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On the growth measures of entire functions focusing their generalized relative order (α, β) and generalized relative type (α, β)

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Abstract. In this paper we introduce the idea of generalized relative order (α, β) and generalized relative type (α, β) of an entire function with respect to an entire function and then study some growth properties of entire functions on the basis of their generalized relative order (α, β) and generalized relative type (α, β), where α, β are non-negative continuous functions defined on $(-\infty, +\infty)$.

1 Introduction, Definitions and Notations

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be an entire function defined on $\mathbb{C}$. The maximum modulus function $M_f(r)$ of $f = \sum\limits_{n=0}^{\infty} a_n z^n$ on $|z| = r$ is defined as $M_f(r) = \max\limits_{|z|=r} |f(z)|$. Moreover, if f is non-constant entire then $M_f(r)$ is also strictly increasing and continuous functions of r. Therefore its inverse $M_f^{-1} : (M_f(0), \infty) \rightarrow (0, \infty)$ exists and is such that $\lim\limits_{s\rightarrow\infty} M_f^{-1}(s) = \infty$. We use the standard notations and definitions of the theory of entire functions which are available in [13] and [14], and therefore we do not explain those in details.

For $x \in [0, \infty)$ and $k \in \mathbb{N}$ where $\mathbb{N}$ is the set of all positive integers, we define iterations of the exponential and logarithmic functions as $\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right)$ and $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$, with convention that $\log^{[0]} x = x$, $\log^{[-1]} x = \exp x$, $\exp^{[0]} x = x$, and $\exp^{[-1]} x = \log x$. Further we assume that p and q always denote positive integers. Now considering this, let us recall that Juneja et al. [9] defined the (p, q)-th order and (p, q)-th lower order of an entire function, respectively, as follows:

Definition 1.1. [9] Let $p \geq q$. The (p, q)-th order $\rho^{(p,q)}(f)$ and (p, q)-th lower order $\lambda^{(p,q)}(f)$ of an entire function f are defined as:

$$\rho^{(p,q)}(f) = \limsup_{r\rightarrow\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} \text{ and } \lambda^{(p,q)}(f) = \liminf_{r\rightarrow\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r}.$$

The function f is said to be of regular (p, q) growth when (p, q)-th order and (p, q)-th lower order of f are the same. Functions which are not of regular (p, q) growth are said to be of irregular (p, q) growth.

Extending the notion (p, q)-th order, recently Shen et al. [10] introduced the new concept of $[p, q]$-φ order of entire function where $p \geq q$. Later on, combining the definitions of (p, q)-order and $[p, q]$-φ order, Biswas (see, e.g., [7]) redefined the (p, q)-order of an entire function without restriction $p \geq q$.

However the above definition is very useful for measuring the growth of entire functions. If $p = l$ and $q = 1$ then we write $\rho^{(l,1)}(f) = \rho^{(l)}(f)$ and $\lambda^{(l,1)}(f) = \lambda^{(l)}(f)$ where $\rho^{(l)}(f)$ and $\lambda^{(l)}(f)$ are respectively known as generalized order and generalized lower order of function f

(see, e.g., [11]). Also for $p = 2$ and $q = 1$, we respectively denote $\rho^{(2,1)}(f)$ and $\lambda^{(2,1)}(f)$ by $\rho(f)$ and $\lambda(f)$ which are classical growth indicators such as order and lower order of entire function f.

Now let L be a class of continuous non-negative functions α defined on $(-\infty, +\infty)$ such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x \to +\infty$ and $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ as $x \to +\infty$. We say that $\alpha \in L^0$, if $\alpha \in L$ and $\alpha(cx) = (1 + o(1))\alpha(x)$ as $x_0 \leq x \to +\infty$ for each $c \in (0, +\infty)$, i.e., α is slowly increasing function. Clearly $L^0 \subset L$. Throughout the present paper we assume that α, β and γ always denote the functions belonging to L^0.

Considering this, the value

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log M_f(r))}{\beta(\log r)} \ (\alpha \in L, \beta \in L)$$

introduced by Sheremeta [12], is called the generalized order (α, β) of an entire function f. Then Biswas et al. [5] has introduced the definitions of the generalized order (α, β) and generalized lower order (α, β) of an entire function in the following way after giving a minor modification to the original definition of generalized order (α, β) of an entire function (e.g. see, [12]) which considerably extend the definition of φ-order introduced by Chyzhykov et al. [8].

Definition 1.2. [5] The generalized order (α, β) denoted by $\rho_{(\alpha,\beta)}[f]$ and generalized lower order (α, β) denoted by $\lambda_{(\alpha,\beta)}[f]$ of an entire function f are defined as:

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to\infty} \frac{\alpha(M_f(r))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f] = \liminf_{r\to\infty} \frac{\alpha(M_f(r))}{\beta(r)}.$$

Definition 1.1 is a special case of Definition 1.2 for $\alpha(r) = \log^{[p]} r$ and $\beta(r) = \log^{[q]} r$.

The function f is said to be of regular generalized (α, β) growth when generalized order (α, β) and generalized lower order (α, β) of f are the same. Functions which are not of regular generalized (α, β) growth are said to be of irregular generalized (α, β) growth.

In order to refine the growth scale namely the generalized order (α, β) of an entire function, Biswas et al. [4] have introduced the definitions of another growth indicators, called generalized type (α, β) and generalized lower type (α, β) respectively of an entire function which are as follows:

Definition 1.3. [4] The generalized type (α, β) denoted by $\sigma_{(\alpha,\beta)}[f]$ and generalized lower type (α, β) denoted by $\overline{\sigma}_{(\alpha,\beta)}[f]$ of an entire function f having finite positive generalized order (α, β) $(0 < \rho_{(\alpha,\beta)}[f] < \infty)$ are defined as :

$$\sigma_{(\alpha,\beta)}[f] = \limsup_{r\to\infty} \frac{\exp(\alpha(M_f(r)))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]}} \text{ and } \overline{\sigma}_{(\alpha,\beta)}[f] = \liminf_{r\to\infty} \frac{\exp(\alpha(M_f(r)))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]}}.$$

It is obvious that $0 \leq \overline{\sigma}_{(\alpha,\beta)}[f] \leq \sigma_{(\alpha,\beta)}[f] \leq \infty$.

