Palestine Journal of Mathematics

Vol. 11(3)(2022), 626–635

COMMON FIXED POINT THEOREMS FOR SET-VALUED MAPS ON MODULAR B-GAUGE SPACES

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Communicated by T. Abdeljawad

MSC 2010 Classifications: Primary 47H10, Secondary 54H25.

Keywords and phrases: Modular b-metric space, Pseudomodular b-metric, Modular b-gauge space, Common fixed point.

Abstract Ali, Dinu, and Petrescu (M. U. Ali, S. Dinu, and L. Petrescu, Existence of fixed points of set-valued maps on modular b-gauge spaces, U.P.B. Sci. Bull., Series A, Vol. 82, Iss. 4, 2020, ISSN 1223-7027) introduced the notion of modular b-gauge spaces induced through the family of pseudomodular b-metrics and showed the existence of fixed point on this space.
In this paper, we employ the notion of modular $b-$gauge spaces to prove the existence of common fixed points of set-valued maps. Precisely, we formulate and prove two different theorems. Also, we state some corollaries. Moreover, we give an example to illustrate the validity of our results.

1 Introduction

The Banach fixed point theorem [1] is the first fixed point in the context of fixed point theory. After than, many authors established many fixed point theorems see [2]-[15]. The notion of a b-metric space was presented by Bakhtin [16] in 1989 as a generalization of a metric space. After four years, Czerwik [17] stated a formula that defined the exact definition of a b-metric space and studied some generalizations of Banach contraction theorems [1] in the context of $b-$metric spaces. Recently, many authors obtained many results on the context $b-$metric spaces as examples see [18]-[28].
In 2010, Chistyakov [29] introduced a new space, called a modular metric space, in such a way that he added a positive parameter on the definition of metric space. Then after, the researchers covered many fixed point theorems on this new space, for examples see ([30]-[32]).
In 2017, Ali [33] expanded the notion of modular metric spaces to modular b-metric spaces. While, Frigon [34] studied the fixed point on gauge spaces. Later on, several researchers extended their study of fixed point theory on gauge spaces, for examples look that ([35]-[37]). Posteriorly, Ali et al [38] presented the notion of modular gauge spaces induced through the family of preudo modular metrics. In 2020, Ali et al [39] defined the concept of modular b-gauge spaces induced through the family of pseudomodular b-metrics and proved the existence of fixed points of set-valued maps on modular b- gauge spaces. In this article, we used the space presented by Ali et al [39] and we proved the existence of a common fixed point for multivalued maps on modular b-gauge space. Precisely, we formulate two theorems, many corollaries and we introduce an example to show the validity our of results.

2 Preliminaries

In this section, we present the most important definitions that will be used in our work.

Definition 2.1. [29] A modular metric on a non empty set X is a function $\omega : (0,\infty)\times X\times X \to [0,\infty)$ that will be written as $\omega_\nu(x,y)=\omega(\nu,x,y)$; for all $x,y,z\in X$ and for all $\nu>0$, satisfies the following three conditions:

(i) $\omega_\nu(x,y)=0$ if and only if $x=y$, $\forall\nu>0$ and $x,y\in X$.

(ii) $\omega_\nu(x,y)=\omega_\nu(y,x)$, $\forall\nu>0$ and $x,y\in X$.

(iii) $\omega_{\nu+\sigma}(x,y) \leq \omega_\nu(x,z) + \omega_\sigma(z,y)$; for all $\nu, \sigma > 0$ and $x,y,z \in X$.

Ali [33] enhanced the notion of the modular metric space to the notion of the modular b-metric space as follows:

Definition 2.2. [33] A modular b-metric on a non empty set X is a function $\omega : (0,\infty)\times X\times X \to [0,\infty)$ that will be written as $\omega_\nu(x,y) = \omega(\nu,x,y)$; for all $x,y,z \in X$ and for all $\nu > 0$, satisfies the following three conditions:

(i) $\omega_\nu(x,y) = 0$ if and only if $x = y$, $\forall \nu > 0$ and $x,y \in X$.

(ii) $\omega_\nu(x,y) = \omega_\nu(y,x)$, $\forall \nu > 0$ and $x,y \in X$.

(iii) $\omega_{\nu+\sigma}(x,y) \leq \omega_{\frac{\nu}{s}}(x,z) + \omega_{\frac{\sigma}{s}}(z,y)$; for all $\nu,\sigma > 0$ and $x,y,z \in X$, here $s \geq 1$ is a fixed real number.

the couple (X,ω_ν) is said to be the modular b-metric space.

