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Uniform Convergence of Successive Approximations for Hybrid Caputo Fractional Differential Equations

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Abstract This paper deals with the global convergence of successive approximations as well as the uniqueness of solutions for a class of hybrid Caputo fractional differential equations. We prove a theorem on the global convergence of successive approximations to the unique solution of our problem. In the last section, an illustrative example is given.

1 Introduction

The subject of fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) has gained considerable popularity and importance during the past three decades [22]. Fractional calculus has developed especially intensively since 1974 when first international conference in the field took place [17]. It was organized by Bertram Ross [17], several results on fractional calculus and fractional differential equations refer to the monographs [3, 4, 22, 24, 31].

Hybrid Caputo fractional differential equations have aroused a lot of interest and attention from several researchers, for some work on hybrid fractional differential equations, we refer to [5, 11, 16, 25].

Recently, the global convergence of successive approximations has been studied in [1, 2]. Next, for some recent works on the stability and existence of the solutions of various fractional and fractal equations with and without delay, we also refer readers to [18, 20, 21, 29].

In this paper we study uniformly convergence of successive approximations for the initial value problem for hybrid Caputo fractional differential equation:

$$ {}^{c}D^{\alpha}_{0^+}\left[\frac{u(t)}{f(t,u(t))}\right] = g(t,u(t)),\ t \in I = [0,1], \tag{1.1}$$

with initial condition

$$u(0) = \phi, \tag{1.2}$$

where $\alpha \in (0,1),^{C} D^{\alpha}_{0^+}$ is the Caputo fractional derivative, $f \in C(I\times\mathbb{R},\mathbb{R}^*)$, $g \in C(I\times\mathbb{R},\mathbb{R})$.

2 Preliminaries

First, we denote by $C(I) := C(I,\mathbb{R})$, the Banach space of continuous functions from I into $\mathbb{R}$ with the supremum (uniform) norm

$$\|u\|_\infty := Sup_{t\in I}|u(t)|.$$

As usual, $L^1(I)$ denotes the space of measurable functions $v : I \to \mathbb{R}$ which are Lebesgue integrable with the norm

$$\|v\|_1 = \int_0^T |v(t)|dt.$$

Now,we will give some essentials definitions and lemmas of fractional calculus theory in this work .

Definition 2.1. [22] Let $\alpha > 0$, for a function $u : [0, \infty) \to \mathbb{R}$. The Riemann-Liouville fractional integral of order α of u is defined by

$$I_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2. [22] Let $\alpha > 0$. The Caputo fractional derivative of order α of a function $u : (0, \infty) \to \mathbb{R}$ is given by

$${}^{C}D_{0^+}^{\alpha} u(t) = I_{0^+}^{n-\alpha} u^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds,$$

where $n = [\alpha] + 1, [\alpha]$ denotes the integer part of real number α, provided that the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 2.3. *[22] Let $\alpha, \beta \geq 0$, and $u \in L^1([0, 1])$. Then,*

$$I_{0^+}^{\alpha} I_{0^+}^{\beta} u(t) = I_{0^+}^{\alpha+\beta} u(t),$$

and

$${}^{C}D_{0^+}^{\alpha} I_{0^+}^{\alpha} u(t) = u(t),$$

for all $t \in [0, 1]$.

Lemma 2.4. *[22] Let $\alpha > 0, n = [\alpha] + 1$, then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) - \sum_{k=0}^{n-1} c_k t^k, \quad c_k \in \mathbb{R}.$$

Lemma 2.5. *Let $f : I \times \mathbb{R} \to \mathbb{R}^*$, $g : I \times \mathbb{R} \to \mathbb{R}$ be continuous functions. Then the problem $(1.1) - (1.2)$ is equivalent to the integral equation*

$$u(t) = f(t, u(t)) \left\{ \frac{\phi}{f(0, \phi)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, u(s)) ds \right\}. \tag{2.1}$$

3 Successive Approximations and Uniqueness Results

In this section, we will present the main result of the global convergence of successive approximation towards a unique solution of our problem.