Analogously, to determine the relative growth of two entire functions having same non zero finite generalized lower order (α, β), Biswas et al. [4] have introduced the definitions of generalized weak type (α, β) and generalized upper weak type (α, β) of an entire function f of finite positive generalized lower order (α, β), $\lambda_{(\alpha,\beta)}[f]$ in the following way:

Definition 1.4. [4] The generalized upper weak type (α, β) denoted by $\tau_{(\alpha,\beta)}[f]$ and generalized weak type (α, β) denoted by $\overline{\tau}_{(\alpha,\beta)}[f]$ of an entire function f having finite positive generalized order (α, β) $(0 < \lambda_{(\alpha,\beta)}[f] < \infty)$ are defined as :

$$\tau_{(\alpha,\beta)}[f] = \limsup_{r\to\infty} \frac{\exp(\alpha(M_f(r)))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]}} \text{ and } \overline{\tau}_{(\alpha,\beta)}[f] = \liminf_{r\to\infty} \frac{\exp(\alpha(M_f(r)))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]}}.$$

It is obvious that $0 \leq \overline{\tau}_{(\alpha,\beta)}[f] \leq \tau_{(\alpha,\beta)}[f] \leq \infty$.

Mainly the growth investigation of entire functions has usually been done through their maximum moduli function in comparison with those of exponential function. But if one is paying attention to evaluate the growth rates of any entire function with respect to a new entire function, the notions of relative growth indicators (see e.g. [1, 2]) will come. Now in order to make some progress in the study of relative order, Biswas et al. [3] have introduced the definitions of generalized relative order (α, β) and generalized relative lower order (α, β) of an entire function with respect to another entire function in the following way:

Definition 1.5. [3] The generalized relative order (α, β) denoted by $\rho_{(\alpha,\beta)}[f]_g$ and generalized relative lower order (α, β) denoted by $\lambda_{(\alpha,\beta)}[f]_g$ of an entire function f with respect to an entire function g are defined as:

$$\rho_{(\alpha,\beta)}[f]_g = \limsup_{r\to\infty} \frac{\alpha(M_g^{-1}(M_f(r)))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f]_g = \liminf_{r\to\infty} \frac{\alpha(M_g^{-1}(M_f(r)))}{\beta(r)}.$$

The previous definitions are easily generated as particular cases, e.g. if $g = z$, then Definition 1.5 reduces to Definition 1.2. If $\alpha(r) = \beta(r) = \log r$, then we get the definition of relative order of entire function f with respect to an entire function g introduced by Bernal [1, 2] and if $g = \exp z$ and $\alpha(r) = \beta(r) = \log r$, then $\rho_{(\alpha,\beta)}[f]_g = \rho(f)$. And if $\alpha(r) = \log^{[p]} r$, $\beta(r) = \log^{[q]} r$ and $g = z$, then Definition 1.5 becomes the classical one given in [7].

Further if generalized relative order (α, β) and the generalized relative lower order (α, β) of an entire function f with respect to an entire function g are the same, then f is called a function of regular generalized relative (α, β) growth with respect to g. Otherwise, f is said to be irregular generalized relative (α, β) growth.with respect to g.

Now in order to refine the above growth scale, Biswas et al. [3] have introduced the definitions of other growth indicators, such as generalized relative type (α, β) and generalized relative lower type (α, β) of entire function with respect to an entire function which are as follows:

Definition 1.6. [3] The generalized relative type (α, β) denoted by $\sigma_{(\alpha,\beta)}[f]_g$ and generalized relative lower type (α, β) denoted by $\overline{\sigma}_{(\alpha,\beta)}[f]_g$ of an entire function f with respect to an entire function g having non-zero finite generalized relative order (α, β) are defined as :

$$\sigma_{(\alpha,\beta)}[f]_g = \limsup_{r\to\infty} \frac{\exp(\alpha(M_g^{-1}(M_f(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]_g}} \text{ and } \overline{\sigma}_{(\alpha,\beta)}[f]_g = \liminf_{r\to\infty} \frac{\exp(\alpha(M_g^{-1}(M_f(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]_g}}$$

Analogously, to determine the relative growth of an entire function f having same non zero finite generalized relative lower order (α, β) with respect to an entire function g, Biswas et al. [3] have introduced the definitions of generalized relative upper weak type (α, β) and generalized relative weak type (α, β) of f with respect to g of finite positive generalized relative lower order (α, β) in the following way:

Definition 1.7. [3] The generalized relative upper weak type (α, β) denoted by $\tau_{(\alpha,\beta)}[f]_g$ and generalized relative weak type (α, β) denoted by $\overline{\tau}_{(\alpha,\beta)}[f]_g$ of an entire function f with respect to an entire function g having non-zero finite generalized relative lower order (α, β) are defined as :

$$\tau_{(\alpha,\beta)}[f]_g = \limsup_{r\to\infty} \frac{\exp(\alpha(M_g^{-1}(M_f(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]_g}} \text{ and } \overline{\tau}_{(\alpha,\beta)}[f]_g = \liminf_{r\to\infty} \frac{\exp(\alpha(M_g^{-1}(M_f(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]_g}}.$$

However the main aim of this paper is to investigate some growth properties of entire functions using generalized relative order (α, β) and generalized relative type (α, β), of an entire function with respect to an entire function which improve and extend some earlier result (see, e.g., [6, 7]). We assume that all the growth indicators are non-zero finite.

2 Main Results

In this section we present the main results of the paper.

