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APPLICATIONS OF SUMUDU TRANSFORM TO ECONOMIC MODELS

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Abstract Sumudu transform technique is implemented to study qualitative solution of nonlinear fractional differential equations. Economic models such as price adjustment equation involving Caputo derivative, Caputo-Fabrizio derivative and Atangana-Balaenu-Caputo derivatives under the assumption that market is in equilibrium are studied. These models are analyzed through simulations for different fractional orders with suitable parameters.

1 Introduction

Mathematical language helps to express meaningful and reviewable in psychology, political science, physics, chemistry and economics. Many economic models get explained using mathematical tools making it possible to solve a wide number of applications including optimization equilibrium static analysis, comparative static and dynamic analysis are observed. A range of new means in differential calculus are used in the matters to foreground economic models. Mathematical economic models are being used by the economists which enable them to predict optimal profit to view the link between demand and supply. The concept of price adjustment is determined by inefficient competitive company lower demand and different price. A market structure in which price adjustment is free from individual behavior and satisfaction consists of a great number of economic agents competing with each other can be referred for more information about price adjustment. Every economic model targets maximum utility for buyers, maximum profit for sellers and liberty of pricing restricted in an equilibrium model has its own elements such as goods, companies, individuals etc. Economics provides the interactions between price, supply and demand, the dependence of supply and demand on price and also how equilibrium point is reached on supply and demand curves. The object of mathematical economics is to formulate an economic process in mathematical form so that the above mentioned interactions can be understood [9, 18].

$$q_d(x) = d_0 - d_1 w(x); \qquad q_v(x) = -v_0 + v_1 w(x)$$

where w the price of goods d_0, v_0, d_1, v_1 are positive constants.
For $q_d(x) = q_v(x)$, where the demanded quantity and supplied quantity are equal, we obtain the equilibrium price as $w = \frac{d_0+v_0}{d_1+v_1}$. In such case, the price disposed to stay stable and in economics there is no shortage and surplus.
Now consider the price adjustment equation as follows:

$$w'(x) = p(q_d - q_v)$$

where $p > 0$, is the speed of adjustment constant. That is,

$$w'(x) + p(d_1 + v_1)w(x) = p(d_0 + v_0)$$

The solution of linear differential equation is

$$w(x) = \frac{(d_0 + v_0)}{(d_1 + v_1)} + \left[w(0) - \frac{(d_0 + v_0)}{(d_1 + v_1)}\right] e^{-p(d_1+v_1)x}$$

where $w(0)$ is the price at the time $x = 0$ and here we do not attention to the expectation of agents in market.
From the above equation, if we consider the prospects of agents, the request and supply functions involving additional factors d_2 and v_2 alters as:

$$q_d(x) = d_0 - d_1 w(x) + d_2 w'(x); \qquad q_v(x) = v_0 - v_1 w(x) - v_2 w'(x).$$

Equating $q_d(x)$ and $q_v(x)$, we obtain

$$w'(x) - \frac{(d_1 + v_1)}{(d_2 + v_2)} w(x) = \frac{(d_0 + v_0)}{(d_2 + v_2)}$$

The solution of above linear differential equation is

$$w(x) = \frac{(d_0 + v_0)}{(d_2 + v_2)} + \left[w(0) - \frac{(d_0 + v_0)}{(d_1 + v_1)} \right] e^{\frac{(d_1+v_1)}{(d_2+v_2)} x}$$

If the price of goods raise, buyers itch to purchase more before price rises to break even and accordingly vendors disposed to offer less in order to make a earn higher prices in later times. The model under study has the condition $q_d(x) = q_v(x)$. Furthermore, when $w'(x) = 0$ for all $x > 0$, market is in a changing economy which means dynamic equilibrium.
In recent years considerable interest in fractional calculus has been stimulated by the applications in different fields of science, including numerical analysis, economics and finance, engineering, physics, nanotechnology, bioengineering etc.[15] which is being used widely applied by researchers such as Caputo[8], Baleanu et.al[4], Miller and Ross, Podlubny[15]. The integral transform method is an efficient method to solve the differential equations [14, 15]. Integral transforms were extensively used in several works on the theory and application of integral transforms such as Laplace, Sumudu, Fourier, Mellin [5, 6, 19, 21]. The Sumudu transform is one of the best integral transform. Birajdar et al. [12] obtained analytical Solution of ψ fractional initial value problems by using sumudu transform. The economic models with Caputo derivative [10], Caputo-Fabrizio [17] and Atangana-Baleanu fractional derivatives [7, 20] have been presented recently. The economic model such as price adjustment equation was studied by Acay et.al. using Laplace transform technique [2]. The Sumudu transform method to solve economic models were studied [13, 19] by Goswami et.al. and Watagula. This motivates us to consider an economic model with Caputo derivative, Caputo-Fabrizio (CFC) and Atangana-Baleanu derivatives (ABC)for price adjustment equation associated with demand and supply in market. In this paper we study economic models which involve Caputo, CFC, ABC type fractional derivatives using Sumudu transform method.
We organize the paper as follows: Some fundamental definitions and theorems are presented in section 2. In section 3 Caputo, CFC, and ABC economic models are solved by impact of the non-local fractional operators. In section 4, comparative analysis and discussion. In section 5, Graphics and finally results concluded some significant discussions.

2 Preliminaries

In this section we consider some definitions, theorems, properties and results required in further sections.

