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FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH DELAY INVOLVING HILFER-HADAMARD TYPE

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Abstract This paper concerns the study of a class of fractional functional differential equations (FFDEs) involving Hilfer-Hadamard fractional derivative. The existence, uniqueness, and Ulam-Hyers-Mittag-Leffler (UHLM) stability of solutions to the problem at hand are investigated. Our discussion is based upon a known fixed point theorem of Banach, Picard operator technique, and Gronwall inequality. An example is also given to show the applicability of our results.

1 Introduction

The theme of fractional calculus (FC) deals with derivatives and integrals of any non-integer order whether a real or complex, it has acquired great attention and increasingly of many mathematicians during the past decades, due mainly to its applications in many fields of applied science, and engineering, etc. For recent developments in this area, we refer to monographs of R. Hilfer (2000, [11]), Kilbas, et al. (2006, [14]), and references therein. In the same framework, the original contribution of the stability of functional equations has been developed by Ulam in [26]). Then follow it, Hyers in [13]. Next, the Ulam-Hyers stability was improved by Rassias in [24].

Over the past few years, considerable attention has widely been given to the existence and stability of solutions for various categories of FDEs, and has been studied by distinct types of fractional derivatives such as Hilfer, Hadamard, Hilfer-Hadamard, and ψ-Hilfer, we refer to the papers [2, 4, 5, 6, 12, 15, 16, 23, 27, 25, 28, 32], and the references therein.

Functional differential equations of fractional order have extensively been deliberated by many researchers. Very briefly, interesting topics in this field are the investigation of some qualitative properties of solutions e.g., existence, uniqueness, and stability, through fixed point techniques. Although there are some articles on the stability results of FDEs in the Ulam–Hyers sense (see to name a few [3, 9, 17, 18, 21, 22]). However, there are a few authors have expanded some of the stability results studied in the Ulam–Hyers concept to another concept so-called Ulam–Hyers- Mittag-Leffler, for further details see [7, 8, 19, 20, 30, 31]. For instance, J. Wang and Y. Zhang in [31] established the Ulam-Hyers-Mittag-Leffler (UHML) stability of delayed FDEs of the form

$$\begin{aligned} {}^{C}D^{\mu}_{0+}\sigma(t) &= \aleph(t,\sigma(t),\sigma(\gamma(t))), \quad 0<\mu<1,\ t\in[0,b], \\ \sigma(t) &= \varphi(t), \quad t\in[-r,0],\ r>0, \end{aligned}$$

where ${}^{C}D^{\mu}_{0+}$ is the Caputo-type fractional derivative of order μ, and $\aleph:[0,b]\times R\times R\to R$, $\gamma:[0,b]\to[-r,b])$ are continuous functions with $\gamma(t)\leq t$.

Unfortunately, UHML stability of FDEs with Hadamard and Hilfer-Hadamard derivatives is still not studied until now. Spurred by the aforementioned works, we will concentrate our attention on the more general problem so-called here Hilfer-Hadamard type for fractional functional differential equation (for short, Hilfer-Hadamard FFDE) containing the initial condition and the initial functional:

$$
\begin{aligned}
{}_HD^{\mu,\nu}_{1^+}\sigma(t) &= \aleph(t,\sigma(t),\sigma(\gamma(t))), \quad 0<\mu<1, \quad 0\le\nu\le 1,\ t\in(1,e], && (1.1)\\
{}_HI^{1-\rho}_{1^+}\sigma(1) &= \sigma_1, \quad \mu\le\rho=\mu+\nu(1-\mu), && (1.2)\\
\sigma(t) &= \varphi(t),\ t\in[-r,1],\ 0<r<\infty, && (1.3)
\end{aligned}
$$

where ${}_HD^{\mu,\nu}_{1^+}$ is the fractional derivative of order μ and parameter ν in the Hilfer-Hadamard sense, ${}_HI^{1-\rho}_{1^+}$ is fractional integral of order $1-\rho$ in the Hadamard sense, $\aleph:(1,e]\times R^2\to R$, $\gamma:(1,e]\to[-r,e]$, $\varphi:[-r,1]\to R$ are continuous with $\gamma(t)\le t$, and $\sigma_1\in R$.

The focus of the paper is the generalization of some results on FFDEs that have been studied in the literature [20, 30, 31], in this work, we obtain the existence, uniqueness and UHML stability of solutions for Hilfer-Hadamard FFDE (1.1)-(1.3). Our discussion mainly depends on the famous fixed point theorem of Banach, the Picard operator technique, and the inequalities of Gronwall. The obtained results can be considered as a contribution to developing UHML stability results for FFDEs involving Hilfer- Hadamard fractional derivative. In this concept and as far as we know, it is the first work concerning FFDEs involving Hilfer-Hadamard fractional derivative.

Besides the aforementioned in the introduction section, the rest distribution of the work is as follows: In Sect. 2, we rendering some needful definitions and results which are applied throughout this work. Sect. 3 studies the existence, uniqueness and UHML stability of solutions to Hilfer-Hadamard FFDE (1.1)-(1.3). In Sect. 4, we give an illustrating example of the applicability of the results obtained. The last Sect. includes the conclusion of the work.

2 Preliminaries

In this segment, we insert some notations and key findings concerning with Hilfer-Hadamard fractional derivative that assist in proving our theories for this work. Let us fix $I=[1,e]$ and let $C[I,R]$ is a Banach space endorsed with the norm $\|\sigma\|_C=\sup\{|\sigma(t)|\,;t\in I\}$, $\sigma\in C[I,R]$. Denote by $L^1[I,R]$ be the space of Lebesgue integrable function σ on I, endowed with the norm $\|\sigma\|_{L^1}=\int_I|\sigma(t)|\,dt<\infty$. As usual, $AC[I,R]$ is the space of functions from I into R which are absolutely continuous, and the spaces $AC^n[I,R]$ and $AC^n_\varrho[I,R]$ are defined by

$$AC^n[I,R]=\{\sigma:I\to R:\sigma^{(n-1)}(t)\in AC[I,R]\};$$

$$AC^n_\varrho[I,R]=\{\sigma:I\to R:\varrho^{(n-1)}\sigma(t)\in AC[I,R],\ \varrho=t\frac{d}{dt}\}.$$