Theorem 2.1. *Let f and g be two entire functions such that* $0 < \lambda_{(\gamma,\beta)}[f] \leq \rho_{(\gamma,\beta)}[f] < \infty$ *and* $0 < \lambda_{(\gamma,\alpha)}[g] \leq \rho_{(\gamma,\alpha)}[g] < \infty$. *Then*

$$\frac{\lambda_{(\gamma,\beta)}[f]}{\rho_{(\gamma,\alpha)}[g]} \leq \lambda_{(\alpha,\beta)}[f]_g \leq \min\left\{\frac{\lambda_{(\gamma,\beta)}[f]}{\lambda_{(\gamma,\alpha)}[g]}, \frac{\rho_{(\gamma,\beta)}[f]}{\rho_{(\gamma,\alpha)}[g]}\right\}$$
$$\leq \max\left\{\frac{\lambda_{(\gamma,\beta)}[f]}{\lambda_{(\gamma,\alpha)}[g]}, \frac{\rho_{(\gamma,\beta)}[f]}{\rho_{(\gamma,\alpha)}[g]}\right\} \leq \rho_{(\alpha,\beta)}[f]_g \leq \frac{\rho_{(\gamma,\beta)}[f]}{\lambda_{(\gamma,\alpha)}[g]}.$$

Proof. From the definitions of $\rho_{(\gamma,\beta)}[f]$ and $\lambda_{(\gamma,\beta)}[f]$ we have for all sufficiently large values of r that

$$M_f(r) \leq \gamma^{-1}((\rho_{(\gamma,\beta)}[f] + \varepsilon)\beta(r)), \tag{2.1}$$
$$M_f(r) \geq \gamma^{-1}((\lambda_{(\gamma,\beta)}[f] - \varepsilon)\beta(r)) \tag{2.2}$$

and also for a sequence of values of r tending to infinity we get that

$$M_f(r) \geq \gamma^{-1}((\rho_{(\gamma,\beta)}[f] - \varepsilon)\beta(r)), \tag{2.3}$$
$$M_f(r) \leq \gamma^{-1}((\lambda_{(\gamma,\beta)}[f] + \varepsilon)\beta(r)). \tag{2.4}$$

Further from the definitions of $\rho_{(\gamma,\alpha)}[g]$ and $\lambda_{(\gamma,\alpha)}[g]$ it follows for all sufficiently large values of r that

$$M_g(r) \leq \gamma^{-1}((\rho_{(\gamma,\alpha)}[g] + \varepsilon)\alpha(r))$$
$$i.e.,\ M_g^{-1}(r) \geq \alpha^{-1}\left(\frac{\gamma(r)}{\rho_{(\gamma,\alpha)}[g] + \varepsilon}\right) \text{ and} \tag{2.5}$$

$$M_g^{-1}(r) \leq \alpha^{-1}\left(\frac{\gamma(r)}{\lambda_{(\gamma,\alpha)}[g] - \varepsilon}\right). \tag{2.6}$$

Also from the definitions of $\rho_{(\gamma,\alpha)}[g]$ and $\lambda_{(\gamma,\alpha)}[g]$, we get for a sequence of values of r tending to infinity we obtain that

$$M_g^{-1}(r) \leq \alpha^{-1}\left(\frac{\gamma(r)}{\rho_{(\gamma,\alpha)}[g] - \varepsilon}\right) \text{ and} \tag{2.7}$$

$$M_g^{-1}(r) \geq \alpha^{-1}\left(\frac{\gamma(r)}{\lambda_{(\gamma,\alpha)}[g] + \varepsilon}\right). \tag{2.8}$$

Now from (2.3) and in view of (2.5), for a sequence of values of r tending to infinity we get that

$$\alpha(M_g^{-1}(M_f(r))) \geq \alpha(M_g^{-1}(\gamma^{-1}((\rho_{(\gamma,\beta)}[f] - \varepsilon)\beta(r))))$$
$$i.e.,\ \alpha(M_g^{-1}(M_f(r))) \geq \alpha\left(\alpha^{-1}\left(\frac{\gamma(\gamma^{-1}((\rho_{(\gamma,\beta)}[f] - \varepsilon)\beta(r)))}{(\rho_{(\gamma,\alpha)}[g] + \varepsilon)}\right)\right)$$

$$i.e.,\ \alpha(M_g^{-1}(M_f(r))) \geq \frac{(\rho_{(\gamma,\beta)}[f] - \varepsilon)\beta(r)}{(\rho_{(\gamma,\alpha)}[g] + \varepsilon)}$$

$$i.e.,\ \frac{\alpha(M_g^{-1}(M_f(r)))}{\beta(r)} \geq \frac{\rho_{(\gamma,\beta)}[f] - \varepsilon}{\rho_{(\gamma,\alpha)}[g] + \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary, it follows that

$$\rho_{(\alpha,\beta)}[f]_g \geq \frac{\rho_{(\gamma,\beta)}[f]}{\rho_{(\gamma,\alpha)}[g]}. \tag{2.9}$$

Analogously from (2.2) and in view of (2.8) it follows that

$$\rho_{(\alpha,\beta)}[f]_g \geq \frac{\lambda_{(\gamma,\beta)}[f]}{\lambda_{(\gamma,\alpha)}[g]}. \tag{2.10}$$

Again from (2.2) and in view of (2.5) we obtain that

$$\lambda_{(\alpha,\beta)}[f]_g \geq \frac{\lambda_{(\gamma,\beta)}[f]}{\rho_{(\gamma,\alpha)}[g]}. \tag{2.11}$$

Now in view of (2.6), we have from (2.1) for all sufficiently large values of r that

$$\begin{aligned} \alpha(M_g^{-1}(M_f(r))) &\leq \alpha(M_g^{-1}(\gamma^{-1}((\rho_{(\gamma,\beta)}[f]+\varepsilon)\beta(r)))) \\ i.e.,\ \alpha(M_g^{-1}(M_f(r))) &\leq \alpha\Big(\alpha^{-1}\Big(\frac{\gamma(\gamma^{-1}((\rho_{(\gamma,\beta)}[f]+\varepsilon)\beta(r)))}{(\lambda_{(\gamma,\alpha)}[g]-\varepsilon)}\Big)\Big) \end{aligned}$$

$$i.e.,\ \alpha(M_g^{-1}(M_f(r))) \leq \frac{(\rho_{(\gamma,\beta)}[f]+\varepsilon)\beta(r)}{(\lambda_{(\gamma,\alpha)}[g]-\varepsilon)}$$

$$i.e.,\ \frac{\alpha(M_g^{-1}(M_f(r)))}{\beta(r)} \leq \frac{\rho_{(\gamma,\beta)}[f]+\varepsilon}{\lambda_{(\gamma,\alpha)}[g]-\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, we obtain that

$$\rho_{(\alpha,\beta)}[f]_g \leq \frac{\rho_{(\gamma,\beta)}[f]}{\lambda_{(\gamma,\alpha)}[g]}. \tag{2.12}$$