Definition 2.1. The classical Mittag-Leffler function with one parameter $E_\alpha(v)$ is

$$E_\alpha(v) = \sum_{k=0}^{\infty} \frac{v^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0$$

Definition 2.2. The classical Mittag-Leffler function with two parameter is $E_{\alpha,\beta}(v)$ is

$$E_{\alpha,\beta}(v) = \sum_{k=0}^{\infty} \frac{v^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in C, \quad Re(\alpha) > 0$$

Definition 2.3. The left-sided and right-sided Caputo-fractional derivatives of order α are defined by

$$ {}_a^cD^{\alpha}w(x) = \frac{1}{\Gamma(1-\alpha)}\int_a^x (x-\tau)^{-\alpha}w'(\tau)d\tau $$

and

$$ {}^cD_b^{\alpha}w(x) = \frac{(-1)}{\Gamma(1-\alpha)}\int_x^b (\tau-x)^{\alpha}w'(\tau)d\tau $$

where $0<\alpha<1$

Theorem 2.4. *[6]The Sumudu transform of Caputo fractional derivative is defined by*

$$ S\big[{}_0^cD^{\alpha}w(x)\big](u) = u^{-\alpha}S[w(x)] - u^{-\alpha}w(0) $$

Definition 2.5. Let $\eta,\psi:[0,\infty)\to R$, then classical convolution product is

$$ (\eta*\psi)(x) = \int_0^x \eta(x-s)\psi(s)ds $$

Proposition 2.6. *Let $\eta,\psi:[0,\infty)\to R$, then the following property is valid*

$$ \begin{aligned} S[(\eta*\psi)(x)] &= uS[\eta(x)]S[\psi(x)] \\ &= u\eta(u).\psi(u) \end{aligned} $$

Definition 2.7. The left-sided and right-sided Caputo-Fabrizio fractional derivatives in the Caputo sense of order α are defined by

$$ {}_a^{CFC}D^{\alpha}w(x) = \frac{M(\alpha)}{(1-\alpha)}\int_a^x w'(\tau)e^{\mu(x-\tau)}d\tau $$

and

$$ {}^{CFC}D_b^{\alpha}w(x) = -\frac{M(\alpha)}{(1-\alpha)}\int_x^b w'(\tau)e^{\mu(\tau-x)}d\tau $$

where $0<\alpha<1, M(\alpha)$ is normalization function and $\mu=-\frac{\alpha}{1-\alpha}$

Theorem 2.8. *[11] The Sumudu transform of CFC fractional derivative $w(x)$ is*

$$ S\bigg[{}_0^{CFC}D_x^{\alpha}w(x)\bigg] = \frac{M(\alpha)}{(1-\alpha+\alpha u)}\bigg[S[w(x)]-w(0)\bigg] $$

Definition 2.9. The left-sided and right-sided Atangana-Baleanu fractional derivatives in the Caputo sense of order α are defined by

$$ {}_a^{ABC}D^{\alpha}w(x) = \frac{B(\alpha)}{(1-\alpha)}\int_a^x w'(\tau)E_{\alpha}[\mu(x-\tau)]d\tau $$

and

$$ {}^{ABC}D_b^{\alpha}w(x) = -\frac{B(\alpha)}{(1-\alpha)}\int_x^b w'(\tau)E_{\alpha}[\mu(\tau-x)]d\tau $$

where $0<\alpha<1, B(\alpha)$ is normalization function and $\mu=-\frac{\alpha}{1-\alpha}$

Theorem 2.10. *[3] The Sumudu transform of ABC fractional derivative is defined by*

$$ S\bigg[{}_0^{ABC}D_x^{\alpha}w(x)\bigg] = \frac{B(\alpha)}{(1-\alpha+\alpha u^{\alpha})}\left[S[w(x)-w(0)]\right] $$

Lemma 2.11. *The Sumudu transform of certain functions holds:*

(i) $S\big[E_\alpha(-ax^\alpha)\big] = \frac{1}{1+au^\alpha}$

(ii) $S\big[1-E_\alpha(-ax^\alpha)\big] = \frac{au^\alpha}{1+au^\alpha}$

(iii) $S\big[x^{\alpha-1}E_{\alpha,\alpha}(-ax^\alpha)\big] = \frac{u^{\alpha-1}}{1+au^\alpha}$

Proof. Since

$$\sum_{k=0}^{\infty}\frac{(k+m)!}{k!}x^k = \frac{m!}{(1-x)^{(m+1)}}$$

(i)

$$\begin{aligned} S[E_\alpha(-ax^\alpha)] &= \sum_{k=0}^{\infty}\frac{(k+m)!}{k!}(-a)^k.\frac{S[x^{\alpha k}]}{\Gamma(\alpha k+1)} \\ &= \sum_{k=0}^{\infty}\frac{(k+m)!}{k!}(-a)^k.\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+1)}.u^{\alpha k} \\ &= \sum_{k=0}^{\infty}\frac{(k+m)!}{k!}(-a)^k.u^{\alpha k} \\ \therefore S[E_\alpha(-ax^\alpha)] &= \frac{1}{1+au^\alpha} \qquad (\because m=0) \end{aligned}$$

(ii)

$$\begin{aligned} S[1-E_\alpha(-ax^\alpha)] &= 1-\frac{1}{1+au^\alpha} \\ &= \frac{au^\alpha}{1+au^\alpha} \end{aligned}$$

(iii)

$$\begin{aligned} S[x^{\alpha-1}E_{\alpha,\alpha}(-ax^\alpha)] &= \sum_{k=0}^{\infty}\frac{(k+m)!}{k!}(-a)^k.\frac{S[x^{\alpha k+\alpha-1}]}{\Gamma(\alpha p+\alpha)} \\ &= \sum_{k=0}^{\infty}\frac{(k+m)!}{k!}(-a)^k.\frac{\Gamma(\alpha k+\alpha)}{\Gamma(\alpha k+\alpha)}.u^{\alpha k+\alpha-1} \\ &= u^{\alpha-1}\sum_{k=0}^{\infty}\frac{(k+m)!}{k!}(-a)^k.u^{\alpha k} \\ \therefore S[x^{\alpha-1}E_{\alpha,\alpha}(-ax^\alpha)] &= \frac{u^{\alpha-1}}{1+au^\alpha} \qquad (\because m=0) \end{aligned}$$

3 Economic models

The purpose of this section is to obtain the solution for economic models involving Caputo derivative, Caputo-Fabrizio in Caputo sense, Atangana-Baleanu in Caputo sense. In addition to get solution of after-said models having non-local properties we also give the special importance to condition when solving the constant coefficient linear differential equation with initial condition in frame of CFC or ABC. However we can find solution for Caputo derivative case.