Further, let $\rho\in(0,1]$ by $C_{1-\rho,\log}[I,R]$ and $C^1_{1-\rho,\log}[I,R]$ we denote the weighted spaces of continuous functions defined by

$$C_{1-\rho,\log}[I,R]=\{\sigma:(1,e]\to R:[\log(t)]^{1-\rho}\sigma(t)\in C[I,R]\},\ \text{and}$$

$$C^1_{1-\rho,\log}[I,R]:=\{\sigma\in C[I,R]:\sigma^{(1)}\in C_{1-\rho,\log}[I,R]\}$$

endowed with the norms

$$\|\sigma\|_{C_{1-\rho,\log}}=\left\|[\log(t)]^{1-\rho}\sigma(t)\right\|_C=\max\{\left|[\log(t)]^{1-\rho}\sigma(t)\right|;t\in I\},\ \text{and}$$

$$\|\sigma\|_{C^1_{1-\rho,\log}}=\|\sigma\|_C+\|\sigma^{(1)}\|_{C_{1-\rho,\log}},$$

respectively. Decidedly, $C_{1-\rho,\log}[I,R]$ and $C^1_{1-\rho,\log}[I,R]$ are Banach spaces with the norms $\|\cdot\|_{C_{1-\rho,\log}}$, and $\|\cdot\|_{C^1_{1-\rho,\log}}$ respectively. Furthermore, $C^0_{1-\rho,\log}[I,R]:=C_{1-\rho,\log}[I,R]$.

Definition 2.1. [1] The left-sided Hadamard fractional integral of order $\mu>0$ for a function $\sigma:I\to R$ is defined by

$${}_HI^{\mu}_{1^+}\sigma(t)=\frac{1}{\Gamma(\mu)}\int_1^t(\log t-\log\tau)^{\mu-1}\sigma(\tau)\frac{d\tau}{\tau},$$

on condition that ${}_HI^{\mu}_{1^+}(\cdot)$ exists, where $\Gamma(\omega)=\int_0^\infty\tau^{\omega-1}e^{-\tau}d\tau;\ \omega>0$ is called the Gamma function, and $\log(\cdot)=\log_e(\cdot)$.

Definition 2.2. [32] Let $n-1<\mu<n$ $(n\in\mathbb{N}_0)$ and $\sigma: I\to R$. The left-sided Hadamard fractional derivative of order μ for a function $\sigma: I\to R$ is defined by

$$ {}_{H}D^{\mu}_{1^+}\sigma(t)=\frac{\varrho^n}{\Gamma(n-\mu)}\int_1^t(\log t-\log\tau)^{n-\mu-1}\sigma(\tau)\frac{d\tau}{\tau}, $$

where $\varrho^n=\left(t\frac{d}{dt}\right)^n$ and $n=[\mu]+1$, here $[\mu]$ denotes the integer part of μ.

Definition 2.3. [32] Let $n-1<\mu<n$ and $\sigma\in AC^n_{\varrho}[I,R]$. The left-sided Caputo-Hadamard fractional derivative of order μ of σ is defined by

$$ {}_{CH}D^{\mu}_{1^+}\sigma(t)={}_{H}D^{\mu}_{1^+}\left(\sigma(t)-\sum_{k=0}^{n-1}\frac{\varrho^k\sigma(1)}{k!}(\log t-\log\tau)^k\right). $$

Moreover, if $\mu\notin\mathbb{N}_0$, then ${}_{CH}D^{\mu}_{1^+}\sigma(t)$ can be represented by

$$ {}_{CH}D^{\mu}_{1^+}\sigma(t)=\frac{1}{\Gamma(n-\mu)}\int_1^t(\log t-\log\tau)^{n-\mu-1}\ \varrho^n\sigma(\tau)\frac{d\tau}{\tau}, $$

While if $\mu\in\mathbb{N}$, then we have ${}_{CH}D^{\mu}_{1^+}\sigma(t)=\varrho^n\sigma(t)$.

Definition 2.4. [15] Let $n-1<\mu<n$, $0\le\nu\le1$, and $\sigma\in AC^n[I,R]$. The left-sided Hilfer-Hadamard fractional derivative of order μ of a function σ is defined by

$$ {}_{H}D^{\mu,\nu}_{1^+}\sigma(t)=\big({}_{H}I^{\nu(n-\mu)}_{1^+}\varrho^n\big({}_{H}I^{(1-\nu)(n-\mu)}_{1^+}\sigma\big)\big)(t), \tag{2.1} $$

One has

$$ {}_{H}D^{\mu,\nu}_{1^+}\sigma(t)=\big({}_{H}I^{\nu(n-\mu)}_{1^+}({}_{H}D^{\rho}_{1^+}\sigma)\big)(t),\quad \rho=\mu+n\nu-\mu\nu, $$

${}_{H}D^{\rho}_{1^+}=\varrho^n\ {}_{H}I^{(1-\nu)(n-\mu)}_{1^+}=\varrho^n\ {}_{H}I^{n-\rho}_{1^+}$.

Remark 2.5.

(i) If $\nu=0$, then Hilfer-Hadamard fractional derivative ${}_{H}D^{\mu,\nu}_{1^+}(\cdot)$ decay to Hadamard fractional derivative ${}_{H}D^{\mu}_{1^+}(\cdot)$.

(ii) If $\nu=1$, then Hilfer-Hadamard fractional derivative ${}_{H}D^{\mu,\nu}_{1^+}(\cdot)$ decay to Caputo–Hadamard fractional derivative ${}_{CH}D^{\mu}_{1^+}(\cdot)$.

(iii) In a special case, if $0<\mu<1$ and $0\le\nu\le1$, then Hilfer-Hadamard fractional derivative ${}_{H}D^{\mu,\nu}_{1^+}(\cdot)$ also can be rewritten as

$$ {}_{H}D^{\mu,\nu}_{1^+}={}_{H}I^{\nu(1-\mu)}_{1^+}\varrho\ {}_{H}I^{(1-\nu)(1-\mu)}_{1^+}={}_{H}I^{\nu(1-\mu)}_{1^+}({}_{H}D^{\rho}_{1^+}),\quad \rho=\mu+\nu-\mu\nu, $$

where ${}_{H}D^{\rho}_{1^+}=\varrho\left({}_{H}I^{(1-\nu)(1-\mu)}_{1^+}\right)=\varrho\ {}_{H}I^{1-\rho}_{1^+}$.