Definition 3.1. By a solution of the problem $(1.1) - (1.2)$ we mean a continuous function $u \in C(I)$ that satisfies the equation (1.1) on I and initial condition (1.2).

Set $I_\sigma := [0, \sigma T]$; for any $\sigma \in [0, 1]$. Let us introduce the following hypotheses.

(H_1) The functions $f : I \times \mathbb{R} \to \mathbb{R}^*$ and $g : I \times \mathbb{R} \to \mathbb{R}$ are continuous,

(H_2) There exist a constant $\rho > 0$ and a continuous function $w : I \times [0, \rho] \to \mathbb{R}_+$ such that $w(t, \cdot)$ is nondecreasing for all $t \in I$, and the inequality

$$|g(t, u) - g(t, \overline{u})| \leq w(t, |u - \overline{u}|), \tag{3.1}$$

holds for all $t \in I$ and $u, \overline{u} \in \mathbb{R}$ such that $|u - \overline{u}| \leq \rho$,

(H_3) $V \equiv 0$ is the only function in $C\left(I_\lambda, [0,\rho]\right)$ satisfying the integral inequality

$$V(t) \leq 2 \sup_{(t,u)\in I_\lambda\times[0,\delta]} |f(t,u)| \left(\left| \frac{\phi}{f(0,\phi)} \right| + \sup_{(t,u)\in I_\lambda\times[0,\delta]} |g(t,u)| \frac{t^\alpha - (t-\lambda t)^\alpha}{\Gamma(\alpha+1)} \right)$$

$$+ \sup_{(t,u)\in I_\lambda\times[0,\delta]} |f(t,u)| \int_0^{\lambda t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} w(s, V(s)) ds, \tag{3.2}$$

with $\sigma \leq \lambda \leq 1$.

Define the successive approximations of the problem (1.1) – (1.2) as follows:

$$u_0(t) = \phi;\ t \in I,$$

$$u_{n+1}(t) = f(t, u_n(t)) \left\{ \frac{\phi}{f(0,\phi)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, u_n(s) ds \right\};\ t \in I.$$

Theorem 3.2. *Assume that the hypotheses $(H_1)-(H_3)$ hold. Then the successive approximations $u_n; n \in \mathbb{N}$ are well defined and converge to the unique solution of the problem (1.1) – (1.2) uniformly on I.*

Proof. Since u_n is in $C(I)$, there exist $\delta > 0$ such that

$$\|u_n\|_\infty \leq \delta.$$

From (H_1); the successive approximations are defined. Now, for each $t_1, t_2 \in I$ with $t_1 < t_2$, and for all $t \in I$,

$$\begin{aligned}
|u_n(t_2) - u_n(t_1)| \leq &\left| f(t_2, u_{n-1}(t_2)) \left\{ \frac{\phi}{f(0,\phi)} + \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2-s)^{\alpha-1} g(s, u_{n-1}(s) ds \right\} \right.\\
&\left. - f(t_1, u_{n-1}(t_1)) \left\{ \frac{\phi}{f(0,\phi)} + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} g(s, u_{n-1}(s) ds \right\} \right| \\
\leq &\left| f(t_2, u_{n-1}(t_2)) \left\{ \frac{\phi}{f(0,\phi)} + \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2-s)^{\alpha-1} g(s, u_{n-1}(s) ds \right\} \right. \\
& - f(t_1, u_{n-1}(t_1)) \left\{ \frac{\phi}{f(0,\phi)} + \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2-s)^{\alpha-1} g(s, u_{n-1}(s) ds \right\} \\
& + f(t_1, u_{n-1}(t_1)) \left\{ \frac{\phi}{f(0,\phi)} + \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2-s)^{\alpha-1} g(s, u_{n-1}(s) ds \right\} \\
&\left. - f(t_1, u_{n-1}(t_1)) \left\{ \frac{\phi}{f(0,\phi)} + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} g(s, u_{n-1}(s) ds \right\} \right| \\
\leq &\left| f(t_2, u_{n-1}(t_2)) - f(t_1, u_{n-1}(t_1)) \right| \left| \frac{\phi}{f(0,\phi)} + \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2-s)^{\alpha-1} g(s, u_{n-1}(s) ds \right| \\
& + |f(t_1, u_{n-1}(t_1))| \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_2-s)^{\alpha-1} g(s, u_{n-1}(s) ds \right. \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} g(s, u_{n-1}(s) ds \\
&\left. - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} g(s, u_{n-1}(s) ds \right|.
\end{aligned}$$

