \in G, \tag{2.9}$$

where

$$N_\sigma(y, x; \lambda) = \left(E \left(\Phi_\sigma(y, x; \lambda) u^0 \right) D^* \left(\frac{\partial}{\partial x} \right) \right) D(t^T).$$

3 Estimation of the stability of the solution to the Cauchy problem

Theorem 3.1. *Let $U(y) \in A(G)$ it satisfy the inequality*

$$|U(y)| \leq 1, \quad y \in T. \tag{3.1}$$

If

$$U_\sigma(x) = \int\limits_S N_\sigma(y, x; \lambda) U(y) ds_y, \quad x \in G, \tag{3.2}$$

then the following estimate is true

$$|U(x) - U_\sigma(x)| \leq C(x) \sigma^{k+1} e^{-\sigma x_m}, \quad \sigma > 1, \quad x \in G. \tag{3.3}$$

Here and below functions bounded on compact subsets of the domain G, we denote by $C(x)$.

Proof. Using the integral formula (2.9) and the equality (3.2), we obtain

$$U(x) = \int\limits_S N_\sigma(y, x; \lambda) U(y) ds_y + \int\limits_T N_\sigma(y, x; \lambda) U(y) ds_y =$$

$$= U_\sigma(x) + \int\limits_T N_\sigma(y, x; \lambda) U(y) ds_y, \quad x \in G.$$

Taking into account the inequality (3.1), we estimate the following

$$\begin{gathered} |U(x) - U_\sigma(x)| \leq \left| \int\limits_T N_\sigma(y, x; \lambda) U(y) ds_y \right| \leq \\ \leq \int\limits_T |N_\sigma(y, x; \lambda)| \, |U(y)| \, ds_y \leq \int\limits_T |N_\sigma(y, x; \lambda)| \, ds_y, \quad x \in G. \end{gathered} \tag{3.4}$$

To do this, we estimate the integrals $\int\limits_T |\Phi_\sigma(y, x; \lambda)| \, ds_y$, $\int\limits_T \left| \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial y_j} \right| ds_y$, $(j = 1, 2, ..., m-1)$ and $\int\limits_T \left| \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial y_m} \right| ds_y$ on the part T of the plane $y_m = 0$.

Separating the imaginary part of (2.7), we obtain

$$\begin{gathered} \Phi_\sigma(y, x; \lambda) = \frac{e^{\sigma(y_m - x_m)}}{c_m} \left[\frac{\partial^{k-1}}{\partial s^{k-1}} \int\limits_0^\infty -\frac{\cos \sigma \sqrt{u^2 + \alpha^2}}{u^2 + r^2} \cos(\lambda u) \, du + \right. \\ \left. + \frac{\partial^{k-1}}{\partial s^{k-1}} \int\limits_0^\infty \frac{(y_m - x_m) \sin \sigma \sqrt{u^2 + \alpha^2}}{u^2 + r^2} \frac{\cos(\lambda u)}{\sqrt{u^2 + \alpha^2}} \, du \right], \quad x_m > 0. \end{gathered} \tag{3.5}$$

Taking into account equality (3.5), we have

$$\int\limits_T |\Phi_\sigma(y, x; \lambda)| \, ds_y \leq C(x) \sigma^{k+1} e^{-\sigma x_m}, \quad \sigma > 1, \quad x \in G, \tag{3.6}$$

To estimate the second integral, we use the equality

$$\begin{gathered} \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial y_j} = \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial s} \frac{\partial s}{\partial y_j} = 2(y_j - x_j) \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial s}, \\ s = \alpha^2, \quad j = 1, 2, \ldots, m-1. \end{gathered} \tag{3.7}$$

Given equality (3.5) and equality (3.7), we obtain

$$\int_T \left| \frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial y_j} \right| ds_y \le C(x)\sigma^{k+1} e^{-\sigma x_m}, \quad \sigma > 1, \quad x \in G. \tag{3.8}$$

Now, we estimate the integral $\int_T \left| \frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial y_m} \right| ds_y$.