Lemma 2.6. *[15] If $\mu>0$, $\nu>0$, then we have*

$$ \big[{}_{H}I^{\mu}_{1^+}(\log\tau)^{\nu-1}\big](t)=\frac{\Gamma(\nu)}{\Gamma(\nu+\mu)}\big(\log(t)\big)^{\nu+\mu-1}, $$

and

$$ \big[{}_{H}D^{\mu}_{1^+}(\log\tau)^{\nu-1}\big](t)=\frac{\Gamma(\nu)}{\Gamma(\nu-\mu)}\big(\log(t)\big)^{\nu-\mu-1} $$

In exceptional condition, if $\nu=1$, then ${}_{H}D^{\mu,\nu}_{1^+}$ of a constant is not equal to zero, i.e.

$$ ({}_{H}D^{\mu}_{1^+}1)(t)=\frac{1}{\Gamma(1-\mu)}(\log t)^{-\mu},\qquad 0<\mu<1. $$

Lemma 2.7. *[15] Let $\mu > 0, \nu > 0$ and $0 < \rho \leq 1$. Then for all $t \in (1, e]$ and $\sigma \in C_{1-\rho,\log}[I, R]$,*

$$ {}_H I_{1^+}^{\mu} \; {}_H I_{1^+}^{\nu} \sigma(t) = \; {}_H I_{1^+}^{\mu+\nu} \sigma(t), \tag{2.2}$$

and

$$ {}_H D_{1^+}^{\mu} \; {}_H I_{1^+}^{\mu} \sigma(t) = \sigma(t). \tag{2.3}$$

Remark 2.8. In particular, if $\sigma \in C[I, R]$, then the relations (2.2) and (2.3) hold at $t \in I$.

Let $0 < \mu < 1$, $0 < \rho \leq 1$. If $\sigma \in C_{1-\rho,\log}[I, R]$ and ${}_H I_{1^+}^{1-\mu} \sigma \in C^1_{1-\rho,\log}[I, R]$, then

$$ {}_H I_{1^+}^{\mu} \; {}_H D_{1^+}^{\mu} \sigma(t) = \sigma(t) - \frac{({}_H I_{1^+}^{1-\mu} \sigma)(1)}{\Gamma(\mu)} (\log t)^{\mu-1}, \qquad \forall t \in (1, e].$$

Now, we are going to present the Picard operator definition, the abstract Gronwall lemma, and the generalized Gronwall inequality in the Hadamard fractional integral sense that be important tools in our subsequent analysis.

Definition 2.9. [31] Let (U, d) be a metric space. We say that the operator $T : U \to U$ is a Picard if there exists $u^* \in U$ such that:

(i) $\Upsilon_T = u^*$ is the fixed points set of the operator T, where $\Upsilon_T = \{u \in U : T(u) = u\}$.

(ii) For all $u_0 \in U$ the sequence $\{T^n(u_0)\}_{n \in \mathbb{N}}$ is a converges to u^*.

Lemma 2.10. *[31] Let $(U, d, \leq)$ be an ordered metric space, and the Picard operator $T : U \to U$ be an increasing with $\Upsilon_T = \{u_T^*\}$. Then for $u \in U$,*

$$ u \leq T(u) \Rightarrow u \leq u_T^* \;\; \textit{while} \;\; u \geq T(u) \Rightarrow u \geq u_T^*.$$

Lemma 2.11. *[32] Let $0 < \mu < 1$, and $\wp, \emptyset : I \to [1, +\infty)$ be continuous functions. If γ is nondecreasing function and there exist a constant $\hbar \geq 0$ such that*

$$ \wp(t) \leq \emptyset(t) + \hbar \int_1^t (\log t - \log \tau)^{\mu-1} \wp(\tau) \frac{d\tau}{\tau}, \quad t \in I,$$

then

$$ \wp(t) \leq \emptyset(t) + \int_1^t \left[\sum_{n=1}^{\infty} \frac{(\hbar \Gamma(\mu))^n}{\Gamma(n\mu)} (\log t - \log \tau)^{n\mu-1} \emptyset(\tau) \right] \frac{d\tau}{\tau}, \quad t \in I.$$

In particular, if the function $\emptyset(t)$ be a nondecreasing on I, then we have

$$ \wp(t) \leq \emptyset(t) \mathbb{E}_{\mu}(\hbar \Gamma(\mu) (\log t)^{\mu}).$$

Here the symbol $\mathbb{E}_{\mu}(\cdot)$ means Mittag-Leffler function defined by

$$ \mathbb{E}_{\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\mu + 1)}, \quad z \in \mathbb{C}.$$

Now we will introduce some concepts of UHML stability analysis.

Let be a positive real number $\epsilon > 0$. Then for $\aleph \in C_{1-\rho,\log}[I, R]$ and $\theta \in C([-r, e], R)$, we consider the Hilfer-Hadamard FFDE (1.1)-(1.3) associated with the following inequality:

$$ |{}_H D_{1^+}^{\mu;\nu} \theta(t) - \aleph(t, \theta(t), \theta(\gamma(t)))| \leq \epsilon \mathbb{E}_{\mu} (\log t)^{\mu}. \tag{2.4}$$

Definition 2.12. Hilfer-Hadamard FFDE (1.1)-(1.3) is UHML stable with respect to $\mathbb{E}_{\mu}(\log t)^{\mu}$ if there exists a real number $c_{\mathbb{E}_{\mu}} > 0$ such that, for each $\epsilon > 0$ and each solution $\theta \in C([-r, e], R)$ satisfying inequality (2.4), there exist a solution $\sigma \in C([-r, e], R)$ of problem (1.1)-(1.3) toe the line

$$ |\theta(t) - \sigma(t)| \leq c_{\mathbb{E}_{\mu}} \epsilon \mathbb{E}_{\mu} (\log t)^{\mu}, \quad t \in [-r, e].$$

Remark 2.13. A function $\theta \in C_{1-\rho,\log}[I, R]$ is a solution of inequality (2.4) if and only if there exists a function depends on θ, let it be $\tilde{h}(t) \in C_{1-\rho,\log}[I, R]$ such that $|\tilde{h}(t)| \leq \epsilon \mathbb{E}_\mu (\log t)^\mu$, for $t \in (1, e]$ and ${}_H D^{\mu,\nu}_{1^+} \theta(t) = \aleph(t, \theta(t), \theta(\gamma(t))) + \tilde{h}(t), \ t \in (1, e]$.

