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NEW IDENTITY CLASSES FOR GENERALISED DEGREE THREE LINEAR RECURRENCE SEQUENCE TERMS

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Abstract We derive, using matrices, new classes of linear recurrence identities linking the general term of a degree three linear recurrence sequence with those of an associated 'cohort' sequence that differs only in its initial values. The class description is seen to recover that previously established for the so called Horadam sequence and its cohort sequence set.

1 Introduction

Denote by $\{w_n\}_{n=0}^{\infty} = \{w_n\}_0^{\infty} = \{w_n(a,b,c;p,q,r)\}_0^{\infty}$, in standard format, the six-parameter sequence arising from the third order linear recursion

$$w_{n+3} = pw_{n+2} + qw_{n+1} + rw_n, \qquad n \geq 0, \tag{1.1}$$

for which $w_0 = a$, $w_1 = b$ and $w_2 = c$ are arbitrary initial values. For any fixed p, q, r, let $\mathcal{C}(p,q,r)$ be the collection, or *cohort*, of associated sequences of individual form $\{v_n\}_0^{\infty} = \{w_n(v_0,v_1,v_2;p,q,r)\}_0^{\infty}$ that each arise as a particular instance of (1.1) with variable start values v_0, v_1 and v_2; any triplet v_0, v_1, v_2 describes a so called cohort sequence within $\mathcal{C}(p,q,r)$, and there are an infinite number of them.

2 Formulation of Result

2.1 Previous Work

Let

$$\mathbf{H} = \mathbf{H}(p,q,r) = \begin{pmatrix} p & q & r \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{2.1}$$

from which the recursion (1.1) readily delivers the matrix power relation

$$\begin{pmatrix} w_n(a,b,c;p,q,r) \\ w_{n-1}(a,b,c;p,q,r) \\ w_{n-2}(a,b,c;p,q,r) \end{pmatrix} = \mathbf{H}^{n-2}(p,q,r) \begin{pmatrix} c \\ b \\ a \end{pmatrix} \tag{2.2}$$

that holds for $n \geq 2$. We seek a matrix $\mathbf{B} = \mathbf{B}(v_0,v_1,v_2,a,b,c;p,q,r)$ which has the following properties (where T denotes transposition):

Property 1: $\mathbf{B}(c,b,a)^T = (v_2,v_1,v_0)^T$;

Property 2: $\mathbf{BH} = \mathbf{HB}$.

Then, repeating the derivation steps seen in [1, (P.4), p. 105]—which deploy both Properties

1 and 2—we arrive at the matrix equation

$$\begin{aligned}
&(w_n, w_{n-1}, w_{n-2})^T \\
&\quad = (w_n(a,b,c;p,q,r), w_{n-1}(a,b,c;p,q,r), w_{n-2}(a,b,c;p,q,r))^T \\
&\quad \vdots \\
&\quad = \mathbf{B}^{-1}(w_n(v_0,v_1,v_2;p,q,r), w_{n-1}(v_0,v_1,v_2;p,q,r), w_{n-2}(v_0,v_1,v_2;p,q,r))^T \\
&\quad = \mathbf{B}^{-1}(v_n, v_{n-1}, v_{n-2})^T,
\end{aligned} \tag{2.3}$$

which contains one independent algebraic relation describing a set of identity classes (the reader is referred to the line of argument set down in [1] for the lower order case therein); we will establish the precise form of the governing result accordingly.

2.2 Details

Initially, we enforce Property 2 which, writing

$$\mathbf{B} = \mathbf{B}(v_0, v_1, v_2, a, b, c; p, q, r) = \begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{pmatrix}, \tag{2.4}$$

yields the nine equations

$$\begin{aligned}
b_2 &= qb_4 + rb_7, \\
qb_1 + b_3 &= pb_2 + qb_5 + rb_8, \\
rb_1 &= pb_3 + qb_6 + rb_9, \\
pb_4 + b_5 &= b_1, \\
qb_4 + b_6 &= b_2, \\
rb_4 &= b_3, \\
pb_7 + b_8 &= b_4, \\
qb_7 + b_9 &= b_5, \\
rb_7 &= b_6,
\end{aligned} \tag{2.5}$$

for $b_1, \ldots, b_9$, from which we express the matrix $\mathbf{B}$ (in terms of b_4, b_5 and b_7) as[1]

$$\mathbf{B}(b_4, b_5, b_7) = \begin{pmatrix} pb_4 + b_5 & qb_4 + rb_7 & rb_4 \\ b_4 & b_5 & rb_7 \\ b_7 & b_4 - pb_7 & b_5 - qb_7 \end{pmatrix}. \tag{2.6}$$