Similarly in view of (2.7), we get from (2.1) that

$$\lambda_{(\alpha,\beta)}[f]_g \leq \frac{\rho_{(\gamma,\beta)}[f]}{\rho_{(\gamma,\alpha)}[g}. \tag{2.13}$$

Again from (2.4) and in view of (2.6) it follows that

$$\lambda_{(\alpha,\beta)}[f]_g \leq \frac{\lambda_{(\gamma,\beta)}[f]}{\lambda_{(\gamma,\alpha)}[g]}. \tag{2.14}$$

The theorem follows from (2.9), (2.10), (2.11), (2.12), (2.13) and (2.14).□

Remark 2.2. From the conclusion of the above result, one may write $\rho_{(\alpha,\beta)}[f]_g = \frac{\rho_{(\gamma,\beta)}[f]}{\rho_{(\gamma,\alpha)}[g]}$ and $\lambda_{(\alpha,\beta)}[f]_g = \frac{\lambda_{(\gamma,\beta)}[f]}{\lambda_{(\gamma,\alpha)}[g]}$ when $\lambda_{(\gamma,\alpha)}[g] = \rho_{(\gamma,\alpha)}[g]$. Similarly $\rho_{(\alpha,\beta)}[f]_g = \frac{\lambda_{(\gamma,\beta)}[f]}{\lambda_{(\gamma,\alpha)}[g]}$ and $\lambda_{(\alpha,\beta)}[f]_g = \frac{\rho_{(\gamma,\beta)}[f]}{\rho_{(\gamma,\alpha)}[g]}$ when $\lambda_{(\gamma,\beta)}[f] = \rho_{(\gamma,\beta)}[f]$.

Theorem 2.3. *Let f and g be two entire functions such that $0 < \rho_{(\gamma,\beta)}[f] < \infty$ and $0 < \lambda_{(\gamma,\alpha)}[g] \leq \rho_{(\gamma,\alpha)}[g] < \infty$. Then*

$$\max\Big\{\Big(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]}{\overline{\tau}_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}, \Big(\frac{\sigma_{(\gamma,\beta)}[f]}{\tau_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}\Big\} \leq \sigma_{(\alpha,\beta)}[f]_g \leq \Big(\frac{\sigma_{(\gamma,\beta)}[f]}{\overline{\sigma}_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}.$$

Proof. Let us consider that $\varepsilon(>0)$ is arbitrary number. Now from the definitions of $\sigma_{(\gamma,\beta)}[f]$ and $\overline{\sigma}_{(\gamma,\beta)}[f]$, we have for all sufficiently large values of r that

$$M_f(r) \leq \gamma^{-1}(\log((\sigma_{(\gamma,\beta)}[f]+\varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]})), \tag{2.15}$$

$$M_f(r) \geq \gamma^{-1}(\log((\overline{\sigma}_{(\gamma,\beta)}[f]-\varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]})) \tag{2.16}$$

and also for a sequence of values of r tending to infinity, we get that

$$M_f(r) \geq \gamma^{-1}(\log((\sigma_{(\gamma,\beta)}[f]-\varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]})), \tag{2.17}$$
$$M_f(r) \leq \gamma^{-1}(\log((\overline{\sigma}_{(\gamma,\beta)}[f]+\varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]})). \tag{2.18}$$

Similarly from the definitions of $\sigma_{(\gamma,\alpha)}[g]$ and $\overline{\sigma}_{(\gamma,\alpha)}[g]$, it follows for all sufficiently large values of r that

$$M_g(r) \leq \gamma^{-1}(\log((\sigma_{(\gamma,\alpha)}[g]+\varepsilon)(\exp(\alpha(r)))^{\rho_{(\gamma,\alpha)}[g]}))$$
$$i.e.,\ M_g^{-1}(r) \geq \alpha^{-1}\Big(\log\Big(\frac{\exp(\gamma(r))}{(\sigma_{(\gamma,\alpha)}[g]+\varepsilon)}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}\Big) \text{ and} \tag{2.19}$$

$$M_g^{-1}(r) \leq \alpha^{-1}\Big(\log\Big(\frac{\exp(\gamma(r))}{(\overline{\sigma}_{(\gamma,\alpha)}[g]-\varepsilon)}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}\Big). \tag{2.20}$$

Also for a sequence of values of r tending to infinity, we obtain that

$$M_g^{-1}(r) \leq \alpha^{-1}\Big(\log\Big(\frac{\exp(\gamma(r))}{(\sigma_{(\gamma,\alpha)}[g]-\varepsilon)}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}\Big) \text{ and} \tag{2.21}$$

$$M_g^{-1}(r) \geq \alpha^{-1}\Big(\log\Big(\frac{\exp(\gamma(r))}{(\overline{\sigma}_{(\gamma,\alpha)}[g]+\varepsilon)}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}\Big). \tag{2.22}$$