Lemma 2.14. *Let* $\aleph : (1, e] \times R \times R \to R$ *be a continuous function. Then the problem* (1.1)-(1.2) *is equivalent to*

$$\sigma(t) = \frac{(\log t)^{\rho-1}}{\Gamma(\rho)} \sigma_1 + \frac{1}{\Gamma(\mu)} \int_1^t (\log t - \log \tau)^{\mu-1} \aleph(\tau, \sigma(\tau), \sigma(\gamma(\tau))) \frac{d\tau}{\tau}, \ t \in (1, e].$$

Lemma 2.15. *Let* $0 < \mu < 1$, $\rho = \mu + \nu(1 - \mu)$, *and* $\theta \in C_{1-\rho,\log}[I, R]$ *satisfying inequality* (2.4). *Then* θ *is a solution of the following integral inequality*

$$\left| \theta(t) - \frac{(\log t)^{\rho-1}}{\Gamma(\rho)} \theta_1 - \frac{1}{\Gamma(\mu)} \int_1^t (\log t - \log \tau)^{\mu-1} \aleph(\tau, \theta(\tau), \theta(\gamma(\tau))) \frac{d\tau}{\tau} \right| \leq \epsilon \mathbb{E}_\mu (\log t)^\mu.$$

Proof. Thanks to Remark 2.13, we have

$${}_H D^{\mu,\nu}_{1^+} \theta(t) = \aleph(t, \theta(t), \theta(\gamma(t))) + \tilde{h}(t), \ t \in (1, e].$$

The Lemma 2 and the condition ${}_H I^{1-\rho}_{1^+} \theta(1) = \theta_1$ lead us to

$$\begin{aligned} \theta(t) &= \frac{(\log t)^{\rho-1}}{\Gamma(\rho)} \theta_1 + \frac{1}{\Gamma(\mu)} \int_1^t (\log t - \log \tau)^{\mu-1} \aleph(\tau, \theta(\tau), \theta(\gamma(\tau))) \frac{d\tau}{\tau} \\ &\quad + \frac{1}{\Gamma(\mu)} \int_1^t (\log t - \log \tau)^{\mu-1} \tilde{h}(\tau) \frac{d\tau}{\tau}, \ \ t \in (1, e]. \end{aligned}$$

From the last equation and using Remark 2.13 again, we get

$$\begin{aligned} &\left| \theta(t) - \frac{(\log t)^{\rho-1}}{\Gamma(\rho)} \theta_1 - \frac{1}{\Gamma(\mu)} \int_1^t (\log t - \log \tau)^{\mu-1} \aleph(\tau, \theta(\tau), \theta(\gamma(\tau))) \frac{d\tau}{\tau} \right| \\ &\leq \frac{1}{\Gamma(\mu)} \int_1^t (\log t - \log \tau)^{\mu-1} \left| \tilde{h}(\tau) \right| \frac{d\tau}{\tau}. \\ &\leq \frac{\epsilon}{\Gamma(\mu)} \int_1^t (\log t - \log \tau)^{\mu-1} \mathbb{E}_\mu (\log \tau)^\mu \frac{d\tau}{\tau} \\ &\leq \frac{\epsilon}{\Gamma(\mu)} \int_1^t (\log t - \log \tau)^{\mu-1} \sum_{k=0}^{\infty} \frac{(\log \tau)^{k\mu}}{\Gamma(k\mu + 1)} \frac{d\tau}{\tau} \\ &= \epsilon \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\mu + 1)} \left[\frac{1}{\Gamma(\mu)} \int_1^t (\log t - \log \tau)^{\mu-1} (\log \tau)^{(k\mu)} \frac{d\tau}{\tau} \right] \\ &= \epsilon \sum_{k=0}^{\infty} \frac{(\log t)^{(k+1)\mu}}{\Gamma((k+1)\mu + 1)} \\ &\leq \epsilon \sum_{n=0}^{\infty} \frac{(\log t)^{n\mu}}{\Gamma((n\mu + 1)} \\ &= \epsilon \mathbb{E}_\mu (\log t)^\mu. \end{aligned}$$

□

The proof of the following lemma is similar to proofs presented in the literature [20, 22, 31].

Lemma 2.16. *Let* $0 < \mu < 1$, $\rho = \mu + \nu(1 - \mu)$ *and assume that* $\aleph : (1, e] \times R \times R \to R$ *be a function such that* $\aleph(., \sigma(.), \sigma(\gamma(.))) \in C_{1-\rho,\log}[I, R]$. *Then a function* $\sigma \in C_{1-\rho,\log}[I, R]$ *is a* *solution of Hilfer-Hadamard FFDE* (1.1) *with the initial condition* ${}_HI_{1^+}^{1-\rho}\sigma(1) = \sigma_1$ *and initial functional* $\sigma(t) = \varphi(t), t \in [-r, 1]$ *if and only if* σ *satisfies the following integral equation*

$$\sigma(t) = \begin{cases} \varphi(t)\,, \quad t \in [-r, 1], \\ \frac{(\log t)^{\rho-1}}{\Gamma(\rho)}\sigma_1 + \frac{1}{\Gamma(\mu)}\int_1^t (\log t - \log \tau)^{\mu-1} \aleph(\tau, \sigma(\tau), \sigma(\gamma(\tau)))\frac{d\tau}{\tau}, \quad t \in (1, e]. \end{cases}$$

3 Existence and uniqueness results

This section gives some sufficient condition to warranty the existence, uniqueness and UHML stability of solutions for the Hilfer Hadamard FFDE (1.1)-(1.3) in $C([-r, e], R) \cap C_{1-\rho,\log}([I, R)$.