3.1 Economic model involving Caputo derivative

Case-I. The price adjustment equation by means of Caputo fractional derivative without considering the expectations of agents is as follows:

$$ {}_a^cD^\alpha w(x) + p(d_1+v_1)w(x) = p(d_0+v_0), \alpha \in (0,1) $$

Applying the Sumudu transform on both sides, we obtain

$$ S\big[{}_a^cD^\alpha w(x)\big] + p(d_1+v_1)S[w(x)] = pS[(d_0+v_0)] $$

By Theorem 2.4, for $a=0$, we obtain

$$ u^{-\alpha}S[w(x)] - u^{-\alpha}w(0) + p(d_1+v_1)S[w(x)] = p[(d_0+v_0)] $$

$$ S[w(x)] = \frac{u^{-\alpha}w(0)}{u^{-\alpha}+p(d_1+v_1)} + \frac{p(d_0+v_0)}{u^{-\alpha}+p(d_1+v_1)} $$

Taking inverse Sumudu transform of both sides

$$ w(x) = S^{-1}\left[\frac{u^{-\alpha}}{u^{-\alpha}+p(d_1+v_1)}\right]w(0) + S^{-1}\left[\frac{p(d_0+v_0)}{u^{-\alpha}+p(d_1+v_1)}\right] $$

$$ = S^{-1}\left[\frac{1}{1+p(d_1+v_1)u^\alpha}\right]w(0) + \frac{(d_0+v_0)}{(d_1+v_1)}S^{-1}\left[\frac{p(d_1+v_1)u^\alpha}{1+p(d_1+v_1)u^\alpha}\right] $$

By Lemma 2.11, we have

$$ w(x) = w(0)E_\alpha[-p(d_1+v_1)x^\alpha] + \frac{(d_0+v_0)}{(d_1+v_1)}[1 - E_\alpha[-p(d_1+v_1)x^\alpha] $$

where $E_\alpha(.)$ is the Mittag-Leffler function.

Case-II. If we consult the expectations of agent, the price adjustment equation with Caputo fractional derivative is given by

$$ {}_a^cD^\alpha w(x) - \frac{(d_1+v_1)}{(d_2+v_2)}w(x) = -\frac{(d_0+v_0)}{(d_2+v_2)} $$

Applying the Sumudu transform of both sides, we obtain

$$ S\big[{}_0^cD^\alpha w(x)\big] - \frac{(d_1+v_1)}{(d_2+v_2)}S[w(x)] = S\big[-\frac{(d_0+v_0)}{(d_2+v_2)}\big] $$

By Theorem 2.4, for $a=0$, we obtain

$$ u^{-\alpha}S[w(x)] - u^{-\alpha}w(0) - \frac{(d_1+v_1)}{(d_2+v_2)}S[w(x)] = \big[-\frac{(d_0+v_0)}{(d_2+v_2)}\big] $$

$$ S[w(x)] = \left[\frac{u^{-\alpha}}{\big[u^{-\alpha} - \frac{(d_1+v_1)}{(d_2+v_2)}\big]}\right]w(0) - \left[\frac{\frac{-\frac{(d_0+v_0)}{(d_2+v_2)}\big]}{u^{-\alpha}}}{\big[u^{-\alpha} - \frac{(d_1+v_1)}{(d_2+v_2)}}\right] $$

$$ = \left[\frac{1}{1-\frac{(d_1+v_1)}{(d_2+v_2)}u^\alpha}\right] - \frac{(d_0+v_0)}{(d_1+v_1)}\left[\frac{\frac{(d_1+v_1)}{(d_2+v_2)}}{1-\frac{(d_1+v_1)}{(d_2+v_2)}u^\alpha}\right] $$

Taking inverse Sumudu transform of both sides, we obtain

$$w(x) = S^{-1}\left[\frac{1}{1-\frac{(d_1+v_1)}{(d_2+v_2)}u^\alpha}\right] - \frac{(d_0+v_0)}{(d_1+v_1)}S^{-1}\left[\frac{\frac{(d_1+v_1)}{(d_2+v_2)}}{1-\frac{(d_1+v_1)}{(d_2+v_2)}u^\alpha}\right]$$

By Lemma 2.11, we have

$$w(x) = w(0)E_\alpha\left[\frac{(d_1+v_1)}{(d_2+v_2)}x^\alpha\right] - \frac{(d_0+v_0)}{(d_1+v_1)}\left[1-E_\alpha\left(\frac{d_1+v_1}{d_2+v_2}x^\alpha\right)\right]$$

where $w(0) = \frac{(d_0+v_0)}{(d_1+v_1)}$

3.2 Economic model with the CFC

Case-I. The price adjustment equation by means of CFC without considering the expectations of agents is as follows:

$${}_a^{CFC}D^\alpha w(x) + p(d_1+v_1)w(x) = p(d_0+v_0)$$

Applying the Sumudu transform of both sides

$$S\left[{}_a^{CFC}D_x^\alpha w(x)\right] + p(d_1+v_1)S[w(x)] = pS[(d_0+v_0)]$$

By Theorem 2.8, for $a = 0$, we obtain

$$\frac{M(\alpha)}{1-\alpha}.\frac{S[w(x)]}{1+\frac{\alpha u}{1-\alpha}} - \frac{M(\alpha)}{1-\alpha}.\frac{w(0)}{1+\frac{\alpha u}{1-\alpha}} + p(d_1+v_1)S[w(x)] = p[(d_0+v_0)]$$

$$S[w(x)]\left[\frac{M(\alpha)}{1-\alpha}.\frac{1}{1+\frac{\alpha u}{1-\alpha}} + p(d_1+v_1)\right] = p(d_0+v_0) + \frac{M(\alpha)}{1-\alpha}.\frac{1}{1+\frac{\alpha u}{1-\alpha}}w(0)$$

$$S[w(x)] = \frac{\frac{M(\alpha)}{1-\alpha}\cdot\frac{1}{1+\frac{\alpha u}{1-\alpha}}w(0)}{\frac{M(\alpha)}{1-\alpha}\cdot\frac{1}{1+\frac{\alpha u}{1-\alpha}} + p(d_1+v_1)} + \frac{p(d_0+v_0)}{\frac{M(\alpha)}{1-\alpha}\cdot\frac{1}{1+\frac{\alpha u}{1-\alpha}} + p(d_1+v_1)}$$