With this form of $\mathbf{B}$ we now apply Property 1, which results in the equations

$$\begin{aligned}
v_2 &= (cp + bq + ar)b_4 + cb_5 + brb_7, \\
v_1 &= cb_4 + bb_5 + arb_7, \\
v_0 &= bb_4 + ab_5 + (c - bp - aq)b_7.
\end{aligned} \tag{2.7}$$

Upon solving for the variables b_4, b_5 and b_7 in terms of $v_0, v_1, v_2, a, b, c, p, q, r$, we in turn back-substitute these into the system of relations (2.5), as appropriate, to find b_1, b_2, b_3, b_6, b_8 and b_9. This, then, establishes the matrix $\mathbf{B}$ (2.4), whence, noting that $|\mathbf{B}| = b_1(b_5b_9 - b_6b_8) - b_2(b_4b_9 - b_6b_7) + b_3(b_4b_8 - b_5b_7)$, with

$$\mathbf{B}^{-1} = \frac{1}{|\mathbf{B}|} \begin{pmatrix} b_5b_9 - b_6b_8 & -(b_2b_9 - b_3b_8) & b_2b_6 - b_3b_5 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \tag{2.8}$$

[1] We do not here use the 2nd, 3rd or 5th equation listed in (2.5), but it is readily checked that they are part of a self-consistent set of relations (simple reader exercise).

we can write down, for $n \geq 2$,

$$|\mathbf{B}|w_n = (b_5b_9 - b_6b_8)v_n - (b_2b_9 - b_3b_8)v_{n-1} + (b_2b_6 - b_3b_5)v_{n-2} \tag{2.9}$$

immediately from (2.3). Moreover, it is found that each element of the matrix $\mathbf{B}$ is a rational algebraic expression with common denominator—that is to say,

$$b_i = \beta_i/\alpha, \qquad i = 1, \ldots, 9, \tag{2.10}$$

where $\alpha = \alpha(a, b, c, p, q, r)$ and $\beta_i = \beta_i(v_0, v_1, v_2, a, b, c, p, q, r)$, whose explicit forms are

$$\begin{aligned} \alpha \; = \; & a^3r^2 + 2a^2bqr + a^2cpr + ab^2pr + ab^2q^2 + abcpq \\ & + b^3pq + b^2cp^2 - 3abcr - ac^2q + b^3r - b^2cq - 2bc^2p + c^3, \end{aligned} \tag{2.11}$$