Let us insert the following hypotheses to aid in proving our main results:

(H$_1$) $\aleph : (1, e] \times R \times R \to R$ be a function such that $\aleph(., \sigma(.), \sigma(\gamma(.))) \in C_{1-\rho,\log}[I, R]$, for any $\sigma \in C_{1-\rho,\log}[I, R]$ and $\gamma \in C([1, e], [-r, e])$ with $\gamma(t) \le t$ and $0 < r < \infty$.

(H$_2$) There exists a constant $K_\aleph > 0$ such that

$$|\aleph(t, p_1, p_2) - \aleph(t, q_1, q_2)| \le K_\aleph \sum_{i=1}^{2} |p_i - q_i|, \forall t \in (1, e], \quad p_i, q_i \in \mathbb{R} \quad i = 1, 2.$$

(H$_3$) The following inequality holds:

$$\frac{2K_\aleph \Gamma(\rho)}{\Gamma(\mu + \rho)} < 1.$$

Theorem 3.1. *Assume that hypotheses* $(H_1) - (H_3)$ *are satisfied. Then*

(i) problem (1.1)-(1.3) *has a unique solution in* $C([-r, e], R) \cap C_{1-\rho,\log}([I, R)$*;*

(ii) Hilfer-Hadamard type FFDE (1.1) *is UHML stable.*

Proof. Thanks to Lemma (2.16), the problem (1.1)-(1.3) can be converted into its equivalent integral model, which takes the form

$$\sigma(t) = \begin{cases} \varphi(t), \qquad t \in [-r, 1], \\ \frac{(\log t)^{\rho-1}}{\Gamma(\rho)}\sigma_1 + \frac{1}{\Gamma(\mu)}\int_1^t (\log t - \log \tau)^{\mu-1} \aleph(\tau, \sigma(\tau), \sigma(\gamma(\tau)))\frac{d\tau}{\tau}, \quad t \in (1, e]. \end{cases} \tag{3.1}$$

To demonstrate our first part, we just show the existence of solution for the model (3.1) which can be transformed into a fixed point problem in the space $C([-r, e], R)$ with respect to an operator $\mathcal{R}_\aleph : C([-r, e], R) \to C([-r, e], R)$ defined by

$$\mathcal{R}_\aleph(\sigma)(t) = \begin{cases} \varphi(t), \qquad t \in [-r, 1], \\ \frac{(\log t)^{\rho-1}}{\Gamma(\rho)}\sigma_1 + \frac{1}{\Gamma(\mu)}\int_1^t (\log t - \log \tau)^{\mu-1} \aleph(\tau, \sigma(\tau), \sigma(\gamma(\tau)))\frac{d\tau}{\tau}, \quad t \in (1, e]. \end{cases} \tag{3.2}$$

Now, we confirm that $\mathcal{R}_\aleph$ is well-define. It is clear that, for any continuous function $\aleph(\cdot, \sigma(\cdot), \sigma(\gamma(\cdot)))$, an operator $\mathcal{R}_\aleph$ is continuous too. In fact,

Case 1. For all $t, t+\varepsilon \in (1, e]$ ($\varepsilon > 0$) and $\sigma \in C([-r, e], R)$, we have

$$\begin{aligned}
&|\mathcal{R}_{\aleph}(\sigma)(t+\varepsilon) - \mathcal{R}_{\aleph}(\sigma)(t)| \\
&= \Bigg| \frac{(\log t+\varepsilon)^{\rho-1}}{\Gamma(\rho)}\sigma_1 + \frac{1}{\Gamma(\mu)}\int_1^{t+\varepsilon}(\log(t+\varepsilon) - \log\tau)^{\mu-1}\aleph(\tau, \sigma(\tau), \sigma(\gamma(\tau)))\frac{d\tau}{\tau} \\
&\quad - \frac{(\log t)^{\rho-1}}{\Gamma(\rho)}\sigma_1 - \frac{1}{\Gamma(\mu)}\int_1^{t}(\log t - \log\tau)^{\mu-1}\aleph(\tau, \sigma(\tau), \sigma(\gamma(\tau)))\frac{d\tau}{\tau}\Bigg| \\
&\to 0, \qquad as\ t+\varepsilon \to t.
\end{aligned}$$

Case 2. For $t, t+\varepsilon \in [-r, 1]$ and $\sigma \in C([-r, e], R)$, we have

$$|\mathcal{R}_{\aleph}(\sigma)(t+\varepsilon) - \mathcal{R}_{\aleph}(\sigma)(t)| = |\varphi(t+\varepsilon) - \varphi(t)| \to 0, \qquad as\ t+\varepsilon \to t.$$

Next, we just need to prove that $\mathcal{R}_{\aleph} : C([-r, e], R) \to C([-r, e], R)$ given by (3.2) is a contraction mapping on $C([-r, e], R)$ w. r. t. $\|.\|_{C_{1-\rho,\log}[I,R]}$. Indeed, the case $t \in [-r, 1]$ is trivial. For each $t \in (1, e]$, and for any $\sigma, \widetilde{\sigma} \in C([-r, e], R)$, we have

$$\begin{aligned}
&|\mathcal{R}_{\aleph}(\sigma)(t) - \mathcal{R}_{\aleph}(\widetilde{\sigma})(t)| \\
&\leq \frac{1}{\Gamma(\mu)}\int_1^t (\log t - \log\tau)^{\mu-1}|\aleph(\tau, \sigma(\tau), \sigma(\gamma(\tau))) - \aleph(\tau, \widetilde{\sigma}(\tau), \widetilde{\sigma}(\gamma(\tau)))|\frac{d\tau}{\tau} \\
&\leq \frac{K_{\aleph}}{\Gamma(\mu)}\int_1^t (\log t - \log\tau)^{\mu-1}(\log\tau)^{\rho-1}\Big\{(\log\tau)^{1-\rho} \\
&\big[|\sigma(\tau) - \widetilde{\sigma}(\tau)| + |\sigma(\gamma(\tau)) - \widetilde{\sigma}(\gamma(\tau))|\big]\Big\}\frac{d\tau}{\tau} \\
&\leq \frac{K_{\aleph}}{\Gamma(\mu)}\int_1^t (\log t - \log\tau)^{\mu-1}(\log\tau)^{\rho-1}\Big[\max_{\tau\in(1,e]}(\log\tau)^{1-\rho}|\sigma(\tau) - \widetilde{\sigma}(\tau)| \\
&+ \max_{\tau\in(1,e]}(\log\tau)^{1-\rho}|\sigma(\gamma(\tau)) - \widetilde{\sigma}(\gamma(\tau))|\Big]\frac{d\tau}{\tau} \\
&\leq 2K_{\aleph}\|\sigma - \widetilde{\sigma}\|_{C_{1-\rho,\log}[I,R]}\frac{1}{\Gamma(\mu)}\int_1^t (\log t - \log\tau)^{\mu-1}(\log\tau)^{\rho-1}\frac{d\tau}{\tau} \\
&= \frac{2K_{\aleph}\Gamma(\rho)(\log t)^{\rho+\mu-1}}{\Gamma(\mu+\rho)}\|\sigma - \widetilde{\sigma}\|_{C_{1-\rho,\log}[I,R]},
\end{aligned} \tag{3.3}$$