Applying inverse Sumudu transform, we have

$$\begin{aligned} w(x) &= S^{-1}\left[\frac{\frac{M(\alpha)}{1-\alpha}\cdot\frac{1}{1+\frac{\alpha u}{1-\alpha}}w(0)}{\frac{M(\alpha)}{1-\alpha}\cdot\frac{1}{1+\frac{\alpha u}{1-\alpha}} + p(d_1+v_1)}\right] + S^{-1}\left[\frac{p(d_0+v_0)}{\frac{M(\alpha)}{1-\alpha}\cdot\frac{1}{1+\frac{\alpha u}{1-\alpha}} + p(d_1+v_1)}\right] \\ &= S^{-1}\left[\frac{\frac{M(\alpha)w(0)}{1-\alpha+\alpha u}}{\frac{M(\alpha)-(\alpha-1-\alpha u)+p(d_1+v_1}{1-\alpha+\alpha u}}\right] + S^{-1}\left[-\frac{p(d_0+v_0)(\alpha-1-\alpha u)}{M(\alpha)-(\alpha-1-\alpha u)+p(d_1+v_1)}\right] \end{aligned}$$

By Lemma 2.11, we have

$$\begin{aligned} w(x) &= \frac{w(0)M(\alpha)e^{\frac{\alpha p(d_1+v_1)x}{-M(\alpha)+(\alpha-1)p(d_1+v_1)}}}{M(\alpha)-(\alpha-1)p(d_1+v_1)} \\ &\quad - \frac{(d_0+v_0)(M(\alpha)(-1+e^{\frac{\alpha p(d_1+v_1)x}{-M(\alpha)+(\alpha-1)p(d_1+v_1)}} + (\alpha-1)p(d_1+v_1))}{(d_1+v_1)(-M(\alpha)+(\alpha-1)p(d_1+v_1))} \\ &= w(0)\left[\frac{M(\alpha)\big(-2e^{\frac{\alpha p(d_1+v_1)x}{\lambda}}+1\big)}{\lambda} - \frac{(\alpha-1)p(d_1+v_1)}{\lambda}\right] \\ &= w(0)\left[\frac{M(\alpha)\big(-2e^{\frac{\alpha p(d_1+v_1)x}{\lambda}}+1\big)-(\alpha-1)p(d_1+v_1)}{\lambda}\right] \end{aligned}$$

where $w(0) = \frac{d_0+v_0}{d_1+v_1}, \lambda = -M(\alpha) + (\alpha-1)p(d_1+v_1)$

Case-II. If we consider the expectations of agents the price adjustment equation with CFC is given by

$$\left[{}^{CFC}_{a} D_x^{\alpha} w(x)\right] - \frac{(d_1+v_1)}{(d_2+v_2)} w(x) = -\frac{(d_0+v_0)}{(d_2+v_2)}$$

Applying the Sumudu transform of both sides, we have

$$S\left[{}^{CFC}_{a} D_x^{\alpha} w(x)\right] - \frac{(d_1+v_1)}{(d_2+v_2)} S[w(x)] = S\left[-\frac{(d_0+v_0)}{(d_2+v_2)}\right]$$

By Theorem 2.8, for $a=0$, we obtain

$$\frac{M(\alpha)}{1-\alpha}.\frac{S[w(x)]}{1+\frac{\alpha u}{1-\alpha}} - \frac{M(\alpha)}{1-\alpha}.\frac{w(0)}{1+\frac{\alpha u}{1-\alpha}} - \frac{(d_1+v_1)}{(d_2+v_2)} S[w(x)] = \left[-\frac{(d_0+v_0)}{(d_2+v_2)}\right]$$

$$S[w(x)]\left[\frac{M(\alpha)}{1-\alpha}.\frac{1}{1+\frac{\alpha u}{1-\alpha}} - \frac{(d_1+v_1)}{(d_2+v_2)}\right] = \frac{M(\alpha)}{1-\alpha}.\frac{w(0)}{1+\frac{\alpha u}{1-\alpha}} - \frac{(d_0+v_0)}{(d_2+v_2)}$$

$$S[w(x)] = \left[\frac{\frac{M(\alpha)}{1-\alpha}\frac{w(0)}{1+\frac{\alpha u}{1-\alpha}}}{\frac{M(\alpha)}{1-\alpha}\frac{1}{1+\frac{\alpha u}{1-\alpha}} - \frac{(d_1+v_1)}{(d_2+v_2)}}\right] + \left[\frac{-\frac{(d_0+v_0)}{(d_2+v_2)}}{\frac{M(\alpha)}{1-\alpha}\frac{1}{1+\frac{\alpha u}{1-\alpha}} - \frac{(d_1+v_1)}{(d_2+v_2)}}\right]$$

Applying inverse Sumudu transform of both sides

$$w(x) = S^{-1}\left[\frac{\frac{M(\alpha)}{1-\alpha}\frac{w(0)}{1+\frac{\alpha u}{1-\alpha}}}{\frac{M(\alpha)}{1-\alpha}\frac{1}{1+\frac{\alpha u}{1-\alpha}} - \frac{(d_1+v_1)}{(d_2+v_2)}}\right] + S^{-1}\left[\frac{-\frac{(d_0+v_0)}{(d_2+v_2)}}{\frac{M(\alpha)}{1-\alpha}\frac{1}{1+\frac{\alpha u}{1-\alpha}} - \frac{(d_1+v_1)}{(d_2+v_2)}}\right]$$