and

$$\begin{aligned}
\beta_1 \; = \; & a^2prv_2 + a^2qrv_1 + a^2r^2v_0 + abpqv_2 + abq^2v_1 + abqrv_0 \\
& + b^2p^2v_2 + b^2pqv_1 + b^2prv_0 - abrv_2 - acqv_2 - acrv_1 \\
& + b^2rv_1 - 2bcpv_2 - bcqv_1 - bcrv_0 + c^2v_2, \\
\beta_2 \; = \; & a^2qrv_2 + a^2r^2v_1 + abq^2v_2 - abr^2v_0 + acprv_1 - acq^2v_1 \\
& - acqrv_0 + b^2pqv_2 - bcpqv_1 - bcprv_0 - acrv_2 + b^2rv_2 \\
& - bcqv_2 - bcrv_1 + c^2qv_1 + c^2rv_0, \\
\beta_3 \; = \; & r(a^2rv_2 + abqv_2 - abrv_1 - acqv_1 - acrv_0 + b^2pv_2 + b^2rv_0 - bcpv_1 - bcv_2 + c^2v_1), \\
\beta_4 \; = \; & a^2rv_2 + abqv_2 - abrv_1 - acqv_1 - acrv_0 + b^2pv_2 + b^2rv_0 - bcpv_1 - bcv_2 + c^2v_1, \\
\beta_5 \; = \; & a^2qrv_1 + a^2r^2v_0 + abprv_1 + abq^2v_1 + abqrv_0 + acpqv_1 \\
& + acprv_0 + b^2pqv_1 + bcp^2v_1 - abrv_2 - acqv_2 - acrv_1 \\
& + b^2rv_1 - bcpv_2 - bcqv_1 - bcrv_0 - c^2pv_1 + c^2v_2, \\
\beta_6 \; = \; & r(a^2rv_1 + abqv_1 - abrv_0 + acpv_1 - b^2qv_0 - bcpv_0 - acv_2 + b^2v_2 - bcv_1 + c^2v_0), \\
\beta_7 \; = \; & a^2rv_1 + abqv_1 - abrv_0 + acpv_1 - b^2qv_0 - bcpv_0 - acv_2 + b^2v_2 - bcv_1 + c^2v_0, \\
\beta_8 \; = \; & -a^2prv_1 - abpqv_1 + abprv_0 - acp^2v_1 + b^2pqv_0 + bcp^2v_0 \\
& + a^2rv_2 + abqv_2 - abrv_1 + acpv_2 - acqv_1 - acrv_0 \\
& + b^2rv_0 - c^2pv_0 - bcv_2 + c^2v_1, \\
\beta_9 \; = \; & a^2r^2v_0 + abprv_1 + 2abqrv_0 + acprv_0 + b^2pqv_1 + b^2q^2v_0 \\
& + bcp^2v_1 + bcpqv_0 - abrv_2 - acrv_1 - b^2qv_2 + b^2rv_1 \\
& - bcpv_2 - bcrv_0 - c^2pv_1 - c^2qv_0 + c^2v_2.
\end{aligned} \tag{2.12}$$

Thus (2.9) (given (2.10)-(2.12)) delivers directly our final result, completing the formulation; evidently, this Governing Identity describes *classes* of identities because it holds for all cohort sequences characterised by p, q, r (which latter variables, as a triplet, give rise to any class instance).

Governing Identity. Let $T(\beta_1, \ldots, \beta_9) = \beta_1(\beta_5\beta_9 - \beta_6\beta_8) - \beta_2(\beta_4\beta_9 - \beta_6\beta_7) + \beta_3(\beta_4\beta_8 - \beta_5\beta_7)$.

Then, noting that $v_n = w_n(v_0, v_1, v_2; p, q, r)$ ($n \geq 0$), for $n \geq 2$,

$$T(\beta_1, \ldots, \beta_9)w_n = \alpha[(\beta_5\beta_9 - \beta_6\beta_8)v_n - (\beta_2\beta_9 - \beta_3\beta_8)v_{n-1} + (\beta_2\beta_6 - \beta_3\beta_5)v_{n-2}].$$

Remark 2.1. As a quick check, we observe that the result is self-satisfying on setting $v_0 = a$, $v_1 = b$ and $v_2 = c$ (for the sequence $\{w_n(a, b, c; p, q, r)\}_0^\infty$ sits within its own set of cohort sequences). In this case β_1, β_5 and β_9 all reduce to $\alpha(a, b, c, p, q, r)$, with $\beta_2 = \beta_3 = \beta_4 = \beta_6 =$

$\beta_7 = \beta_8 = 0$ so that $\mathbf{B}$ (2.4) contracts to $\mathbf{I}_3$, the 3×3 identity matrix. Proposition 1 is satisfied trivially, as is Proposition 2 ($\mathbf{I}_3$ is the simplest matrix commuting with $\mathbf{H}$), whereupon both sides of the Governing Identity become $\alpha^3 w_n$ by inspection (the $\{v_n\}_0^\infty$ cohort sequence instance now co-incides with the sequence $\{w_n(a,b,c;p,q,r)\}_0^\infty$, as noted).