which implies

$$\begin{aligned}
\|\mathcal{R}_{\aleph}(\sigma) - \mathcal{R}_{\aleph}(\widetilde{\sigma})\|_{C_{1-\rho,\log}[I,R]} &\leq \frac{2K_{\aleph}\Gamma(\rho)(\log e)^{\mu}}{\Gamma(\mu+\rho)}\|\sigma - \widetilde{\sigma}\|_{C_{1-\rho,\log}[I,R]}. \\
&= \frac{2K_{\aleph}\Gamma(\rho)}{\Gamma(\mu+\rho)}\|\sigma - \widetilde{\sigma}\|_{C_{1-\rho,\log}[I,R]}.
\end{aligned}$$

Due to (H_3), $\mathcal{R}_{\aleph}$ is contraction mapping via the norm $\|\cdot\|_{C_{1-\rho,\log}[I,R]}$ on $C_{1-\rho,\log}([I, R)$. Hence, we infer that $\mathcal{R}_{\aleph}$ has a unique fixed point according to the outstanding fixed point theorem of Banach. Claim (1) was substantiated.

Now, we demonstrate the claim (2). Assume that solution $\theta \in C([-r, e], R) \cap C_{1-\rho,\log}([I, R)$ satisfying inequality (2.4). We denote by $\sigma \in C([-r, e], R) \cap C_{1-\rho,\log}([I, R)$ the unique solution of the following problem

$$\begin{cases}
{}_H D_{1^+}^{\mu,\nu}\sigma(t) = \aleph(t, \sigma(t), \sigma(\gamma(t))), \quad 0 < \mu < 1, \quad 0 \leq \nu \leq 1,\ t \in (1, e], \\
{}_H I_{1^+}^{1-\rho}\sigma(1) = {}_H\, I_{1^+}^{1-\rho}\theta(1), \quad \mu \leq \rho = \mu + \nu(1-\mu), \\
\sigma(t) = \theta(t), \qquad t \in [-r, 1].
\end{cases}$$

It follows from Lemma 2.16 that

$$\sigma(t)=\begin{cases}\theta(t)\,, & t\in[-r,1],\\ \frac{(\log t)^{\rho-1}}{\Gamma(\rho)}\theta_1+\frac{1}{\Gamma(\mu)}\int_1^t(\log t-\log\tau)^{\mu-1}\aleph(\tau,\sigma(\tau),\sigma(\gamma(\tau)))\frac{d\tau}{\tau}, & t\in(1,e],\end{cases} \tag{3.4}$$

where we used the fact

$${}_HI_{1+}^{1-\rho}\sigma(1)={}_H I_{1+}^{1-\rho}\theta(1) \text{ which implies } \sigma_1=\theta_1.$$

In light of Lemma 2.15, we have

$$\left|\theta(t)-\frac{(\log t)^{\rho-1}}{\Gamma(\rho)}\theta_1-\frac{1}{\Gamma(\mu)}\int_1^t(\log t-\log\tau)^{\mu-1}\aleph(\tau,\theta(\tau),\theta(\gamma(\tau)))\frac{d\tau}{\tau}\right| \\ \leq \epsilon\mathbb{E}_\mu(\log t)^\mu,\ t\in(1,e]. \tag{3.5}$$

Note that $|\sigma(t)-\theta(t)|=0$ for $t\in[-r,1]$. For all $t\in(1,e]$, we have from (H_2), (3.5) and (3.4) that

$$\begin{aligned}&|\theta(t)-\sigma(t)|\\ &\leq\left|\theta(t)-\frac{(\log t)^{\rho-1}}{\Gamma(\rho)}\theta_1-\frac{1}{\Gamma(\mu)}\int_1^t(\log t-\log\tau)^{\mu-1}\aleph(\tau,\theta(\tau),\theta(\gamma(\tau)))\frac{d\tau}{\tau}\right|\\ &\quad+\frac{1}{\Gamma(\mu)}\int_1^t(\log t-\log\tau)^{\mu-1}\left|\aleph(\tau,\theta(\tau),\theta(\gamma(\tau)))-\aleph(\tau,\sigma(\tau),\sigma(\gamma(\tau)))\right|\frac{d\tau}{\tau}\\ &\leq\epsilon\mathbb{E}_\mu(\log t)^\mu\\ &\quad+\frac{K_\aleph}{\Gamma(\mu)}\int_1^t(\log t-\log\tau)^{\mu-1}\Big[|\theta(\tau)-\sigma(\tau)|+|\theta(\gamma(\tau))-\sigma(\gamma(\tau))|\Big]\frac{d\tau}{\tau}.\end{aligned} \tag{3.6}$$

According to the last inequality for $w\in C([-r,e],R^+)$, we consider the operator $\mathcal{R}_1: C([-r,e],R^+)\to C([-r,e],R^+)$ defined by

$$\mathcal{R}_1(w)(t)=\begin{cases}0\,, & t\in[-r,1],\\ \epsilon\mathbb{E}_\mu(\log t)^\mu+\frac{K_\aleph}{\Gamma(\mu)}\Big[\int_1^t(\log t-\log\tau)^{\mu-1}w(\tau)\frac{d\tau}{\tau} \\ \quad+\int_1^t(\log t-\log\tau)^{\mu-1}w(\gamma(\tau))\frac{d\tau}{\tau}\Big], & t\in(1,e].\end{cases}$$