Example 2.3. [33] Consider $X = [0,\infty)$ and $\omega_\nu(x,y) = \frac{x^2+y^2-2xy}{\nu}$. Then (X,ω_ν) is a modular b-metric space with $s = 2$ but not a modular metric space.

Definition 2.4. [33] A regular modular b-metric on a non empty set X is a function $\omega : (0,\infty)\times X\times X \to [0,\infty)$ that will be written as $\omega_\nu(x,y) = \omega(\nu,x,y)$; for all $x,y,z \in X$ and for all $\nu > 0$, satisfies the following three conditions:

(i) $\omega_\nu(x,y) = 0$ if and only if $x = y$, for some $\nu > 0$.

(ii) $\omega_\nu(x,y) = \omega_\nu(y,x)$, $\forall \nu > 0$ and $x,y \in X$.

(iii) $\omega_{\nu+\sigma}(x,y) \leq \omega_{\frac{\nu}{s}}(x,z) + \omega_{\frac{\sigma}{s}}(z,y)$; for all $\nu,\sigma > 0$ and $x,y,z \in X$, here $s \geq 1$ is a fixed real number.

A pseudomodular b-metric on X is obtained by replacing axiom (1) of a modular b-metric with the following condition:
(4): For each $x \in X$, $\omega_\nu(x,x) = 0$,$\forall \nu > 0$.

Remark 2.5. Let ω_ν be a modular b-metric on a set X. Then for given $x,y \in X$, the function $0 < \nu \to \omega_\nu(x,y)$ is non increasing on $(0,\infty)$.
In fact if $0 < \frac{\nu}{s} < \sigma$, then by above definition

$$\omega_\sigma(x,y) \leq \omega_{\frac{\sigma-\nu}{s}}(x,x) + \omega_{\frac{\nu}{s}}(x,y) = \omega_{\frac{\nu}{s}}(x,y)$$

for all $x,y \in X$.

Khamsi [40] defined the concept of ω_ν-convergent sequences, ω_ν-Cauchy sequences, ω_ν-closed sets and ω_ν-complete sets in modular b-metric spaces as follows:

Definition 2.6. [40] Given a modular b-metric ω_ν on X, let $\{x_n\}_{n\in N}$ in X_ω and $x \in X_\omega$. Then:

(i) The sequence $\{x_n\}$ is said to be ω_ν-convergent to x if $\lim_{n\to\infty} \omega_\nu(x_n,x) = 0$, for some $\nu > 0$.

(ii) The sequence $\{x_n\}$ is said to be ω_ν-Cauchy if $\lim_{n,m\to\infty} \omega_\nu(x_n,x_m) = 0$, for some $\nu > 0$.

(iii) A subset A of X_ω is said to ω_ν-complete if each ω_ν-Cauchy sequence in A is ω_ν-convergent in A.

(iv) A subset A of X_ω is said to be ω_ν-closed if it contains the limit point of each ω_ν-convergent sequence contained in A.

(v) A subset A of X_ω is said to be ω_ν-bounded if we have
$\delta_{\omega_\nu}(A) = sup\{\omega_1(x,y) : x,y \in A\} < \infty$

The Δ_b-condition and Fatou property are given in a modular $b-$metric space as follows:

Definition 2.7. [33] A modular b-metric ω_ν on X satisfies:

(i) The Δ_b-condition, if the following axioms hold:

a. for each $\{x_n\}$ in X satisfying $\omega_\nu(x_n, x_{n+1}) \leq t^n K$ for some $\nu > 0$ and for each $n \in N$, where $t \in [0, \frac{1}{s})$ and $K > 0$ is some fixed real numbers, then we have $\omega_\sigma(x_n, x_{n+1}) \leq t^n K$ for each $\sigma > 0$ and for each $n \in N$, and

b. for each $\{x_n\}$ in X and $x \in X$ with $\lim_{n\to\infty} \omega_\nu(x_n, x) = 0$, for some $\nu > 0$ implies that $\lim_{n\to\infty} \omega_\sigma(x_n, x) = 0$, for all $\sigma > 0$.

(ii) The Fatou property if for each $\{x_n\}$ ω_ν-convergent to x and $\{y_n\}$ ω_ν-convergent to y, we have $\omega_1(x, y) \leq \liminf_{n\to\infty} \omega_1(x_n, y_n) = 0$.