By Lemma 2.11, we have

$$\begin{aligned} w(x) &= \frac{w(0)M(\alpha)e^{\frac{\alpha(d_1+v_1)x}{(\alpha-1)(d_1+v_1)+M(\alpha)(d_2+v_2)}}(d_2+v_2)}{(\alpha-1)(d_1+v_1)+M(\alpha)(d_2+v_2)} \\ &\quad + \frac{(d_0+v_0)(\alpha-1)(d_1+v_1) - M(\alpha)(-1+e^{\frac{\alpha(d_1+v_1)x}{(\alpha-1)(d_1+v_1)+M(\alpha)(d_2+v_2)}}(d_2+v_2))}{(d_1+v_1)((\alpha-1)(d_1+v_1)+M(\alpha)(d_2+v_2))} \\ &= w(0)\left[\frac{M(\alpha)e^{\frac{\alpha(d_1+v_1)x}{\sigma}}(d_2+v_2)}{\sigma} + \frac{(\alpha-1)(d_1+v_1)}{\sigma} + \frac{M(\alpha)}{\sigma} - \frac{M(\alpha)e^{\frac{\alpha(d_1+v_1)x}{\sigma}}(d_2+v_2)}{\sigma}\right] \\ &= w(0)\left[\frac{(\alpha-1)(d_1+v_1)+M(\alpha)}{\sigma}\right] \end{aligned}$$

where $w(0) = \frac{(d_0+v_0)}{(d_1+v_1)}, \sigma = (\alpha-1)(d_1+v_1) + M(\alpha)$

3.3 Economic model with the ABC

Case-I. The price of adjustment equation by means of ABC without considering the expectations of agent is as follows

$$\left[{}^{ABC}_{a} D_x^{\alpha} w(x)\right] + p(d_1+v_1)w(x) = p(d_0+v_0)$$

Applying Sumudu transform, we have

$$S\left[{}^{ABC}_{a} D_x^{\alpha} w(x)\right] + p(d_1+v_1)S[w(x)] = S[p(d_0+v_0)]$$

By Theorem 2.10, when $a = 0$, we obtain

$$\frac{B(\alpha)}{1-\alpha}\left[\frac{S[w(x)]}{1+\frac{\alpha u^{\alpha}}{1-\alpha}} - \frac{w(0)}{1+\frac{\alpha u^{\alpha}}{1-\alpha}}\right] + p(d_1+v_1)S[w(x)] = p(d_0+v_0)$$

$$S[w(x)]\left[\frac{B(\alpha)}{1-\alpha}\frac{1}{1+\frac{\alpha u^{\alpha}}{1-\alpha}} + p(d_1+v_1)\right] = \left[\frac{B(\alpha)}{1-\alpha}\frac{w(0)}{1+\frac{\alpha u^{\alpha}}{1-\alpha}} + p(d_0+v_0)\right]$$

$$\begin{aligned} S[w(x)] &= \frac{\frac{B(\alpha)}{1-\alpha}\frac{w(0)}{1+\frac{\alpha u^{\alpha}}{1-\alpha}}}{\frac{B(\alpha)}{1-\alpha}\frac{1}{1+\frac{\alpha u^{\alpha}}{1-\alpha}} + p(d_1+v_1)} + \frac{p(d_0+v_0)}{\frac{B(\alpha)}{1-\alpha}\frac{w(0)}{1+\frac{\alpha u^{\alpha}}{1-\alpha}} + p(d_1+v_1)} \\ &= \frac{\frac{B(\alpha)w(0)}{1-\alpha+\alpha u^{\alpha}}}{\frac{B(\alpha)+p(d_1+v_1)(1-\alpha+\alpha u^{\alpha})}{1-\alpha+\alpha u^{\alpha}}} + \frac{p(d_0+v_0)(1-\alpha+\alpha u^{\alpha})}{B(\alpha)+p(d_1+v_1)(1-\alpha+\alpha u^{\alpha})} \end{aligned}$$

$$\begin{aligned} S[w(x)] = &\frac{B(\alpha)w(0)}{B(\alpha)+p(d_1+v_1)(1-\alpha)}.\frac{1}{\left(1+\frac{p(d_1+v_1)\alpha u^{\alpha}}{B(\alpha)+p(d_1+v_1)(1-\alpha)}\right)} \\ &+ \frac{p(d_0+v_0)(1-\alpha)}{B(\alpha)+p(d_1+v_1)(1-\alpha)}.\frac{1}{\left(1+\frac{p(d_1+v_1)\alpha u^{\alpha}}{B(\alpha)+p(d_1+v_1)(1-\alpha)}\right)} \\ &+ \frac{(d_0+v_0)\alpha}{B(\alpha)+p(d_1+v_1)(1-\alpha)}.\frac{u^{\alpha}}{\left(1+\frac{p(d_1+v_1)\alpha u^{\alpha}}{B(\alpha)+p(d_1+v_1)(1-\alpha)}\right)} \end{aligned}$$

Applying inverse Sumudu transform, we have

$$\begin{aligned} w(x) = &\frac{B(\alpha)w(0)}{B(\alpha)+p(d_1+v_1)(1-\alpha)}S^{-1}\left[\frac{1}{1+\frac{p(d_1+v_1)\alpha u^{\alpha}}{B(\alpha)+p(d_1+v_1)(1-\alpha)}}\right] \\ &+ \frac{p(d_0+v_0)(1-\alpha)}{B(\alpha)+p(d_1+v_1)(1-\alpha)}S^{-1}\left[\frac{1}{1+\frac{p(d_1+v_1)\alpha u^{\alpha}}{B(\alpha)+p(d_1+v_1)(1-\alpha)}}\right] \\ &+ \frac{(d_0+v_0)}{(d_1+v_1)}S^{-1}\left[\frac{\frac{p(d_1+v_1)\alpha u^{\alpha}}{B(\alpha)+p(d_1+v_1)(1-\alpha)}}{1+\frac{p(d_1+v_1)\alpha u^{\alpha}}{B(\alpha)+p(d_1+v_1)(1-\alpha)}}\right] \end{aligned}$$