Further from the definitions of $\tau_{(\gamma,\beta)}[f]$ and $\overline{\tau}_{(\gamma,\beta)}[f]$, we have for all sufficiently large values of r that

$$M_f(r) \leq \gamma^{-1}(\log((\tau_{(\gamma,\beta)}[f]+\varepsilon)(\exp(\beta(r)))^{\lambda_{(\gamma,\beta)}[f]})), \tag{2.23}$$
$$M_f(r) \geq \gamma^{-1}(\log((\overline{\tau}_{(\gamma,\beta)}[f]-\varepsilon)(\exp(\beta(r)))^{\lambda_{(\gamma,\beta)}[f]})) \tag{2.24}$$

and also for a sequence of values of r tending to infinity, we get that

$$M_f(r) \geq \log(\gamma^{-1}(\log((\tau_{(\gamma,\beta)}[f]-\varepsilon)(\exp(\beta(r)))^{\lambda_{(\gamma,\beta)}[f]}))), \tag{2.25}$$
$$M_f(r) \leq \log(\gamma^{-1}(\log((\overline{\tau}_{(\gamma,\beta)}[f]+\varepsilon)(\exp(\beta(r)))^{\lambda_{(\gamma,\beta)}[f]}))). \tag{2.26}$$

Similarly from the definitions of $\tau_{(\gamma,\alpha)}[g]$ and $\overline{\tau}_{(\gamma,\alpha)}[g]$, it follows for all sufficiently large values of r that

$$M_g^{-1}(r) \geq \alpha^{-1}\Big(\log\Big(\frac{\exp(\gamma(r))}{(\tau_{(\gamma,\alpha)}[g]+\varepsilon)}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}\Big) \text{ and} \tag{2.27}$$

$$M_g^{-1}(r) \leq \alpha^{-1}\Big(\log\Big(\frac{\exp(\gamma(r))}{(\overline{\tau}_{(\gamma,\alpha)}[g]-\varepsilon)}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}\Big) \tag{2.28}$$

Also for a sequence of values of r tending to infinity, we obtain that

$$M_g^{-1}(r) \leq \alpha^{-1}\Big(\log\Big(\frac{\exp(\gamma(r))}{(\tau_{(\gamma,\alpha)}[g]-\varepsilon)}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}\Big) \text{ and} \tag{2.29}$$

$$M_g^{-1}(r) \geq \alpha^{-1}\Big(\log\Big(\frac{\exp(\gamma(r))}{(\overline{\tau}_{(\gamma,\alpha)}[g]+\varepsilon)}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}\Big). \tag{2.30}$$

Now from (2.17) and in view of (2.27), we get for a sequence of values of r tending to infinity that

$$\exp(\alpha(M_g^{-1}(M_f(r)))) \geq \Big(\frac{(\sigma_{(\gamma,\beta)}[f]-\varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]}}{(\tau_{(\gamma,\alpha)}[g]+\varepsilon)}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}$$

$$i.e., \ \frac{\exp(\alpha(M_g^{-1}(M_f(r))))}{(\exp(\beta(r)))^{\frac{\rho_{(\gamma,\beta)}[f]}{\lambda_{(\gamma,\alpha)}[g]}}} \geq \left(\frac{\sigma_{(\gamma,\beta)}[f]-\varepsilon}{\tau_{(\gamma,\alpha)}[g]+\varepsilon}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}.$$

Since in view of Theorem 2.1 ,$\frac{\rho_{(\gamma,\beta)}[f]}{\lambda_{(\gamma,\alpha)}[g]} \geq \rho_{(\alpha,\beta)}[f]_g$, and as $\varepsilon\,(>0)$ is arbitrary, therefore it follows from above that

$$\limsup_{r\to\infty} \frac{\exp(\alpha(M_g^{-1}(M_f(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]_g}} \geq \left(\frac{\sigma_{(\gamma,\beta)}[f]-\varepsilon}{\tau_{(\gamma,\alpha)}[g]+\varepsilon}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}$$

$$i.e., \ \sigma_{(\alpha,\beta)}[f]_g \geq \left(\frac{\sigma_{(\gamma,\beta)}[f]}{\tau_{(\gamma,\alpha)}[g]}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}. \tag{2.31}$$

Analogously from (2.16) and (2.30), we get that

$$\sigma_{(\alpha,\beta)}[f]_g \geq \left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]}{\overline{\tau}_{(\gamma,\alpha)}[g]}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}, \tag{2.32}$$

as in view of Theorem 2.1 it follows that $\frac{\rho_{(\gamma,\beta)}[f]}{\lambda_{(\gamma,\alpha)}[g]} \geq \rho_{(\alpha,\beta)}[f]_g$.

Again in view of (2.20), we have from (2.15) for all sufficiently large values of r that

$$\exp(\alpha(M_g^{-1}(M_f(r)))) \leq \left(\frac{\left(\sigma_{(\gamma,\beta)}[f]+\varepsilon\right)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]}}{\left(\overline{\sigma}_{(\gamma,\alpha)}[g]-\varepsilon\right)}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}$$

$$i.e., \ \frac{\exp(\alpha(M_g^{-1}(M_f(r))))}{(\exp(\beta(r)))^{\frac{\rho_{(\gamma,\beta)}[f]}{\rho_{(\gamma,\alpha)}[g]}}} \leq \left(\frac{\sigma_{(\gamma,\beta)}[f]+\varepsilon}{\overline{\sigma}_{(\gamma,\alpha)}[g]-\varepsilon}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}.$$

Since in view of Theorem 2.1 it follows that $\frac{\rho_{(\gamma,\beta)}[f]}{\rho_{(\gamma,\alpha)}[g]} \leq \rho_{(\alpha,\beta)}[f]_g$ and $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to\infty} \frac{\exp(\alpha(M_g^{-1}(M_f(r))))}{(\exp(\beta(r)))^{\frac{\rho_{(\gamma,\beta)}[f]}{\rho_{(\gamma,\alpha)}[g]}}} \leq \left(\frac{\sigma_{(\gamma,\beta)}[f]+\varepsilon}{\overline{\sigma}_{(\gamma,\alpha)}[g]-\varepsilon}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}$$

$$i.e., \ \sigma_{(\alpha,\beta)}[f]_g \leq \left(\frac{\sigma_{(\gamma,\beta)}[f]}{\overline{\sigma}_{(\gamma,\alpha)}[g]}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}. \tag{2.33}$$

Thus the theorem follows from (2.31), (2.32) and (2.33).□

The conclusion of the following theorem can be carried out from (2.20) and (2.23); (2.23) and (2.28) respectively after applying the same technique of Theorem 2.3 and with the help of Theorem 2.1. Therefore its proof is omitted.