Definition 2.8. [39] Let ω_ν be a pseudomodular b-metric on X. Then the ω_ν-ball having the radius $\sigma > 0$ with $x \in X$ as a center is the set

$$B[x, \omega_\nu, \sigma] = \{z \in X : \forall \nu > 0, \omega_\nu(x, y) < \sigma\}.$$

Example 2.9. [39] Consider $X = [0, \infty)$ with the pseudomodular b-metric $\omega_\nu(x, y) = \frac{x^2+y^2-2xy}{\nu}$ for each $x, y \in X$ and $\sigma > 0$, where $s = 2$. Then

$$B[x_0, \sigma, 1] = \{z \in X : \forall \sigma > 0, x_0^2 + z^2 - 2x_0 z < \sigma\} = \{x_0\}.$$

Definition 2.10. [39] A collection $\tau = \{\omega_\nu, \text{ with } s_\eta \geq 1 : \eta \in \Lambda\}$ of pseudomodular b-metrics on X is called separating if for every pair (x, y) with $x \neq y$, we have atleast one $\omega_\nu \in \tau$ with $\omega_\nu(x, y) \neq 0, \forall \nu > 0$.

Definition 2.11. [39] Take a collection $\tau = \{\omega_\nu, \text{ with } s_\eta \geq 1 : \eta \in \Lambda\}$ of pseudomodular b-metrics on $X \neq \emptyset$. The topology $\varsigma(\tau)$ with a collection of subbases

$$B(\tau) = \{B[z, \omega_\nu, \sigma] : z \in X, \omega_\nu \in \tau \text{ and } \sigma > 0\}$$

of the balls is a modular topology induced by the collection τ of pseudomodular b-metrics.

The pair $(X, \varsigma(\tau))$ is said to be a modular b-gauge space.
Note that $X_\tau = \{x \in X : \forall \eta \in \Lambda, \omega_\nu(x_0, x) \to 0 \text{ as } \nu \to \infty\}$, where x_0 is fixed in X.

Definition 2.12. [39] Take a modular b-gauge space $(X, \varsigma(\tau))$ with respect to the collection

$$\tau = \{\omega_\nu, \text{ with } s_\eta \geq 1 : \eta \in \Lambda\}$$

of pseudomodular b-metrics on X and let $\{x_n\}_{n\in N}$ in X_τ and $x \in X_\tau$.
Then:

(i) The sequence $\{x_n\}$ is said to be ω_ν-convergent to x if for every $\eta \in \Lambda$, we have $\lim_{n\to\infty} \omega_\nu(x_n, x) = 0$ for some $\nu > 0$. We denote it as $x_n \to x$.

(ii) The sequence $\{x_n\}$ is said to be ω_ν-Cauchy if for every $\eta \in \Lambda$, we have $\lim_{n,m\to\infty} \omega_\nu(x_n, x_m) = 0$ for some $\nu > 0$.

(iii) X_τ is said to be ω_ν-complete if each ω_ν-Cauchy sequence in X_τ is ω_ν-convergent in X_τ.

(iv) A subset F of X_τ is said to be ω_ν-closed if it contains the limit point of each ω_ν-convergent sequence of its elements.

(v) A subset F of X_τ is said to be ω_ν-bounded if we have
$\delta_\tau(F) := sup\{\omega_1(x, y) : x, y \in F, \eta \in \Lambda\} < \infty$.

Take a separating modular b-gauge space induced through the collection of pseudomodular b-metrics $\tau = \{\omega_\nu, \text{ with } s_\eta \geq 1 : \eta \in \Lambda\}$ on $X \neq \emptyset$ and $\{x_n\}$ is ω_ν-convergent in X_τ, then $\{x_n\}$ is ω_ν-convergent to a unique limit point.
Assume not; that is (x_n) converges to different elements, say $x_n \to a$ and $x_n \to b$. Then for every $\eta \in \Lambda$, there are $\sigma_1, \sigma_2 > 0$ such that

$\lim_{n\to\infty}\omega_{\sigma_1}(x_n,a)=0$ and $\lim_{n\to\infty}\omega_{\sigma_2}(x_n,b)=0$. By the triangular axiom, we obtain

$$\omega_{s_\eta\sigma_1+s_\eta\sigma_2}(a,b)\leq\omega_{\sigma_1}(a,x_n)+\omega_{\sigma_2}(x_n,b)$$

$\forall n\in N$ and $\eta\in\Lambda$. Thus $\lim_{n\to\infty}\omega_{s_\eta\sigma_1+s_\eta\sigma_2}(a,b)=0$. Since $\tau=\{\omega_\nu$, with $s_\eta\geq 1:\eta\in\Lambda\}$ is separating, hence we get $a=b$.