By Lemma 2.11, we have

$$\begin{aligned} w(x) = &\frac{B(\alpha)w(0)}{B(\alpha)+p(d_1+v_1)(1-\alpha)}E_{\alpha}\left(-\frac{p(d_1+v_1)\alpha}{B(\alpha)+p(d_1+v_1)(1-\alpha)}x^{\alpha}\right) \\ &+ \frac{p(d_0+v_0)(1-\alpha)}{B(\alpha)+p(d_1+v_1)(1-\alpha)}E_{\alpha}\left(-\frac{p(d_1+v_1)\alpha}{B(\alpha)+p(d_1+v_1)(1-\alpha)}x^{\alpha}\right) \\ &+ \frac{(d_0+v_0)}{(d_1+v_1)}\left[1-E_{\alpha}\left(-\frac{p(d_1+v_1)\alpha}{B(\alpha)+p(d_1+v_1)(1-\alpha)}x^{\alpha}\right)\right] \\ = &\frac{B(\alpha)w(0)}{B(\alpha)+p(d_1+v_1)(1-\alpha)}E_{\alpha}\left(-\delta x^{\alpha}\right) + \frac{p(d_0+v_0)(1-\alpha)}{B(\alpha)+p(d_1+v_1)(1-\alpha)}E_{\alpha}\left(-\delta x^{\alpha}\right) \\ &+ \frac{(d_0+v_0)}{(d_1+v_1)}\left[1-E_{\alpha}\left(-\delta x^{\alpha}\right)\right] \end{aligned}$$

where $w(0)=\frac{(d_0+v_0)}{(d_1+v_1)}, \delta=\frac{p(d_1+v_1)\alpha}{B(\alpha)+p(d_1+v_1)(1-\alpha)}$

Case-II. If we consider the expectations of agents the price adjustment equation with ABC is as follows:

$$\left[{}^{ABC}_{a} D_x^{\alpha} w(x)\right]-\frac{(d_1+v_1)}{(d_2+v_2)} w(x)=-\frac{(d_0+v_0)}{(d_2+v_2)}$$

Applying the Sumudu transform, we have

$$S\left[{}^{ABC}_{a} D_x^{\alpha} w(x)\right]-\frac{(d_1+v_1)}{(d_2+v_2)} S[w(x)]=S\left[-\frac{(d_0+v_0)}{(d_2+v_2)}\right]$$

By Theorem 2.10, when $a=0$, we obtain

$$\frac{B(\alpha)}{1-\alpha}\left[\frac{S[w(x)]}{1+\frac{\alpha u^{\alpha}}{1-\alpha}}-\frac{w(0)}{1+\frac{\alpha u^{\alpha}}{1-\alpha}}\right]-\frac{(d_1+v_1)}{(d_2+v_2)} S[w(x)]=-\frac{(d_0+v_0)}{(d_2+v_2)}$$

$$S[w(x)]\left[\frac{B(\alpha)}{1-\alpha} \frac{1}{1+\frac{\alpha u^{\alpha}}{1-\alpha}}-\frac{(d_1+v_1)}{(d_2+v_2)}\right]=\left[\frac{B(\alpha)}{1-\alpha} \frac{w(0)}{1+\frac{\alpha u^{\alpha}}{1-\alpha}}-\frac{(d_0+v_0)}{(d_2+v_2)}\right]$$

$$\begin{aligned} S[w(x)] &= \left[\frac{\frac{B(\alpha)}{1-\alpha} \frac{w(0)}{1+\frac{\alpha u^{\alpha}}{1-\alpha}}}{\frac{B(\alpha)}{1-\alpha} \frac{1}{1+\frac{\alpha u^{\alpha}}{1-\alpha}}-\frac{(d_1+v_1)}{(d_2+v_2)}}\right]-\left[\frac{\frac{(d_0+v_0)}{(d_2+v_2)}}{\frac{B(\alpha)}{1-\alpha} \frac{1}{1+\frac{\alpha u^{\alpha}}{1-\alpha}}-\frac{(d_1+v_1)}{(d_2+v_2)}}\right] \\ &= \left[\frac{\frac{B(\alpha) w(0)}{1-\alpha+\alpha u^{\alpha}}}{\frac{B(\alpha)}{1-\alpha+\alpha u^{\alpha}}-\frac{(d_1+v_1)}{(d_2+v_2)}}\right]-\left[\frac{\frac{(d_0+v_0)}{(d_2+v_2)}}{\frac{B(\alpha)}{1-\alpha+\alpha u^{\alpha}}-\frac{(d_1+v_1)}{(d_2+v_2)}}\right] \end{aligned}$$

$$\begin{aligned} &= \left[\frac{B(\alpha) w(0)}{B(\alpha)-\frac{(d_1+v_1)}{(d_2+v_2)}(1-\alpha+\alpha u^{\alpha})}\right]-\left[\frac{\frac{(d_0+v_0)}{(d_2+v_2)}(1-\alpha+\alpha u^{\alpha})}{B(\alpha)-\frac{(d_1+v_1)}{(d_2+v_2)}(1-\alpha+\alpha u^{\alpha})}\right] \\ &= \left[\frac{\frac{B(\alpha) w(0)}{1-\alpha}}{\frac{B(\alpha)}{1-\alpha}-\frac{(d_1+v_1)}{(d_2+v_2)}-\frac{(d_1+v_1)}{(d_2+v_2)} \frac{\alpha u^{\alpha}}{1-\alpha}}\right]-\frac{(d_0+v_0)}{(d_2+v_2)}\left[\frac{1}{\frac{B(\alpha)}{1-\alpha}-\frac{(d_1+v_1)}{(d_2+v_2)}-\frac{(d_1+v_1)}{(d_2+v_2)} \frac{\alpha u^{\alpha}}{1-\alpha}}\right] \\ &\quad -\frac{(d_0+v_0)}{(d_2+v_2)}\left[\frac{\frac{\alpha u^{\alpha}}{1-\alpha}}{\frac{B(\alpha)}{1-\alpha}-\frac{(d_1+v_1)}{(d_2+v_2)}-\frac{(d_1+v_1)}{(d_2+v_2)} \frac{\alpha u^{\alpha}}{1-\alpha}}\right] \end{aligned}$$