Theorem 2.4. *Let f and g be two entire functions such that $0 < \lambda_{(\gamma,\beta)}[f] < \infty$ and $0 < \lambda_{(\gamma,\alpha)}[g] \leq \rho_{(\gamma,\alpha)}[g] < \infty$. Then*

$$\sigma_{(\alpha,\beta)}[f]_g \leq \min\left\{\left(\frac{\tau_{(\gamma,\beta)}[f]}{\overline{\tau}_{(\gamma,\alpha)}[g]}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}, \left(\frac{\tau_{(\gamma,\beta)}[f]}{\overline{\sigma}_{(\gamma,\alpha)}[g]}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}\right\}.$$

Similarly in the line of Theorem 2.3 and with the help of Theorem 2.1, one may easily carry out the following theorem from pairwise inequalities numbers (2.24) and (2.27) ; (2.21) and (2.23); (2.20) and (2.26) respectively and therefore its proof is omitted:

Theorem 2.5. *Let f and g be two entire functions such that $0 < \lambda_{(\gamma,\beta)}[f] \leq \rho_{(\gamma,\beta)}[f] < \infty$ and $0 < \lambda_{(\gamma,\alpha)}[g] \leq \rho_{(\gamma,\alpha)}[g] < \infty$. Then*

$$\left(\frac{\overline{\tau}_{(\gamma,\beta)}[f]}{\tau_{(\gamma,\alpha)}[g]}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}} \leq \overline{\tau}_{(\alpha,\beta)}[f]_g \leq \min\left\{\left(\frac{\overline{\tau}_{(\gamma,\beta)}[f]}{\overline{\sigma}_{(\gamma,\alpha)}[g]}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}, \left(\frac{\tau_{(\gamma,\beta)}[f]}{\sigma_{(\gamma,\alpha)}[g]}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}\right\}.$$

Theorem 2.6. *Let f and g be two entire functions such that $0 < \rho_{(\gamma,\beta)}[f] < \infty$ and $0 < \lambda_{(\gamma,\alpha)}[g] \leq \rho_{(\gamma,\alpha)}[g] < \infty$. Then*

$$\overline{\tau}_{(\alpha,\beta)}[f]_g \geq \max\left\{\left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]}{\sigma_{(\gamma,\alpha)}[g]}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}, \left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]}{\tau_{(\gamma,\alpha)}[g]}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}\right\}.$$

With the help of Theorem 2.1, the conclusion of the above theorem can be carried out from (2.16), (2.19) and (2.16), (2.27) respectively after applying the same technique of Theorem 2.3 and therefore its proof is omitted.

Theorem 2.7. *Let f and g be two entire functions such that $0 < \rho_{(\gamma,\beta)}[f] < \infty$ and $0 < \lambda_{(\gamma,\alpha)}[g] \leq \rho_{(\gamma,\alpha)}[g] < \infty$. Then*

$$\left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]}{\tau_{(\gamma,\alpha)}[g]}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}} \leq \overline{\sigma}_{(\alpha,\beta)}[f]_g \leq \min\left\{\left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]}{\overline{\sigma}_{(\gamma,\alpha)}[g]}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}, \left(\frac{\sigma_{(\gamma,\beta)}[f]}{\sigma_{(\gamma,\alpha)}[g]}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}\right\}.$$

Proof. From (2.16) and in view of (2.27), we get for all sufficiently large values of r that

$$\exp(\alpha(M_g^{-1}(M_f(r)))) \geq \left(\frac{(\overline{\sigma}_{(\gamma,\beta)}[f]-\varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]}}{(\tau_{(\gamma,\alpha)}[g]+\varepsilon)}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}$$

$$i.e., \ \frac{\exp(\alpha(M_g^{-1}(M_f(r))))}{(\exp(\beta(r)))^{\frac{\rho_{(\gamma,\beta)}[f]}{\lambda_{(\gamma,\alpha)}[g]}}} \geq \left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]-\varepsilon}{\tau_{(\gamma,\alpha)}[g]+\varepsilon}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}.$$

Since in view of Theorem 2.1, $\frac{\rho_{(\gamma,\beta)}[f]}{\lambda_{(\gamma,\alpha)}[g]} \geq \rho_{(\alpha,\beta)}[f]_g$, and $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\liminf_{r\to\infty}\frac{\exp(\alpha(M_g^{-1}(M_f(r))))}{(\exp(\beta(r)))^{\frac{\rho_{(\gamma,\beta)}[f]}{\lambda_{(\gamma,\alpha)}[g]}}} \geq \left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]-\varepsilon}{\tau_{(\gamma,\alpha)}[g]+\varepsilon}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}$$

$$i.e., \ \overline{\sigma}_{(\alpha,\beta)}[f]_g \geq \left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]}{\tau_{(\gamma,\alpha)}[g]}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}. \tag{2.34}$$

Further in view of (2.21), we get from (2.15) for a sequence of values of r tending to infinity that

$$\exp(\alpha(M_g^{-1}(M_f(r)))) \leq \left(\frac{(\sigma_{(\gamma,\beta)}[f]+\varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]}}{(\sigma_{(\gamma,\alpha)}[g]-\varepsilon)}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}$$

$$i.e., \ \frac{\exp(\alpha(M_g^{-1}(M_f(r))))}{(\exp(\beta(r)))^{\frac{\rho_{(\gamma,\beta)}[f]}{\rho_{(\gamma,\alpha)}[g]}}} \leq \left(\frac{\sigma_{(\gamma,\beta)}[f]+\varepsilon}{\sigma_{(\gamma,\alpha)}[g]-\varepsilon}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}.$$