In the rest of this paper, we let Λ be an indexed set and $(X,\varsigma(\tau))$ be a modular b-gauge space and $\tau=\{\omega_\nu$, with $s_\eta\geq 1:\eta\in\Lambda\}$ satisfies the Fatou property and Δ_b-condition. Also, we let A be an ω_ν-bounded set which is ω_ν-complete of X_τ with respect to $\varsigma(\tau)$. Also, β denoted to a mapping from $A\times A$ into $[0,\infty)$. We denote the collection of nonempty ω_ν-closed subsets of A under the above modular b-gauge space by $CL(A)$.

Notation:
Consider the following:

- Φ_1 to be the family of all functions $\phi:[0,+\infty)\to[0,+\infty)$ such that: ϕ is continuous, nondecreasing and $\phi(t)\leq t$ for all $t\geq 0$.
- Φ_2 to be the family of all functions $\psi:[0,+\infty)\to[0,+\infty)$ such that: ψ is continuous, nondecreasing and $\psi(t)=0$ if and only if $t=0$.

3 Mains results

Theorem 3.1. *Let $T,S:A\to CL(A)$ be two maps. Suppose that for all $x,y\in A$ with $\beta(x,y)\geq 1$:*
If $u\in Tx$, there exists $v\in Sy$, or if $u\in Sx$, there exists $v\in Ty$ such that:

$$\omega_1(u,v)\leq C\max\Big\{\phi(\omega_1(x,y)),\phi(\omega_1(x,u)),\phi(\omega_1(y,v)),\psi\Big(\frac{\omega_{2s_\eta}(x,v)+\omega_1(y,u)}{2}\Big)\Big\}\quad(3.1)$$

for all $C\in[0,\frac{1}{s_\eta})$, $\forall\eta\in\Lambda$, where $(\phi,\psi)\in(\Phi_1,\Phi_2)$, $\psi(l)\leq\phi(l)$, $\forall l>0$. Assume we have the following hypotheses:

(i) There are three elements $x_0\in A$, $x_1\in Tx_0$ and $x_2\in Sx_1$ with $\beta(x_0,x_1)\geq 1$, $\beta(x_1,x_2)\geq 1$.

(ii) For $x\in A$, $y\in Tx$ and $z\in Sy$, we have $\beta(z,w)\geq 1$, $\beta(w,k)\geq 1$, for each $w\in Tz$, $k\in Sw$.

(iii) If $\{x_n\}$ is a sequence in A, with $x_n\to x\in A$ and $\beta(x_n,x_{n+1})\geq 1$ $\forall n\in N$, then $\beta(x_n,x)\geq 1$, $\forall n\in N$.

Then S and T have at least one common fixed point.

Proof. By hypothesis (i), there are three elements $x_0\in A$, $x_1\in Tx_0$ and $x_2\in Sx_1$ with $\beta(x_0,x_1)\geq 1$, $\beta(x_1,x_2)\geq 1$. By (3.1) we have

$$\omega_1(x_1,x_2)\leq C\max\Big\{\phi(\omega_1(x_0,x_1)),\phi(\omega_1(x_0,x_1)),\phi(\omega_1(x_1,x_2)),\psi\Big(\frac{\omega_{2s_\eta}(x_0,x_2)+\omega_1(x_1,x_1)}{2}\Big)\Big\}$$

$$\leq C\max\Big\{\phi(\omega_1(x_0,x_1)),\phi(\omega_1(x_1,x_2)),\psi\Big(\frac{\omega_1(x_0,x_1)+\omega_1(x_1,x_2)}{2}\Big)\Big\}.$$

Since ψ is a non decreasing function and $\psi(l)\leq\phi(l)$, $\forall l>0$, we get

$$\omega_1(x_1,x_2)\leq C\max\{\phi(\omega_1(x_0,x_1)),\phi(\omega_1(x_1,x_2))\}.\quad(3.2)$$

If we take $\max\{\phi(\omega_1(x_0,x_1)),\phi(\omega_1(x_1,x_2))\}=\phi(\omega_1(x_1,x_2))$, we obtain

$$\omega_1(x_1,x_2)\leq C\phi(\omega_1(x_1,x_2))<\omega_1(x_1,x_2),$$

a contradiction. Thus

$$\max\{\phi(\omega_1(x_0,x_1)),\phi(\omega_1(x_1,x_2))\}=\phi(\omega_1(x_0,x_1)).$$