Applying inverse Sumudu transform, we have

$$\begin{aligned} w(x) = & \frac{B(\alpha) w(0)(d_2+v_2)}{(\alpha-1)(d_1+v_1)+B(\alpha)(d_2+v_2)} S^{-1}\left[\frac{1}{1+\left(\frac{-\alpha(d_1+v_1) u^{\alpha}}{(\alpha-1)(d_1+v_1)+B(\alpha)(d_2+v_2)}\right)}\right] \\ & -\frac{(\alpha-1)(d_0+v_0)}{(\alpha-1)(d_1+v_1)+B(\alpha)(d_2+v_2)} S^{-1}\left[\frac{1}{1+\left(\frac{-\alpha(d_1+v_1) u^{\alpha}}{(\alpha-1)(d_1+v_1)+B(\alpha)(d_2+v_2)}\right)}\right] \\ & +\frac{(\alpha-1)(d_0+v_0)}{\alpha(d_1+v_1)} S^{-1}\left[\frac{\frac{\alpha(d_1+v_1)}{(\alpha-1)(d_1+v_1)+B(\alpha)(d_2+v_2)}}{1+\left(\frac{-\alpha(d_1+v_1) u^{\alpha}}{(\alpha-1)(d_1+v_1)+B(\alpha)(d_2+v_2)}\right)}\right] \end{aligned}$$

By Lemma 2.11, we have

$$
\begin{aligned}
w(x) = &\frac{B(\alpha)w(0)(d_2+v_2)}{(\alpha-1)(d_1+v_1)+B(\alpha)(d_2+v_2)}E_\alpha\left(\frac{\alpha(d_1+v_1)x^\alpha}{(\alpha-1)(d_1+v_1)+B(\alpha)(d_2+v_2)}\right)\\
&-\frac{(\alpha-1)(d_0+v_0)}{(\alpha-1)(d_1+v_1)+B(\alpha)(d_2+v_2)}E_\alpha\left(\frac{\alpha(d_1+v_1)x^\alpha}{(\alpha-1)(d_1+v_1)+B(\alpha)(d_2+v_2)}\right)\\
&+\frac{(\alpha-1)(d_0+v_0)}{\alpha(d_1+v_1)}\left[1-E_\alpha\left(\frac{\alpha(d_1+v_1)x^\alpha}{(\alpha-1)(d_1+v_1)+B(\alpha)(d_2+v_2)}\right)\right]\\
= &\frac{B(\alpha)w(0)(d_2+v_2)-(\alpha-1)(d_0+v_0)}{(\alpha-1)(d_1+v_1)+B(\alpha)(d_2+v_2)}E_\alpha\left(\Delta x^\alpha\right)\\
&+\frac{w(0)(\alpha-1)}{\alpha}\left[1-E_\alpha\left(\Delta x^\alpha\right)\right]
\end{aligned}
$$

where $w(0)=\frac{(d_0+v_0)}{(d_1+v_1)}, \Delta=\frac{\alpha(d_1+v_1)}{(\alpha-1)(d_1+v_1)+B(\alpha)(d_2+v_2)}$

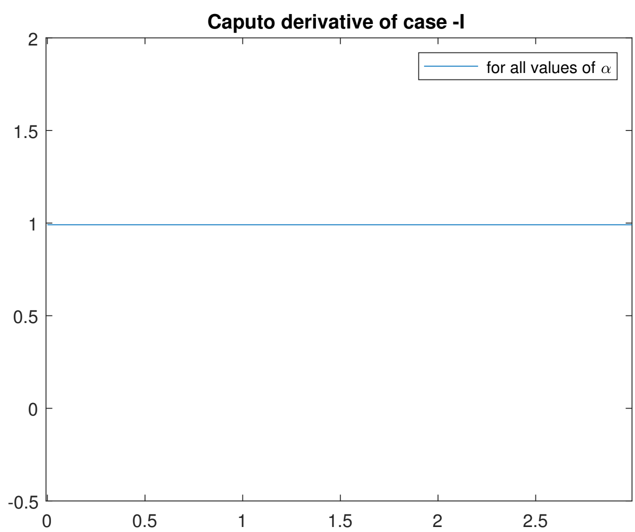
4 Comparative Analysis

We assign some values of constants $d_0, v_0, d_1, v_1, d_2, v_2$ influencing the market equilibrium as $d_0=10, v_0=100, d_1=14, v_1=97, d_2=18, v_2=94$ and $M(\alpha)=1=B(\alpha)$.

4.1 Economic Model with Caputo Derivative:

Case-I:

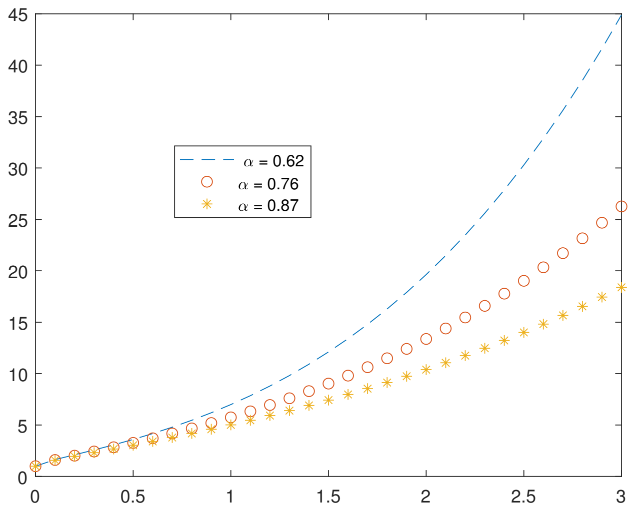
$$w(x)=\frac{110}{111}\Big[E_\alpha[-111px^\alpha]+[1-E_\alpha(-111px^\alpha)]\Big].$$



Graphical representation of $w(x)$ for $\alpha=0.62, 0.76, 0.87$

Case-II:

$$w(x)=\frac{110}{111}\left[2E_\alpha\left(\frac{111}{112}\right)x^\alpha-1\right].$$



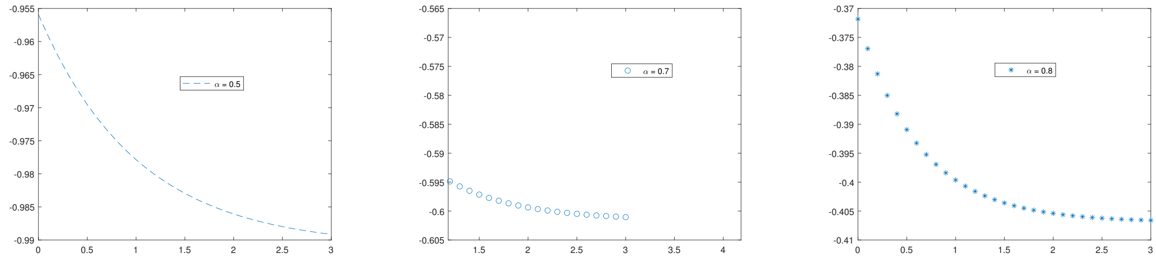
Graphical representation of $w(x)$ for $\alpha = 0.62, 0.76, 0.87$

4.2 Economic Model with CFC Derivative:

Case-I:

$$w(x) = \frac{110}{111}\left[\frac{M(\alpha)\big(-2e^{\frac{111\alpha px}{\lambda}}+1\big)-111(\alpha-1)p}{\lambda}\right]$$

where $\lambda = -M(\alpha) + 111(\alpha - 1)p$.