Thus

$$|u_n(t_2) - u_n(t_1)| \leq \left|f(t_2, u_{n-1}(t_2)) - f(t_1, u_{n-1}(t_1))\right| \left|\frac{\phi}{f(0,\phi)}\right.$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} g(s, u_{n-1}(s) ds\right|$$
$$+ sup(t,u) \in I \times [0,\delta] |f(t,u)| \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_1} \Big((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \Big) g(s, u_{n-1}(s) ds\right.$$
$$+ \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} g(s, u_{n-1}(s) ds\right|$$
$$\leq \left|f(t_2, u_{n-1}(t_2)) - f(t_1, u_{n-1}(t_1))\right| \left(\left|\frac{\phi}{f(0,\phi)}\right| + \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} |g(s, u_{n-1}(s)| ds \right)$$
$$+ sup(t,u) \in I \times [0,\delta] |f(t,u)| \frac{1}{\Gamma(\alpha)} \left(\int_0^{t_1} \left| (t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right| |g(s, u_{n-1}(s)| ds\right.$$
$$\left. + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} |g(s, u_{n-1}(s)| ds \right)$$
$$\leq \left|f(t_2, u_{n-1}(t_2)) - f(t_1, u_{n-1}(t_1))\right| \left(\left|\frac{\phi}{f(0,\phi)}\right| + \sup_{(t,u)\in I\times[0,\delta]} |g(t,u)| \int_0^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} ds \right)$$
$$+ \sup_{(t,u)\in I\times[0,\delta]} |f(t,u)|$$
$$\times \sup_{(t,u)\in I\times[0,\delta]} |g(t,u)| \left(\int_0^{t_1} \frac{\left|(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\right|}{\Gamma(\alpha)} ds + \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} ds \right).$$

From the continuity of the fonction f, we obtain

$$\begin{aligned}
|u_n(t_2) - u_n(t_1)| \leq & \left|f(t_2, u_{n-1}(t_2)) - f(t_1, u_{n-1}(t_1))\right| \\
& \times \left(\left|\frac{\phi}{f(0,\phi)}\right| + \sup_{(t,u)\in I\times[0,\delta]} |g(t,u)| \frac{t_2^\alpha}{\Gamma(\alpha+1)} \right) \\
& + \sup_{(t,u)\in I\times[0,\delta]} |f(t,u)| \\
& \times \sup_{(t,u)\in I\times[0,\delta]} |g(t,u)| \left(\int_0^{t_1} \frac{\left|(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\right|}{\Gamma(\alpha)} ds \right. \\
& \left. + \frac{(t_2 - t_1)^\alpha}{\Gamma(\alpha+1)} \right) \\
& \longrightarrow 0, \ as \ t_1 \to t_2.
\end{aligned}$$

Hence

$$|u_n(t_2) - u_n(t_1)| \longrightarrow 0, \ \ as \ \ t_1 \to t_2.$$

The sup $\{u_n(t);\ n \in \mathbb{N}\}$ is equi-continuous on I.

Let

$$\tau := sup\{\sigma \in [0,1] : \{u_n(t)\} \text{ converges uniformly on } I_\sigma\}.$$

If $\tau = 1$, then we have the global convergence of successive approximations. Suppose that $\tau < 1$, then this sequence is equi-continuous, so it converges uniformly to a continuous function $\tilde{u}(t)$. If we prove that there exists $\lambda \in (\tau, 1]$ such that $\{u_n(t)\}$ converges uniformly on I_λ, this will yield a contradiction.