From (3.2), we have

$$\omega_1(x_1,x_2)\le C\omega_1(x_0,x_1). \tag{3.3}$$

As $x_0\in A$, $x_1\in Tx_0$ and $x_2\in Sx_1$ with $\beta(x_0,x_1)\ge 1$, $\beta(x_1,x_2)\ge 1$. Then by hypothesis (ii) for $x_1\in Tx_0$ and $x_2\in Sx_1$, we have $\beta(x_2,x_3)\ge 1$, $\beta(x_3,x_4)\ge 1$ for each $x_3\in Tx_2$, $x_4\in Sx_3$. From (3.1), for $x_2\in Sx_1$ and $x_3\in Tx_2$ we have

$$\omega_1(x_2,x_3)\le C\max\Big\{\phi(\omega_1(x_1,x_2)),\phi(\omega_1(x_1,x_2)),\phi(\omega_1(x_2,x_3)),\psi\Big(\frac{\omega_{2s_\eta}(x_1,x_3)+\omega_1(x_2,x_2)}{2}\Big)\Big\}.$$

By the same method, we get

$$\omega_1(x_2,x_3)\le C\max\{\phi(\omega_1(x_1,x_2)),\phi(\omega_1(x_2,x_3))\}.$$

So

$$\omega_1(x_2,x_3)\le C\omega_1(x_1,x_2). \tag{3.4}$$

From (3.3) and (3.4), we have

$$\omega_1(x_2,x_3)\le C^2\omega_1(x_0,x_1).$$

Continuing this process, we construct a sequence $\{x_n\}$ in $\in A$ such that $x_{2n+1}\in Tx_{2n}$ and $x_{2n+2}\in Sx_{2n+1}$ with $\beta(x_{n+1},x_n)\ge 1$ and

$$\omega_1(x_n,x_{n+1})\le C^n\omega_1(x_0,x_1) \tag{3.5}$$

for each $n\in N$ and $\eta\in\Lambda$. By definition of Δ_b-condition and (3.5), we get $\omega_\nu(x_n,x_{n+1})\le C^n\omega_1(x_0,x_1)$ for all $\nu>0$ and $n\in N$. For each $i,j\in N$, we get

$$\omega_j(x_i,x_{i+j})\le\sum_{k=i}^{i+j-1}\omega_{\frac{1}{s_\eta^k}}(x_k,x_{k+1})\le\sum_{k=i}^{i+j-1}C^k\omega_1(x_0,x_1)$$
$$\le\sum_{k=i}^{\infty}C^k\omega_1(x_0,x_1)\to 0$$

as $i\to\infty$, $\forall\eta\in\Lambda$.

Hence $\{x_n\}$ is ω_ν-cauchy sequence in A.
Since A is ω_ν-complete, then there exist u^* such that $\forall\eta\in\Lambda$ we have $\lim_{n\to\infty}\omega_\nu(x_n,u^*)=0$ for some $\nu>0$. By definition of Δ_b-condition on X, we get $\lim_{n\to\infty}\omega_\sigma(x_n,u^*)=0$, for all $\sigma>0$.
Hypothesis (iii) yields $\beta(x_n,u^*)\ge 1$, $\forall n\in N$. From (3.1), for $\beta(x_n,u^*)\ge 1$ and $x_{2n+1}\in Tx_{2n}$ there is $v^*\in Su^*$ such that

$$\omega_1(x_{2n+1},v^*)\le C\max\Big\{\phi(\omega_1(x_{2n},u^*)),\phi(\omega_1(x_{2n},x_{2n+1})),\phi(\omega_1(u^*,v^*)),\psi\Big(\tfrac{\omega_{2s_\eta}(x_{2n},v^*)+\omega_1(u^*,x_{2n+1})}{2}\Big)\Big\}$$

$$\le C\max\Big\{\phi(\omega_1(x_{2n},u^*)),\phi(\omega_1(x_{2n},x_{2n+1})),\phi(\omega_1(u^*,v^*)),\psi\Big(\frac{\omega_1(x_{2n},u^*)+\omega_1(u^*,v^*)+\omega_1(u^*,x_{2n+1})}{2}\Big)\Big\}$$