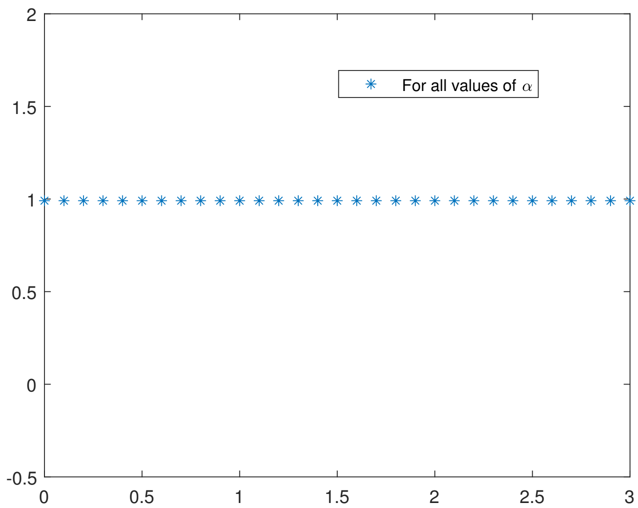


Graphical representation of $w(x)$ for $\alpha = 0.5, 0.7, 0.8$

Case-II:

$$w(x) = \frac{110}{111}\left[\frac{(\alpha-1)(d_1+v_1)+M(\sigma)}{\sigma}\right]$$

where $\sigma = 111(\alpha - 1) + 112M(\alpha)$.



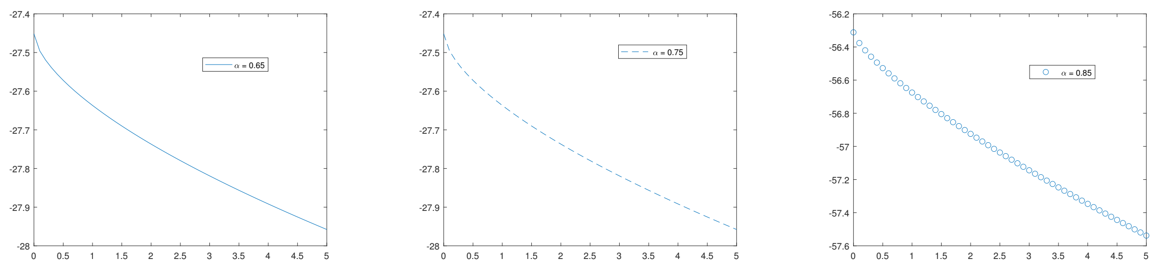
Graphical representation of $w(x)$ for $\alpha = 0.5, 0.7, 0.8$

4.3 Economic Model with ABC Derivative:

Case-I:

$$w(x) = \frac{110B(\alpha)}{111B(\alpha)+111(1-\alpha)p}E_\alpha\big(-\delta x^\alpha\big) + \frac{110(1-\alpha)p}{B(\alpha)+111(1-\alpha)p}E_\alpha\big(-\delta x^\alpha\big) + \frac{110}{111}\Big[1-E_\alpha\big(-\delta x^\alpha\big)\Big]$$

where $\delta = \frac{111p\alpha}{B(\alpha)+111(1-\alpha)p}$.

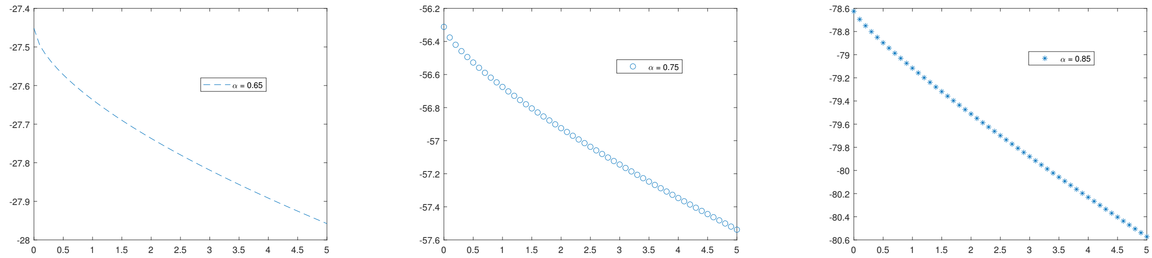


Graphical representation of $w(x)$ for $\alpha = 0.65, 0.75, 0.85$

Case-II:

$$w(x) = \frac{110}{111}\frac{112B(\alpha)}{111(\alpha-1)+112B(\alpha)}E_\alpha\big(\Delta x^\alpha\big) - \frac{110B(\alpha-1)}{111(\alpha-1)+112B(\alpha)}E_\alpha\big(\Delta x^\alpha\big) + \frac{110}{111}\frac{(\alpha-1)}{\alpha}\Big[1-E_\alpha\big(\Delta x^\alpha\big)\Big]$$

where $\Delta = \frac{111\alpha}{111(\alpha-1)+112B(\alpha)}$



Graphical representation of $w(x)$ for $\alpha = 0.65, 0.75, 0.85$

It is observed that the price of goods remains stable if we do not expect role of agent and expect role of agent respectively in economic model with Caputo derivative and CFC derivative for any choice of α. However price decreases in any case for all economic models except economic model involving CFC derivative.

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