Again as in view of Theorem 2.1, $\frac{\rho_{(\gamma,\beta)}[f]}{\rho_{(\gamma,\alpha)}[g]} \leq \rho_{(\alpha,\beta)}[f]_g$ and $\varepsilon\,(>0)$ is arbitrary, therefore we get from above that

$$\liminf_{r\to\infty}\frac{\exp(\alpha(M_g^{-1}(M_f(r))))}{(\exp(\beta(r)))^{\frac{\rho_{(\gamma,\beta)}[f]}{\rho_{(\gamma,\alpha)}[g]}}} \leq \left(\frac{\sigma_{(\gamma,\beta)}[f]+\varepsilon}{\sigma_{(\gamma,\alpha)}[g]-\varepsilon}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}$$

$$i.e., \ \overline{\sigma}_{(\alpha,\beta)}[f]_g \leq \left(\frac{\sigma_{(\gamma,\beta)}[f]}{\sigma_{(\gamma,\alpha)}[g]}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}. \tag{2.35}$$

Similarly from (2.18) and (2.20), we get that

$$i.e., \ \overline{\sigma}_{(\alpha,\beta)}[f]_g \leq \left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]}{\overline{\sigma}_{(\gamma,\alpha)}[g]}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}, \tag{2.36}$$

as in view of Theorem 2.1 it follows that $\frac{\rho_{(\gamma,\beta)}[f]}{\rho_{(\gamma,\alpha)}[g]} \leq \rho_{(\alpha,\beta)}[f]_g$.

Thus the theorem follows from (2.34), (2.35) and (2.36).□

Theorem 2.8. *Let f and g be two entire functions such that $0 < \lambda_{(\gamma,\beta)}[f] < \infty$ and $0 < \lambda_{(\gamma,\alpha)}[g] \le \rho_{(\gamma,\alpha)}[g] < \infty$. Then*

$$\overline{\sigma}_{(\alpha,\beta)}[f]_g \le \min\Big\{\Big(\frac{\overline{\tau}_{(\gamma,\beta)}[f]}{\overline{\tau}_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}, \Big(\frac{\tau_{(\gamma,\beta)}[f]}{\tau_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}, \Big(\frac{\tau_{(\gamma,\beta)}[f]}{\sigma_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}, \Big(\frac{\overline{\tau}_{(\gamma,\beta)}[f]}{\overline{\sigma}_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}\Big\}.$$

The conclusion of the above theorem can be carried out from pairwise inequalities numbered (2.20) and (2.26) ; (2.21) and (2.23) ; (2.26) and (2.28); (2.23) and (2.29) respectively after applying the same technique of Theorem 2.7 and with the help of Theorem 2.1. Therefore its proof is omitted.

Similarly in the line of Theorem 2.3 and with the help of Theorem 2.1, one may easily carry out the following theorem from pairwise inequalities numbered (2.25) and (2.27) ; (2.24) and (2.30); (2.20) and (2.23) respectively and therefore its proof is omitted:

Theorem 2.9. *Let f and g be two entire functions such that $0 < \lambda_{(\gamma,\beta)}[f] < \infty$ and $0 < \lambda_{(\gamma,\alpha)}[g] \le \rho_{(\gamma,\alpha)}[g] < \infty$. Then*

$$\max\Big\{\Big(\frac{\tau_{(\gamma,\beta)}[f]}{\tau_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}, \Big(\frac{\overline{\tau}_{(\gamma,\beta)}[f]}{\overline{\tau}_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}\Big\} \le \tau_{(\alpha,\beta)}[f]_g \le \Big(\frac{\tau_{(\gamma,\beta)}[f]}{\overline{\sigma}_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}.$$

Theorem 2.10. *Let f and g be two entire functions such that $0 < \lambda_{(\gamma,\beta)}[f] \le \rho_{(\gamma,\beta)}[f] < \infty$ and $0 < \lambda_{(\gamma,\alpha)}[g] \le \rho_{(\gamma,\alpha)}[g] < \infty$. Then*

$$\tau_{(\alpha,\beta)}[f]_g \ge \max\Big\{\Big(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]}{\overline{\sigma}_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}, \Big(\frac{\sigma_{(\gamma,\beta)}[f]}{\sigma_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}, \Big(\frac{\sigma_{(\gamma,\beta)}[f]}{\tau_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}, \Big(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]}{\overline{\tau}_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}\Big\}.$$

The conclusion of the above theorem can be carried out from pairwise inequalities numbered (2.17) and (2.19) ; (2.16) and (2.22) ; (2.17) and (2.27); (2.16) and (2.30) respectively after applying the same technique of Theorem 2.7 and with the help of Theorem 2.1. Therefore its proof is omitted.

Now we state the following two theorems without their proofs as because those can be derived easily using the same technique or with some easy reasoning with the help of Remark 2.2 and therefore left to the readers.

Theorem 2.11. *Let f and g be two entire functions such that $0 < \rho_{(\gamma,\beta)}[f] < \infty$ and $0 < \rho_{(\gamma,\alpha)}[g]\big(= \lambda_{(\gamma,\alpha)}[g]\big) < \infty$. Then*

$$\begin{aligned}\Big(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]}{\sigma_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}} &\le \overline{\sigma}_{(\alpha,\beta)}[f]_g \\ &\le \min\Big\{\Big(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]}{\overline{\sigma}_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}, \Big(\frac{\sigma_{(\gamma,\beta)}[f]}{\sigma_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}\Big\} \\ &\le \max\Big\{\Big(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]}{\overline{\sigma}_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}, \Big(\frac{\sigma_{(\gamma,\beta)}[f]}{\sigma_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}\Big\} \\ &\le \sigma_{(\alpha,\beta)}[f]_g \le \Big(\frac{\sigma_{(\gamma,\beta)}[f]}{\overline{\sigma}_{(\gamma,\alpha)}[g]}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]}}.\end{aligned}$$

Remark 2.12. In Theorem 2.11, if we will replace the conditions "$0 < \rho_{(\gamma,\beta)}[f] < \infty$ and $0 < \rho_{(\gamma,\alpha)}[g]\big(= \lambda_{(\gamma,\alpha)}[g]\big) < \infty$" by "$0 < \rho_{(\gamma,\beta)}[f]\big(= \lambda_{(\gamma,\beta)}[f]\big) < \infty$ and $0 < \rho_{(\gamma,\alpha)}[g] < \infty$" respectively, then Theorem 2.11 remains valid with $\overline{\tau}_{(\alpha,\beta)}[f]_g$ and $\tau_{(\alpha,\beta)}[f]_g$ replacing $\overline{\sigma}_{(\alpha,\beta)}[f]_g$ and $\sigma_{(\alpha,\beta)}[f]_g$ respectively.