$\forall\eta\in\Lambda$.
Letting $n\to\infty$. Then Fatou property implies that

$$\omega_1(u^*,v^*)\le C\frac{\omega_1(u^*,v^*)}{2},$$

this occurs only if $\omega_1(u^*,v^*)=0$. Since the collection $\tau=\{\omega_\nu$, with $s_\eta\ge 1:\forall\eta\in\Lambda\}$ is separating, then $u^*=v^*$. So $u^*\in Tu^*$. Thus T and S have at least a common fixed point. □

Theorem 3.2. *Let $T, S : A \to CL(A)$ be two maps. Suppose that for all $x, y \in A$ with $\beta(x, y) \geq 1$:*
If $u \in Tx$, there exists $v \in Sy$, or if $u \in Sx$, there exists $v \in Ty$ such that:

$$\omega_1(u,v) \leq a(\omega_1(x,y))\omega_1(x,y)+b(\omega_1(x,y))\left[\omega_1(x,u)+\omega_1(y,v)\right]+c(\omega_1(x,y))\left[\omega_{2s_\eta}(x,v)+\omega_1(y,u)\right] \tag{3.6}$$

$\forall \eta \in \Lambda$, where $a, b, c : R \to [0, 1)$ are functions, with $b(t) + c(t) < 1$, $\lim_{t\to 0} b(t) \neq 0$, $\lim_{t\to 0} c(t) \neq 0$, $\limsup_{s\to t} \frac{a(s)+b(s)+c(s)}{1-b(s)-c(s)} < 1$, $\forall t > 0$. Assume we have the following hypotheses:

- *(i) There are three elements $x_0 \in A$, $x_1 \in Tx_0$ and $x_2 \in Sx_1$ with $\beta(x_0, x_1) \geq 1$, $\beta(x_1, x_2) \geq 1$.*
- *(ii) For $x \in A$, $y \in Tx$ and $z \in Sy$, we have $\beta(z, w) \geq 1$, $\beta(w, k) \geq 1$, for each $w \in Tz$, $k \in Sw$.*
- *(iii) if $\{x_n\}$ is a sequence in A, with $x_n \to x \in A$ and $\beta(x_n, x_{n+1}) \geq 1$ $\forall n \in N$, then $\beta(x_n, x) \geq 1$, $\forall n \in N$.*

Then S and T have at least one common fixed point.

Proof. By hypothesis (i), there are three elements $x_0 \in A$, $x_1 \in Tx_0$ and $x_2 \in Sx_1$ with $\beta(x_0, x_1) \geq 1$, $\beta(x_1, x_2) \geq 1$. By (3.6), we have

$$\omega_1(x_1,x_2) \leq a(\omega_1(x_0,x_1))\omega_1(x_0,x_1) + b(\omega_1(x_0,x_1))\left[\omega_1(x_0,x_1)+\omega_1(x_1,x_2)\right]$$
$$+c(\omega_1(x_0,x_1))\left[\omega_{2s_\eta}(x_0,x_2)+\omega_1(x_1,x_1)\right]$$

$\forall \eta \in \Lambda$.

$$\leq \left[a(\omega_1(x_0,x_1))+b(\omega_1(x_0,x_1))+c(\omega_1(x_0,x_1))\right]\omega_1(x_0,x_1)+\left[b(\omega_1(x_0,x_1))+c(\omega_1(x_0,x_1))\right]\omega_1(x_1,x_2)$$

$$\omega_1(x_1,x_2) \leq \frac{a(\omega_1(x_0,x_1))+b(\omega_1(x_0,x_1))+c(\omega_1(x_0,x_1))}{1-b(\omega_1(x_0,x_1))-c(\omega_1(x_0,x_1))}\omega_1(x_0,x_1).$$

Let
$$K = \frac{a(\omega_1(x_n,x_{n+1}))+b(\omega_1(x_n,x_{n+1}))+c(\omega_1(x_n,x_{n+1}))}{1-b(\omega_1(x_n,x_{n+1}))-c(\omega_1(x_n,x_{n+1}))}$$
for all $n \in N$.
Then
$$\omega_1(x_1,x_2) \leq K\omega_1(x_0,x_1), \tag{3.7}$$
here $K < 1$.