Put $u(t) = \tilde{u}(t)$; for $t \in I_\tau$. From (H_2), there exist a constant $\rho > 0$ and a continuous function $w : I \times [0, \rho] \to \mathbb{R}_+$ satisfying inequality (3.1). Also, there exist $\lambda \in [\tau, 1]$ and $n_0 \in \mathbb{N}$, such that, for all $t \in I_\lambda$ and $n, m > n_0$, we have

$$|u_n(t) - u_m(t)| \leq \rho.$$

For any $t \in I_\lambda$, put

$$V^{(n,m)}(t) = sup|u_n(t) - u_m(t)|,$$

$$V_k(t) = sup_{n,m\geq k} V^{(n,m)}(t).$$

Since the sequence $V_k(t)$ is non-increasing, it is convergent to a function $V(t)$ for each $t \in I_\lambda$. From the equi-continuity of $\{V_k(t)\}$ it follows that $Lim_{k\to\infty} V_k(t) = V(t)$ uniformly on I_λ. Furthermore, for $t \in I_\lambda$ and $n, m \geq k$, we have

$$\begin{aligned}
&V^{(n,m)}(t) = sup|u_n(t) - u_m(t)| \\
&\leq sup_{s\in[0,t]}|u_n(s) - u_m(s)| \\
&\leq \left| f(t, u_{n-1}(t)) \left\{ \frac{\phi}{f(0,\phi)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, u_{n-1}(s)ds \right\} \right. \\
&\left. - f(t, u_{m-1}(t)) \left\{ \frac{\phi}{f(0,\phi)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, u_{m-1}(s)ds \right\} \right|.
\end{aligned}$$

Then,

$$\begin{aligned}
&V^{(n,m)}(t) = |u_n(t) - u_m(t)| \\
&\leq \left| f(t, u_{n-1}(t)) \left\{ \frac{\phi}{f(0,\phi)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, u_{n-1}(s)ds \right\} \right. \\
&- f(t, u_{m-1}(t)) \left\{ \frac{\phi}{f(0,\phi)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, u_{n-1}(s)ds \right\} \\
&+ f(t, u_{m-1}(t)) \left\{ \frac{\phi}{f(0,\phi)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, u_{n-1}(s)ds \right\} \\
&\left. - f(t, u_{m-1}(t)) \left\{ \frac{\phi}{f(0,\phi)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, u_{m-1}(s)ds \right\} \right| \\
&\leq \left| f(t, u_{n-1}(t)) - f(t, u_{m-1}(t) \right| \left| \frac{\phi}{f(0,\phi)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, u_{n-1}(s)ds \right| \\
&+ |f(t, u_{m-1}(t))| \\
&\times \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, u_{n-1}(s)ds - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, u_{m-1}(s)ds \right|.
\end{aligned}$$

Thus

$$\begin{aligned}
&V^{(n,m)}(t) = |u_n(t) - u_m(t)| \\
&\leq \left| f(t, u_{n-1}(t)) - f(t, u_{m-1}(t) \right| \left(\left| \frac{\phi}{f(0,\phi)} \right| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |g(s, u_{n-1}(s)| ds \right) \\
&+ |f(t, u_{m-1}(t))| \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \right| g(s, u_{n-1}(s) - g(s, u_{m-1}(s) \Big| ds \\
&\leq \left| f(t, u_{n-1}(t)) - f(t, u_{m-1}(t) \right| \left(\left| \frac{\phi}{f(0,\phi)} \right| + \sup_{(t,u)\in I_\lambda\times[0,\delta]} |g(t,u)| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \right) \\
&+ \sup_{(t,u)\in I_\lambda\times[0,\delta]} |f(t,u)| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big| g(s, u_{n-1}(s) - g(s, u_{m-1}(s) \Big| ds.
\end{aligned}$$