Taking into account equality (3.5), we obtain

$$\int_T \left| \frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial y_m} \right| ds_y \le C(x)\sigma^{k+1} e^{-\sigma x_m}, \quad \sigma > 1, \quad x \in G, \tag{3.9}$$

From inequalities (3.6), (3.8) and (3.9), we obtain an estimate (3.3).
Theorem 3.1 is proved. □

Corollary 3.2. *The limiting equality*

$$\lim_{\sigma \to \infty} U_\sigma(x) = U(x),$$

holds uniformly on each compact set from the domain G

Theorem 3.3. *Let* $U(y) \in A(G)$ *satisfy condition (3.1), and on a smooth surface* S *the inequality*

$$|U(y)| \le \delta, \quad 0 < \delta < e^{-\sigma \overline{y}_m}, \tag{3.10}$$

where $\overline{y}_m = \max_{y \in S} y_m$.

Then the following estimate is true

$$|U(x)| \le C(x)\sigma^{k+1} \delta^{\frac{x_m}{\overline{y}_m}}, \quad \sigma > 1, \quad x \in G. \tag{3.11}$$

Proof. Using the integral formula (2.9), we have

$$U(x) = \int_S N_\sigma(y,x;\lambda) U(y) ds_y + \int_T N_\sigma(y,x;\lambda) U(y) ds_y, \quad x \in G.$$

We estimate the following

$$|U(x)| \le \left| \int_S N_\sigma(y,x;\lambda) U(y) ds_y \right| + \left| \int_T N_\sigma(y,x;\lambda) U(y) ds_y \right|, \quad x \in G. \tag{3.12}$$

Given inequality (3.10), we estimate the first integral of inequality (3.12).

$$\left| \int_S N_\sigma(y,x;\lambda) U(y) ds_y \right| \le \int_S |N_\sigma(y,x;\lambda)|\, |U(y)|\, ds_y \le$$
$$\le \delta \int_S |N_\sigma(y,x;\lambda)|\, ds_y, \quad x \in G. \tag{3.13}$$

To do this, we estimate the integrals $\int_S |\Phi_\sigma(y,x;\lambda)|\, ds_y$, $\int_S \left| \frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial y_j} \right| ds_y$, $(j = 1, 2, \dots, m-1)$ and $\int_S \left| \frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial y_m} \right| ds_y$ on a smooth surface S.

Taking into account equality (3.5), we have

$$\int_S |\Phi_\sigma(y,x;\lambda)|\, ds_y \leq C(x)\sigma^{k+1} e^{\sigma(y_m - x_m)}, \quad \sigma > 1, \quad x \in G. \tag{3.14}$$

To estimate the second integral, using equalities (3.5) and (3.7), we obtain

$$\int_S \left| \frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial y_j} \right| ds_y \leq C(x)\sigma^{k+1} e^{\sigma(y_m - x_m)}, \quad \sigma > 1, \quad x \in G, \tag{3.15}$$

To estimate the integral $\int_S \left| \frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial y_m} \right| ds_y$, using equality (3.5), we obtain

$$\int_S \left| \frac{\partial \Phi_\sigma(y,x;\lambda)}{\partial y_m} \right| ds_y \leq C(x)\sigma^{k+1} e^{\sigma(y_m - x_m)}, \quad \sigma > 1, \quad x \in G. \tag{3.16}$$

From (3.14) - (3.16), we obtain

$$\left| \int_S N_\sigma(y,x;\lambda) U(y) ds_y \right| \leq C(x)\sigma^{k+1} \delta\, e^{\sigma(y_m - x_m)}, \quad \sigma > 1, \quad x \in G. \tag{3.17}$$

The following is known

$$\left| \int_T N_\sigma(y,x;\lambda) U(y) ds_y \right| \leq C(x)\sigma^{k+1} e^{-\sigma x_m}, \quad \sigma > 1, \quad x \in G. \tag{3.18}$$

Now taking into account (3.17) - (3.18), we have

$$|U(x)| \leq \frac{C(x)\sigma^{k+1}}{2} (\delta\, e^{\sigma \bar{y}_m} + 1) e^{-\sigma x_m}, \quad \sigma > 1, \quad x \in G. \tag{3.19}$$

Choosing σ from the equality

$$\sigma = \frac{1}{\bar{y}_m} \ln \frac{1}{\delta}, \tag{3.20}$$

we obtain an estimate (3.11).