Remark 2.2. The previous lower order case of [1] is reproduced in the Appendix, to further validate the Governing Identity, for the interested reader.

To finish, it is noted that the Governing Identity has been checked for a host of values for $n \geq 2$ using the fully symbolic form of terms of the sequences $\{w_n(a,b,c;p,q,r)\}_0^\infty = \{a, b, c, ar + bq+cp, apr+b(pq+r)+c(p^2+q), \ldots\}$ and $\{v_n\}_0^\infty = \{w_n(v_0, v_1, v_2; p, q, r)\}_0^\infty = \{v_0, v_1, v_2, v_0 r + v_1 q + v_2 p, v_0 pr + v_1(pq + r) + v_2(p^2 + q), \ldots\}$; algebraic computation has been executed up to a value $n = 20$, in which each side of the result in expanded form contains almost 20,000 matching terms (with w_{20}, v_{20} themselves each containing over 100 terms). The author wishes to extend his thanks to Dr. James Clapperton for undertaking all of the computing aspects of the work presented here.

Appendix

In this Appendix, the lower order case presented in [1] is reproduced by recovering the corresponding $\mathbf{B}$ matrix therein, to which $\mathbf{B}$ (2.4) collapses. That recent paper deals with identity classes, formulated in the same way, for the classic Horadam sequence which is, subject to arbitrary initial values $w_0 = a$, $w_1 = b$, generated from the recurrence equation

$$w_{n+2} = pw_{n+1} - qw_n, \qquad n \geq 0, \tag{A.1}$$

along with all of its cohort sequences having start values v_0, v_1.

Comparing (A.1) with (1.1) we set $q \to -q$ and $r = 0$, whereupon we observe $\beta_3 = \beta_6 = 0$ (immediate from (2.12)) $\Rightarrow b_3 = b_6 = 0$ (by (2.10)) and, in addition, we see contractions

$$\begin{aligned} b_1 &= \frac{(c - bp)v_2 + bqv_1}{b^2 q - bcp + c^2}, \\ b_2 &= \frac{q(bv_2 - cv_1)}{b^2 q - bcp + c^2}, \\ b_4 &= \frac{cv_1 - bv_2}{b^2 q - bcp + c^2}, \\ b_5 &= \frac{cv_2 + (bq - cp)v_1}{b^2 q - bcp + c^2}, \end{aligned} \tag{A.2}$$

which reduce further, on setting $c = pb - qa$ and $v_2 = pv_1 - qv_0$ (c, v_2 assuming their naturally generated forms, via (A.1), in terms of recurrence characterising parameters p, q and respective start value pairs for the Horadam sequence and its cohort sequence set),

$$\begin{aligned} b_1 &= \frac{bv_1 - a(pv_1 - qv_0)}{qa^2 + b^2 - pab}, \\ b_2 &= \frac{q(av_1 - bv_0)}{qa^2 + b^2 - pab}, \\ b_4 &= \frac{bv_0 - av_1}{qa^2 + b^2 - pab}, \\ b_5 &= \frac{bv_1 - (pb - qa)v_0}{qa^2 + b^2 - pab}; \end{aligned} \tag{A.3}$$

thus we see that these four simplified matrix terms of $\mathbf{B}$ (2.4) agree, as expected, with those comprising the 2×2 matrix of [1, (P.3), p. 105] (which latter are given through (2.1),(2.2) on p. 104 therein).

Finally, note that each of b_7, b_8 and b_9 become indeterminate, of form $0/0$, upon applying the reducing conditions as stated (for $q, r, c = c(p, q, a, b), v_2 = v_2(p, q, v_0, v_1)$; this is left as a routine reader exercise)—these, together with the wipeout of b_3 and b_6, reflect the reduction of the dimension of both the matrix $\mathbf{H}$ (2.1) (as well as $\mathbf{B}$ (2.4)), and the associated power relation (2.2), from 3 to 2. It is always pleasing to recover a particular result from a more general one to gain confidence in the latter.

References

[1] P. J. Larcombe and E. J. Fennessey, New classes of generalised linear recurrence Horadam sequence term identities, *Palest. J. Math.* **10**, 104–108 (2021).

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