This gives

$$
\begin{aligned}
V^{(n,m)}(t) &= |u_n(t) - u_m(t)| \\
&\leq 2 \sup_{(t,u)\in I_\lambda\times[0,\delta]} |f(t,u)| \left(\left| \frac{\phi}{f(0,\phi)} \right| + \sup_{(t,u)\in I_\lambda\times[0,\delta]} |g(t,u)| \int_0^{\lambda t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \right) \\
&+ \sup_{(t,u)\in I_\lambda\times[0,\delta]} |f(t,u)| \int_0^{\lambda t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big| g(s,u_{n-1}(s) - g(s,u_{m-1}(s) \Big| ds.
\end{aligned}
$$

Next, by (3.1) we get

$$
\begin{aligned}
V^{(n,m)}(t) &\leq 2 \sup_{(t,u)\in I_\lambda\times[0,\delta]} |f(t,u)| \left(\left| \frac{\phi}{f(0,\phi)} \right| + \sup_{(t,u)\in I_\lambda\times[0,\delta]} |g(t,u)| \frac{t^\alpha - (t-\lambda t)^\alpha}{\Gamma(\alpha+1)} \right) \\
&+ \sup_{(t,u)\in I_\lambda\times[0,\delta]} |f(t,u)| \int_0^{\lambda t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} w(s,|u_{n-1} - u_{m-1}|) ds \\
&\leq 2 \sup_{(t,u)\in I_\lambda\times[0,\delta]} |f(t,u)| \left(\left| \frac{\phi}{f(0,\phi)} \right| + \sup_{(t,u)\in I_\lambda\times[0,\delta]} |g(t,u)| \frac{t^\alpha - (t-\lambda t)^\alpha}{\Gamma(\alpha+1)} \right) \\
&+ \sup_{(t,u)\in I_\lambda\times[0,\delta]} |f(t,u)| \int_0^{\lambda t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} w(s, V^{(n-1,m-1)}(s)) ds.
\end{aligned}
$$

Hence

$$
\begin{aligned}
V_k(t) &\leq 2 \sup_{(t,u)\in I_\lambda\times[0,\delta]} |f(t,u)| \left(\left| \frac{\phi}{f(0,\phi)} \right| + \sup_{(t,u)\in I_\lambda\times[0,\delta]} |g(t,u)| \frac{t^\alpha - (t-\lambda t)^\alpha}{\Gamma(\alpha+1)} \right) \\
&\quad + \sup_{(t,u)\in I_\lambda\times[0,\delta]} |f(t,u)| \int_0^{\lambda t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} w(s, V_{k-1}(s)) ds.
\end{aligned}
$$

By the Lebesgue dominated convergence theorem we get

$$
\begin{aligned}
V(t) &\leq 2 \sup_{(t,u)\in I_\lambda\times[0,\delta]} |f(t,u)| \left(\left| \frac{\phi}{f(0,\phi)} \right| + \sup_{(t,u)\in I_\lambda\times[0,\delta]} |g(t,u)| \frac{t^\alpha - (t-\lambda t)^\alpha}{\Gamma(\alpha+1)} \right) \\
&\quad + \sup_{(t,u)\in I_\lambda\times[0,\delta]} |f(t,u)| \int_0^{\lambda t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} w(s, V(s)) ds.
\end{aligned}
$$

Then, by (H_1) and (H_3) we get $V \equiv 0$ on I_λ, which yields that $Lim_{k\to\infty} V_k(t) = 0$ uniformly on I_λ. Thus $\{u_k(t)\}_{k=1}^\infty$ is a Cauchy sequence on I_λ. Consequently $\{u_k(t)\}_{k=1}^\infty$ is uniformly convergent on I_λ which yields the contradiction.