Here we have to prove that $\mathcal{R}_1$ is Picard operator. For each $t\in(1,e]$ with the same arguments in the relationship (3.3), one obtain

$$|\mathcal{R}_1(w)(t)-\mathcal{R}_1(z)(t)|\leq\frac{2K_\aleph\Gamma(\rho)(\log t)^{\rho+\mu-1}}{\Gamma(\mu+\rho)}\|w-z\|_{C_{1-\rho,\log}[I,R^+]},\ \ w,z\in C_{1-\rho,\log}[1,e].$$

This implies that

$$\|\mathcal{R}_1(w)-\mathcal{R}_1(z)\|\leq\frac{2K_\aleph\Gamma(\rho)}{\Gamma(\mu+\rho)}\|w-z\|_{C_{1-\rho,\log}[I,R^+]},\ \ w,z\in C_{1-\rho,\log}[I,R^+].$$

Assumption (H_3) shows that $\mathcal{R}_1$ is a contraction mapping in $C([-r,e],R^+)$ w. r. t. $\|.\|_{C_{1-\rho,\log}[I,R^+]}$. By applying the contractive theorem of Banach, $\mathcal{R}_1$ is a Picard operator and $\Upsilon_{\mathcal{R}_1}=\{w^*\}$. Then, for all $t \in (1, e]$, we have

$$\begin{aligned} w^*(t) &= \mathcal{R}_1 (w^*)(t) \\ &= \epsilon \mathbb{E}_\mu (\log t)^\mu + \frac{K_\aleph}{\Gamma(\mu)} \left[\int_1^t (\log t - \log \tau)^{\mu-1} w^*(\tau) \frac{d\tau}{\tau} \right. \\ &+ \left. \int_1^t (\log t - \log \tau)^{\mu-1} w^*(\gamma(\tau)) \frac{d\tau}{\tau} \right]. \end{aligned} \tag{3.7}$$

By the following assumption

$$m := \min_{\tau \in [1,e]} [w^*(\tau) + w^*(\gamma(\tau))] \in \mathbb{R}^+$$

with some straightforward computations, we infer that, for all $1 < t_1 \leq t_2 \leq e$,

$$w^*(t_2) > w^*(t_1),$$

which indicating that w^* is an increasing, since $\gamma(t) \leq t$, we get $w^*(\gamma(t)) \leq w^*(t)$, it follows from (3.7) that

$$w^*(t) \leq \epsilon \mathbb{E}_\mu (\log t)^\mu + \frac{2K_\aleph}{\Gamma(\mu)} \int_1^t (\log t - \log \tau)^{\mu-1} w^*(\tau) \frac{d\tau}{\tau}.$$

By taking advantage of Lemma 2.11 and Remark 2, then for $t \in (1, e]$, we obtain

$$\begin{aligned} w^*(t) &\leq \epsilon \mathbb{E}_\mu (\log t)^\mu \mathbb{E}_\mu (2K_\aleph (\log t)^\mu) \\ &\leq \epsilon \mathbb{E}_\mu (\log t)^\mu \mathbb{E}_\mu (2K_\aleph) \\ &= c_{\mathbb{E}_\mu} \epsilon \mathbb{E}_\mu (\log t)^\mu, \end{aligned}$$

where $c_{\mathbb{E}_\mu} := \mathbb{E}_\mu(2K_\aleph)$. So, in short, if $w = |\theta - \sigma|$, it follows from (3.6) that $w \leq \mathcal{R}_1 w$. Due to the increasing property of the Picard operator $\mathcal{R}_1$, Lemma 2.10 shows that $w \leq w^*$. As a conclusion, we can see that

$$|\theta(t) - \sigma(t)| \leq c_{\mathbb{E}_\mu} \epsilon \mathbb{E}_\mu (\log t)^\mu, \quad t \in [-r, e].$$

Hence the equation (1.1) is UHML stable. The results are proved completely.□ □

Theorem 3.2. *Assume that the hypotheses* (H_1) *and* (H_2) *are satisfied. If*

$$\frac{2K_\aleph e^\lambda \Gamma(\rho)}{\Gamma(\rho + \mu)} < 1, \quad \lambda > 0. \tag{3.8}$$

then

(i) problem (1.1)-(1.3) has a unique solution in $C([-r, e], R) \cap C_{1-\rho,\log}([1, e], R)$.

(ii) Hilfer-Hadamard FFDE (1.1) is UHML stable.

Proof: As in Theorem (3.1), we need only show that $\mathcal{R}_\aleph$ defined by (3.2) is a contraction mapping on the space $C([-r, e], R)$ w. r. t. the norm $\|.\|_B$, where

$$\|\sigma\|_B = \max_{t \in (1,e]} \left| [\log(t)]^{1-\rho} \sigma(t) \right| e^{-\lambda \log(t)}, \ \lambda > 0.$$

Since the technique of proof will be identical to the previous parts in Theorem (3.1), here we will give the main difference represented in the following. For each $t \in (1, e]$ and $\sigma, \widetilde{\sigma} \in$

$C([-r,e],R)$, we have

$$\begin{aligned}
&|\mathcal{R}_{\aleph}(\sigma)(t)-\mathcal{R}_{\aleph}(\widetilde{\sigma})(t)|\\
\leq\ & \frac{1}{\Gamma(\mu)}\int_1^t(\log t-\log\tau)^{\mu-1}|\aleph(\tau,\sigma(\tau),\sigma(\gamma(\tau)))-\aleph(\tau,\widetilde{\sigma}(\tau),\widetilde{\sigma}(\gamma(\tau)))|\frac{d\tau}{\tau}\\
\leq\ & \frac{K_{\aleph}}{\Gamma(\mu)}\int_1^t(\log t-\log\tau)^{\mu-1}\Big[|\sigma(\tau)-\widetilde{\sigma}(\tau)|+|\sigma(\gamma(\tau))-\widetilde{\sigma}(\gamma(\tau))|\Big]\frac{d\tau}{\tau}\\
\leq\ & \frac{K_{\aleph}}{\Gamma(\mu)}\int_1^t(\log t-\log\tau)^{\mu-1}(\log\tau)^{\rho-1}e^{\lambda\log(\tau)}\\
&\quad\left[\max_{\tau\in(1,e]}(\log\tau)^{1-\rho}e^{-\lambda\log(\tau)}\Big(|\sigma(\tau)-\widetilde{\sigma}(\tau)|+|\sigma(\gamma(\tau))-\widetilde{\sigma}(\gamma(\tau))|\Big)\right]\frac{d\tau}{\tau}\\
\leq\ & 2K_{\aleph}\|\sigma-\widetilde{\sigma}\|_B\frac{1}{\Gamma(\mu)}\int_1^t(\log t-\log\tau)^{\mu-1}(\log\tau)^{\rho-1}e^{\lambda\log(\tau)}\frac{d\tau}{\tau}\\
\leq\ & 2K_{\aleph}\|\sigma-\widetilde{\sigma}\|_Be^{\lambda\log(e)}\frac{1}{\Gamma(\mu)}\int_1^t(\log t-\log\tau)^{\mu-1}(\log\tau)^{\rho-1}\frac{d\tau}{\tau}\\
=\ & \frac{2K_{\aleph}e^{\lambda}\Gamma(\rho)(\log t)^{\rho+\mu-1}}{\Gamma(\mu+\rho)}\|\sigma-\widetilde{\sigma}\|_B,
\end{aligned}$$