Theorem 2.13. *Let f and g be two entire functions such that $0<\rho_{(\gamma,\beta)}[f]\left(=\lambda_{(\gamma,\beta)}[f]\right)<\infty$ and $0<\lambda_{(\gamma,\alpha)}[g]<\infty$. Then*

$$\begin{aligned}\left(\frac{\overline{\tau}_{(\gamma,\beta)}[f]}{\tau_{(\gamma,\alpha)}[g]}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}} &\leq \overline{\sigma}_{(\alpha,\beta)}[f]_g \\ &\leq \min\left\{\left(\frac{\overline{\tau}_{(\gamma,\beta)}[f]}{\overline{\tau}_{(\gamma,\alpha)}[g]}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}},\left(\frac{\tau_{(\gamma,\beta)}[f]}{\tau_{(\gamma,\alpha)}[g]}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}\right\} \\ &\leq \max\left\{\left(\frac{\overline{\tau}_{(\gamma,\beta)}[f]}{\overline{\tau}_{(\gamma,\alpha)}[g]}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}},\left(\frac{\tau_{(\gamma,\beta)}[f]}{\tau_{(\gamma,\alpha)}[g]}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}\right\} \\ &\leq \sigma_{(\alpha,\beta)}[f]_g \leq \left(\frac{\tau_{(\gamma,\beta)}[f]}{\overline{\tau}_{(\gamma,\alpha)}[g]}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]}}.\end{aligned}$$

Remark 2.14. In Theorem 2.13, if we will replace the conditions "$0<\rho_{(\gamma,\beta)}[f]\left(=\lambda_{(\gamma,\beta)}[f]\right)<\infty$ and $0<\lambda_{(\gamma,\alpha)}[g]<\infty$" by "$0<\lambda_{(\gamma,\beta)}[f]<\infty$ and $0<\rho_{(\gamma,\alpha)}[g]\left(=\lambda_{(\gamma,\alpha)}[g]\right)<\infty$" respectively, then Theorem 2.13 remains valid with $\overline{\tau}_{(\alpha,\beta)}[f]_g$ and $\tau_{(\alpha,\beta)}[f]_g$ replacing $\overline{\sigma}_{(\alpha,\beta)}[f]_g$ and $\sigma_{(\alpha,\beta)}[f]_g$ respectively.

3 Conclusion

After introducing the idea of generalized relative order (α,β) and generalized relative type (α,β) of an entire function with respect to an entire function where α,β are non-negative continuous functions defined on $(-\infty,+\infty)$, here in this paper we study some growth properties of entire functions. This assumption is also used to modify the ideas of relative order and relative type of an entire function with respect to another entire function. Taking these modifications, we derived some results which will no doubt inspire the future researchers to increase their domain of knowledge in this field.

References

[1] L. Bernal-Gonzaléz, *Crecimiento relativo de funciones enteras. Aportaciones al estudio de las funciones enteras con índice exponencial finito*, Doctoral Thesis, Universidad de Sevilla, Spain, (1984).

[2] L. Bernal, Orden relative de crecimiento de funciones enteras, *Collect. Math.*, **39**, 209–229 (1988).

[3] T. Biswas, C. Biswas and B. Saha, Sum and product theorems relating to generalized relative order (α,β) and generalized relative type (α,β) of entire functions, *J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math.*, **28**(2), 155–185 (2021).

[4] T. Biswas and C. Biswas, On some growth properties of composite entire and meromorphic functions from the view point of their generalized type (α,β) and generalized weak type (α,β), *South East Asian J. Math. Math. Sci.*, **17**(1), 31–44 (2021).

[5] T. Biswas, C. Biswas and R. Biswas, A note on generalized growth analysis of composite entire functions, *Poincare J. Anal. Appl.*, **7**(2), 257–266 (2020).

[6] T. Biswas, On some growth analysis of entire and meromorphic functions in the light of their relative $(p,q,t)L$-th order with respect to another entire function, *An. Univ. Oradea, fasc. Mat.*, **26**(1), 59–80 (2019).

[7] T. Biswas, On some inequalities concerning relative (p,q)-φ type and relative (p,q)-φ weak type of entire or meromorphic functions with respect to an entire function, *J. Class. Anal.*, **13**(2), 107–122 (2018).

[8] I. Chyzhykov and N. Semochko, Fast growing entire solutions of linear differential equations, *Math. Bull. Shevchenko Sci. Soc.*, **13**, 1–16 (2016).

[9] O. P. Juneja, G. P. Kapoor and S. K. Bajpai, On the (p,q)-order and lower (p,q)-order of an entire function, *J. Reine Angew. Math.*, **282**, 53–67 (1976).

[10] X. Shen, J. Tu and H. Y. Xu,, Complex oscillation of a second-order linear differential equation with entire coefficients of $[p,q]$-φ order, *Adv. Difference Equ.*, **2014:200**, 14 pp., http://www.advancesindifferenceequations.com/content/2014/1/200, (2014).

[11] D. Sato, On the rate of growth of entire functions of fast growth, *Bull. Amer. Math. Soc.*, **69**, 411–414 (1963).

[12] M. N. Sheremeta, Connection between the growth of the maximum of the modulus of an entire function and the moduli of the coefficients of its power series expansion, *Izv. Vyssh. Uchebn. Zaved Mat.*, **2**, 100–108 (1967) (in Russian).

[13] G. Valiron, *Lectures on the general theory of integral functions*, Chelsea Publishing Company, New York, (1949).

[14] L. Yang, *Value distribution theory*, Springer-Verlag, Berlin, (1993).

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