Thus $\{u_k(t)\}_{k=1}^\infty$ converges uniformly on I to a continuous function $u_*(t)$. By the Lebesgue dominated convergence theorem, we get

$$
\begin{aligned}
&Lim_{k\to\infty} f(t,u_k(t)) \left\{ \frac{\phi}{f(0,\phi)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s,u_k(s) ds \right\} \\
&= f(t,u_*(t)) \left\{ \frac{\phi}{f(0,\phi)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s,u_*(s) ds \right\},
\end{aligned}
$$

for each $t \in I$. This yields that u_* is a solution of the problem (1.1)-(1.2).

Finally, we show the uniqueness of solutions of the problem (1.1)-(1.2). Let u_1 and u_2 be two solutions of (2.1). Put

$$\tau := sup\{\sigma \in [0,1] : u_1(t) = u_2(t) \ for \ t \in I_\sigma\},$$

and suppose that $\tau < 1$. There exist a constant $\rho > 0$ and a comparison function $w : I_\tau \times [0,\rho] \to \mathbb{R}_+$ satisfying inequality (2.1). We choose $\lambda \in (\sigma, 1)$ such that

$$|u_1(t) - u_2(t)| \leq \rho\,;$$

for $t \in I_\lambda$. Then for all $t \in I_\lambda$ we obtain

$$
\begin{aligned}
|u_1(t) - u_2(t)| \le 2 \sup_{(t,u)\in I_\lambda\times[0,\delta]} |f(t,u)| \Bigg(\left| \frac{\phi}{f(0,\phi)} \right| \\
&+ \sup_{(t,u)\in I_\lambda\times[0,\delta]} |g(t,u)| \int_0^{\lambda t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \Bigg) \\
&+ \sup_{(t,u)\in I_\lambda\times[0,\delta]} |f(t,u)| \int_0^{\lambda t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Big| g(s,u_0(s) - g(s,u_1(s) \Big| ds \\
&\le 2 \sup_{(t,u)\in I_\lambda\times[0,\delta]} |f(t,u)| \Bigg(\left| \frac{\phi}{f(0,\phi)} \right| \\
&+ \sup_{(t,u)\in I_\lambda\times[0,\delta]} |g(t,u)| \frac{t^\alpha - (t-\lambda t)^\alpha}{\Gamma(\alpha+1)} \Bigg) \\
&+ \sup_{(t,u)\in I_\lambda\times[0,\delta]} |f(t,u)| \int_0^{\lambda t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} w(s,|u_0 - u_1|) ds.
\end{aligned}
$$

Again, by (H_1) and (H_3) we get $u_1 - u_2 \equiv 0$ on I_λ. This gives $u_1 = u_2$ on I_λ, which yields a contradiction. Consequently, $\tau = 1$ and the solution of the problem (1.1)-(1.2) is unique on I.

4 An Example

Consider the following Hybrid Caputo fractional differential equation

$$
\begin{cases}
{}^cD_{0^+}^{\frac{1}{2}} \left[\frac{u(t)}{\sqrt{1+|u(t)|}} \right] = \frac{te^t}{1+|u(t)|};\ t \in [0,1], \\
u(0) = 3.
\end{cases}
\tag{4.1}
$$

For each $u,\ \overline{u} \in \mathbb{R}$ and $t \in [0,1]$ we have

$$|g(t,u) - g(t,\overline{u})| \le te^t(|u - \overline{u}|).$$

This means that condition (3.1) holds with any $t \in [0,1],\ \rho > 0$ and the comparison function $w : [0,1] \times [0,\rho] \to [0,\infty)$ given by

$$w(t,u) = te^t|u|.$$

Consequently, Theorem 3.2 implies that the successive approximations $u_n;\ n \in \mathbb{N}$, defined by

$$u_0(t) = 3;\ t \in [0,1],$$

$$u_{n+1}(t) = f(t,u_n(t)) \left\{ \frac{3}{f(0,3)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s,u_n(s)ds \right\};\ t \in [0,1],$$

converges uniformly on $[0,1]$ to the unique solution of the problem (4.1).

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