Since $x_0 \in A$, $x_1 \in Tx_0$ and $x_2 \in Sx_1$ with $\beta(x_0, x_1) \geq 1$, $\beta(x_1, x_2) \geq 1$, then by hypothesis (ii) for $x_1 \in Tx_0$ and $x_2 \in Sx_1$, we have $\beta(x_2, x_3) \geq 1$, $\beta(x_3, x_4) \geq 1$ for each $x_3 \in Tx_2$, $x_4 \in Sx_3$. From (3.6), for $x_2 \in Sx_1$ and $x_3 \in Tx_2$ we have

$$\omega_1(x_2,x_3) \leq \left[a(\omega_1(x_1,x_2))+b(\omega_1(x_1,x_2))+c(\omega_1(x_1,x_2))\right]\omega_1(x_1,x_2)$$
$$+\left[b(\omega_1(x_1,x_2))+c(\omega_1(x_1,x_2))\right]\omega_1(x_2,x_3)$$

$$\omega_1(x_2,x_3) \leq \frac{a(\omega_1(x_1,x_2))+b(\omega_1(x_1,x_2))+c(\omega_1(x_1,x_2))}{1-b(\omega_1(x_1,x_2))-c(\omega_1(x_1,x_2))}\omega_1(x_1,x_2)$$

So
$$\omega_1(x_2,x_3) \leq K^2\omega_1(x_0,x_1)$$

Continuing this process, we get $\{x_n\} \in A$ such that $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$ with $\beta(x_{n+1}, x_n) \geq 1$. and
$$\omega_1(x_n,x_{n+1}) \leq K^n\omega_1(x_0,x_1) \tag{3.8}$$

for each $n \in N$, and $\eta \in \Lambda$.
By defintion of Δ_b-condition and (3.8), we get $\omega_\nu(x_n, x_{n+1}) \leq k^n \omega_1(x_0, x_1)$, for all $\nu > 0$ and $n \in N$. For each $i, j \in N$, we get

$$\omega_j(x_i, x_{i+j}) \leq \sum_{k=i}^{i+j-1} \omega_{\frac{1}{s_\eta^k}}(x_k, x_{k+1}) \leq \sum_{k=i}^{i+j-1} K^k \omega_1(x_0, x_1)$$

$$\leq \sum_{k=i}^{\infty} K^k \omega_1(x_0, x_1) \to 0$$

as $i \to \infty$, $\forall \eta \in \Lambda$.

Hence $\{x_n\}$ is ω_ν-cauchy sequence in A.
Since A is ω_ν-complete, there exist u^* such that $\forall \eta \in \Lambda$ we have $\lim_{n\to\infty} \omega_\nu(x_n, u^*) = 0$ for some $\nu > 0$. Also, Δ_b-condition on X implies that $\lim_{n\to\infty} \omega_\sigma(x_n, u^*) = 0$ for all $\sigma > 0$. Hypothesis (iii) yields $\beta(x_n, u^*) \geq 1$ $\forall n \in N$. From (3.6), for $\beta(x_n, u^*) \geq 1$ and $x_{2n+1} \in Tx_{2n}$ there is $v^* \in Su^*$ such that

$$\omega_1(x_{2n+1}, v^*) \leq a(\omega_1(x_{2n}, u^*))\omega_1(x_{2n}, u^*) + b(\omega_1(x_{2n}, u^*)) \left[\omega_1(x_{2n}, x_{2n+1}) + \omega_1(u^*, v^*)\right]$$

$$+c(\omega_1(x_{2n}, u^*)) \left[\omega_{2s_\eta}(x_{2n}, v^*) + \omega_1(u^*, x_{2n+1})\right]$$

$\forall \eta \in \Lambda$.

$n \to \infty$. Then Fatou property implies that

$$\omega_1(u^*, v^*) \leq (b(s) + c(s))\, \omega_1(u^*, v^*).$$

The last inequality is true only if $\omega_1(u^*, v^*) = 0$. Since the collection $\tau = \{\omega_\nu$, with $s_\eta \geq 1 : \forall \eta \in \Lambda\}$ is separating, then $u^* = v^*$. So $u^* \in Tu^*$. So, we conclude that T and S have at least one common fixed point.

□

If we take $T, S : A \to A$, $\beta(x, y) = 1$ in the above theorems we get the following corollaries:

Corollary 3.3. *Let $T, S : A \to A$ be two maps. Suppose there exists $C \in [0, \frac{1}{s_\eta})$ such that have*

$$\omega_1(Tx, Sy) \leq C \max\left\{\phi(\omega_1(x, y)), \phi(\omega_1(x, Tx)), \phi(\omega_1(y, Sy)), \psi\Big(\frac{\omega_{2s_\eta}(x, Sy) + \omega_1(y, Tx)}{2}\Big)\right\}$$

holds for all $x, y \in A$, $\forall \eta \in \Lambda$, where $(\phi, \psi) \in (\Phi_1, \Phi_2)$, $\psi(l) \leq \phi(l)$, $\forall l > 0$.
Then S and T have at least one common fixed point.