Theorem 3.3 is proved. □

Let $U(y) \in A(G)$ and instead $U(y)$ on S with its approximation $f_\delta(y)$ are given, respectively, with an error $0 < \delta < e^{-\sigma \bar{y}_m}$,

$$\max_S |U(y) - f_\delta(y)| \leq \delta. \tag{3.21}$$

We put

$$U_{\sigma(\delta)}(x) = \int_S N_\sigma(y,x;\lambda) f_\delta(y) ds_y, \quad x \in G. \tag{3.22}$$

The following is true

Theorem 3.4. *Let $U(y) \in A(G)$ on the part of the plane $y_m = 0$ satisfy condition (3.1). Then the following estimate is true*

$$\left| U(x) - U_{\sigma(\delta)}(x) \right| \leq C(x)\sigma^{k+1} \delta^{\frac{x_m}{\bar{y}_m}}, \quad \sigma > 1, \quad x \in G. \tag{3.23}$$

Proof. From the integral formulas (2.9) and (3.22), we have

$$U(x) - U_{\sigma(\delta)}(x) = \int\limits_{\partial G} N_\sigma(y, x; \lambda) U(y) ds_y - \int\limits_S N_\sigma(y, x; \lambda) f_\delta(y) ds_y =$$

$$= \int\limits_S N_\sigma(y, x; \lambda) U(y) ds_y + \int\limits_T N_\sigma(y, x; \lambda) U(y) ds_y - \int\limits_S N_\sigma(y, x; \lambda) f_\delta(y) ds_y =$$

$$= \int\limits_S N_\sigma(y, x; \lambda) \{U(y) - f_\delta(y)\} \, ds_y + \int\limits_T N_\sigma(y, x; \lambda) U(y) ds_y.$$

Using conditions (3.1) and (3.21), we estimate the following:

$$\left|U(x) - U_{\sigma(\delta)}(x)\right| = \left| \int\limits_S N_\sigma(y, x; \lambda) \{U(y) - f_\delta(y)\} \, ds_y \right| +$$

$$+ \left| \int\limits_T N_\sigma(y, x; \lambda) U(y) ds_y \right| \leq \int\limits_S |N_\sigma(y, x; \lambda)| \, |\{U(y) - f_\delta(y)\}| \, ds_y +$$

$$+ \int\limits_T |N_\sigma(y, x; \lambda)| \, |U(y)| \, ds_y \leq \delta \int\limits_S |N_\sigma(y, x; \lambda)| \, ds_y + \int\limits_T |N_\sigma(y, x; \lambda)| \, ds_y.$$

Now, repeating the proof of Theorems 3.2 and 3.3, we obtain

$$\left|U(x) - U_{\sigma(\delta)}(x)\right| \leq \frac{C(\lambda, x)\sigma^{k+1}}{2} (\delta \, e^{\sigma \bar{y}_m} + 1) e^{-\sigma x_m}.$$

From here, choosing σ from equality (3.20), we obtain an estimate (3.23).
Theorem 3.4 is proved. □

Corollary 3.5. *The limit equality*

$$\lim_{\delta \to 0} U_{\sigma(\delta)}(x) = U(x),$$

holds uniformly on every compact set from the domain G*.*

4 Conclusion

The article obtained the following results:

Using the Carleman function, a formula is obtained for the continuation of the solution of linear elliptic systems of the first order with constant coefficients in a spatial bounded domain $\mathbb{R}^m$, $(m = 2k + 1,\ k \geq 1)$. The resulting formula is an analogue of the classical formula of B. Riemann, W. Voltaire and J. Hadamard, which they constructed to solve the Cauchy problem in the theory of hyperbolic equations. An estimate of the stability of the solution of the Cauchy problem in the classical sense for matrix factorizations of the Helmholtz equation is given. The problem it is considered when, instead of the exact data of the Cauchy problem, their approximations with a given deviation in the uniform metric are given and under the assumption that the solution of the Cauchy problem is bounded on part T, of the boundary of the domain G, an explicit regularization formula is obtained.

Thus, functional $U_{\sigma(\delta)}(x)$ determines the regularization of the solution of problem (1.1) - (2.1).

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