which results in

$$\|\mathcal{R}_{\aleph}(\sigma)-\mathcal{R}_{\aleph}(\widetilde{\sigma})\|_B\leq\frac{2K_{\aleph}e^{\lambda}\Gamma(\rho)}{\Gamma(\mu+\rho)}\|\sigma-\widetilde{\sigma}\|_B. \tag{3.9}$$

The assumption (3.8) shows that $\mathcal{R}_{\aleph}$ is contraction mapping via the norm $\|\cdot\|_B$ on $C_{1-\rho,\log}([I,R)$. So, we drow that $\mathcal{R}_{\aleph}$ has a unique fixed point according to the fixed point theorem of Banach. This proves the first allegation.

The proof of the second allegation which deals with UHML stability is similar to the proof of the Theorem 3.1, so we skip it here. □

Example 3.3. Consider the following Hilfer-Hadamard FFDE

$$\begin{cases} {}^HD_{1^+}^{\frac{1}{2},0}\sigma(t)=\frac{1}{20}\log(\sqrt{t})+\frac{1}{8}\sin(2\sigma(t))+\frac{1}{8}\sigma(t-3),\ t\in(1,e],\\ {}^HI_{1+}^{\frac{1}{2}}\sigma(1)=1,\\ \sigma(t)=t,\ t\in[-1,1]. \end{cases} \tag{3.10}$$

Here $\mu=\frac{1}{2},\beta=0,\rho=\frac{1}{2},\aleph(t,\sigma(t),\sigma(\gamma(t))=\frac{1}{20}\log(\sqrt{t})+\frac{1}{8}\sin(2\sigma(t))+\frac{1}{8}\sigma(t-3),\ t\in(1,e]$, and $\gamma(t)=t-3$. Note that, for all $\sigma\in\mathbb{R}$ and $t\in(1,e]$,

$$C_{\frac{1}{2},\log}[I,R]=\{h\colon(1,e]\to R:[\log(t)]^{\frac{1}{2}}h(t)\in C[I,R]\}.$$

Clearly, the function $\aleph(t,\sigma(t),\sigma(\gamma(t))\in C_{\frac{1}{2};\log t}(I,\mathbb{R})$ due to $[\log(t)]^{\frac{1}{2}}\aleph(t,\sigma(t),\sigma(\gamma(t))\in C([I,R]$. In addition, let $\sigma,\theta\in\mathbb{R}$ and $t\in(1,e]$, it is easy to see that $\gamma(t)=t-3\leq t$ and

$$|\aleph(t,\sigma,\sigma(\gamma)-\aleph(t,\theta,\theta(\gamma)|\leq\frac{1}{4}\left[|\sigma-\theta|+|\sigma(\gamma)-\theta(\gamma)|\right].$$

Hence the hypotheses (H_1) and (H_2) hold with $K_{\aleph}=\frac{1}{4}$. Through straightforward computations, the condition

$$\frac{2K_{\aleph}\Gamma(\rho)}{\Gamma(\mu+\rho)}=\frac{\sqrt{\pi}}{2}<1$$

is satisfied. Therefore, the whole suppositions in Theorem 3.1 are satisfied. It follows from Theorem 3.1 part (1) that the problem (3.10) has a unique solution in $C([-1,e],R)$.

Further, as shown in Theorem 3.1 part (2), for every $\epsilon=\frac{1}{2}>0$ if $\theta\in C_{\frac{1}{2};\log t}([I,R])$ satisfies

$$\left|D_{1^+}^{\mu;\nu}\theta(t)-\aleph(t,\theta(t),\theta(\gamma(t))\right|\leq\frac{1}{2}\mathbb{E}_{\mu}(\log t)^{\mu},\quad t\in[-1,e],$$

there exists a unique solution $\sigma \in C_{\frac{1}{2};\log t}([I, R])$ comply with

$$|\theta(t) - \sigma(t)| \leq \frac{1}{2} c_{\mathbb{E}_\mu} \mathbb{E}_\mu (\log t)^\mu, \quad t \in [-1, e].$$

where

$$c_{\mathbb{E}_\mu} = \mathbb{E}_\mu(2K_\aleph) = \mathbb{E}_{\frac{1}{2}}(\frac{1}{2}) = e^{\frac{1}{4}} \left[1 + erf\left(\frac{1}{2}\right)\right] \simeq 2 > 0.$$

Hence the problem (3.10) is UHML stable.

Remark 3.4. The formula provided for the solution to the Hilfer-Hadamard FFDE (1.1)-(1.3) includes other formulas containing some operator's fractional derivatives, among them Hadamard ($\nu \to 0$), and Caputo-Hadamard ($\nu \to 1$).

Conclusion

This paper mainly investigated some existence, uniqueness, and UHML stability results of a class of initial value problems for Hilfer-Hadamard type FFDE with initial and functional conditions. The main key findings to our analysis included the fixed point theorem of Banach, the Picard operator technique and the generalized inequality of Gronwall. We are confident that the aforementioned results will have positive effects on the growth fractional nonlinear analysis and development of further applications in engineering and applied sciences.

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