By taking $a(t) = q$, $b(t) = c(t) = \frac{q}{2}$, $q \in R^*$, and $q \leq \frac{1}{4}$ in Theorem (3.2), then we have:

Corollary 3.4. *Let $T, S : A \to A$ be two maps. Suppose for all $x, y \in A$, we have*

$$\omega_1(Tx, Sy) \leq q\left[\omega_1(x, y) + \frac{\omega_1(x, Tx) + \omega_1(y, Sy) + \omega_{2s_\eta}(x, Sy) + \omega_1(y, Tx)}{2}\right]$$

$\forall \eta \in \Lambda$. *Then S and T have at least one common fixed point.*

Example 3.5. Consider A the collection of real sequence with $\omega_\nu(x, y) = \frac{1}{[\nu]}|x_n - y_n|^2$ for all $n \in N$ and $\nu > 0$, such that $x = \{x_n\}$, $y = \{y_n\}$. Take $T, S : A \to CL(A)$ the mappings defined as follows:

$$T(\{x_n\}_{n\in N}) = \begin{cases} \{\frac{x_n}{3}\} & \text{for} \quad \{x_n\}_{n\in N} \subseteq [0, \infty) \\ 0 & otherwise \end{cases}$$

$$S(\{x_n\}_{n\in N}) = \begin{cases} \{\frac{x_n}{2}\} & \text{for} \quad \{x_n\}_{n\in N} \subseteq [0, \infty) \\ 0 & otherwise \end{cases}$$

and $\beta : A \times A \to [0, \infty)$ such that:

$$\beta(\{x_n\}_{n\in N}, \{y_n\}_{n\in N}) = \begin{cases} 1 & \text{for} \quad \{x_n\}_{n\in N}, \{y_n\}_{n\in N} \subseteq [0,\infty) \\ 0 & otherwise \end{cases}$$

We have (3.1) satisfied for $\phi(t) = \psi(t) = t$, for all $x, y \in A$ with $\beta(x, y) = 1$, where $C = \frac{1}{9}$, $s_\eta = 2$. Also for $x_0 = \{n\}_{n\in N} \in A$, we get $x_1 = \{\frac{n}{3}\}_{n\in N} \in Tx_0$, we have $\{x_2 = \frac{n}{6}\}_{n\in N} \in Sx_1$, with $\beta(\{n\}, \{\frac{n}{3}\}) = 1$, $\beta(\{\frac{n}{6}\}, \{\frac{n}{3}\}) = 1$ such that:

$\omega_1(u,v) = \omega_1(\frac{n}{3}, \frac{n}{6}) = |\frac{n}{3} - \frac{n}{6}|^2 = \frac{1}{9}|n - \frac{n}{2}|^2 \leq \frac{1}{9}|n - \frac{n}{3}|^2 = \frac{1}{9}\phi(\omega_1(x,y))$.
$\omega_1(u,v) = \omega_1(\frac{n}{3}, \frac{n}{6}) = \frac{1}{9}|n - \frac{n}{2}|^2 \leq \frac{1}{9}|n - \frac{n}{3}|^2 = \frac{1}{9}\phi(\omega_1(x,u))$.
$\omega_1(u,v) = \omega_1(\frac{n}{3}, \frac{n}{6}) = \phi(\omega_1(y,v))$.

$$\begin{aligned} \omega_1(u,v) &= \frac{1}{9}|n - \frac{n}{2}|^2 \\ &\leq \frac{1}{9}|n - \frac{n}{3}|^2 \\ &\leq \frac{1}{8}|n - \frac{n}{3}|^2 = \frac{1}{2}(\frac{1}{4}|n - \frac{n}{3}|^2) \\ &\leq \frac{1}{2}(\frac{1}{4}|n - \frac{n}{6}|^2) = \frac{1}{2}\omega_{2s}(x,v) = \psi\Big(\frac{\omega_{2s_\eta}(x,v) + \omega_1(y,u)}{2}\Big). \end{aligned}$$

Then

$$\omega_1(u,v) \leq C \max\Big\{\phi(\omega_1(x,y)), \phi(\omega_1(x,u)), \phi(\omega_1(y,v)), \psi\Big(\frac{\omega_{2s_\eta}(x,v) + \omega_1(y,u)}{2}\Big)\Big\}.$$

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Received: November 15, 2020
Accepted: January 16, 2021