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REFLECTIONS ON WHAT MATHEMATICS IS AND ISN'T: HALMOS, KEYSER, AND OTHERS

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Abstract Two outstanding thinkers and writers about mathematics each had very interesting things to say about the subject. Their words—although from bygone eras—still carry resonance today, as do those of some other mathematicians which are also included in context.

1 Introduction

Paul R. Halmos (1916–2006) was a Hungarian-born American mathematician, well known as an informed philosopher and ferociously active expositor who contributed hugely to the mathematical community with dedicated service across a number of fronts. Cassius J. Keyser (1862–1947) was an American mathematician also, he too in possession of his own pronounced philosophical leanings who likewise wrote extensively on mathematical matters. I wish, in this encomium, to highlight some of their thoughts on what mathematics is, and isn't, as a subject, if only to at least pique interest among readers that we may ourselves reflect on the topic a little (which is no bad thing). These are concerns very much subjective, of course, but I feel the deliberations set down here are worth sharing. They contain differences in both nature and tone—whose roots lie, I believe, in the fact that Halmos was a researcher, which as such Keyser wasn't—and similarities despite a birth separation of over half a century; perhaps more importantly, when put together alongside sentiments of others the creeds and ideologies on show are evidently premised on an enduring love for, and unshakeable delight in, mathematics that will stir something in us all.

2 What Mathematics Is (Halmos and Keyser)

In answer to a previous suggestion of his that all mathematics comes to us, is suggested to us, by the physical universe, Halmos (in an extensive interview published in [2]) stated that while there is of course truth in this, it was shallow, uninformative and meaningless to say that

> "... we are human beings with eyes, and we can see things that we think are outside of us. Our mathematics—our instinctive, unformulated, undefined terms—come from our sense impressions; ..." (p. 13),

and was prompted to clarify what the subject meant to him:

> "It is security. Certainty. Truth. Beauty. Insight. Structure. Architecture. I see mathematics, the part of human knowledge that I call mathematics, as one thing—one great, glorious thing. Whether it is differential topology, or functional analysis, or homological algebra, it is all one thing. They all have to do with each other, and even though a differential topologist may not know any functional analysis, every little bit he hears, every rumor that comes to him about that other subject, sounds like something that he does know. They are intimately interconnected, and they are all facets of the same thing. That interconnection, that architecture, is secure truth and is beauty. That's what mathematics is to me."

There is a commonality of sentiment which, at a fundamental level of consciousness, is surely part and parcel of our endeavours as professionals. Pressing the point, he said

> "It all hangs together. Be it topology, or algebra, or functional analysis, or combinatorics, it is the same subject with the same facets of the same diamond; it's beautiful and it's a work of art. In all parts of the subject the language is the same; the attitude is the same; the way the researcher feels when he sits down at his desk is the same; the way he feels when he starts a problem is the same." (p. 17).

Unconsciously, as well, there are some profound sensibilities that many, I know, nurture:

> "I'm not a religious man, but it's almost like being in touch with God when you're thinking about mathematics. God is keeping secrets from us, and it's fun to try to learn some of the secrets." (p. 21),

and it is no surprise, therefore, that he was of the opinion that

> "All educable human beings should know what mathematics *is* because their souls would grow by that. They would enjoy life more, they would understand life more, they would have greater insight." (p. 15).

Halmos achieved much in his lifetime, living and breathing mathematics—and championing its cause—to the extent that he must be accorded serious agency. His picture of mathematics as a complete and edifying entity is comforting.

This leads us on to Keyser—one time school teacher and principal, and later professor at Columbia University for over twenty years—who gave an address to a meeting of the Michigan School Masters' Club on 28th March, 1912, at Ann Arbor. Published in the journal *Science* the following month, he set out a detailed experiential testimony on what he termed the humanisation of the teaching of mathematics in schools and colleges that reveals his emotional attachment to the discipline. I draw on a passage of his measured and wide ranging transcript article, where he litters his pronouncements with metaphor and analogy so as to capture the essence of his convictions; it is a lovely piece to have come across.

> "I do not forget, . . . , that, of all theory, mathematical theory is the most abstract. I do not forget that mathematics therefore lends especial sharpness to the contrast in the Mephistophelian warning:
>
> *Gray, my dear friend, is all theory,*
> *Green the golden tree of life.*
>
> Yet I know that one who loves not the gray of a naked woodland has much to learn of the esthetic resources of our northern clime. A mathematical doctrine, taken in its purity, is indeed gray. Yet such a doctrine, a world-filling theory woven of gray relationships finer than gossamer but stronger than cables of steel, leaves upon an intersecting plane a tracery surpassing in fineness and beauty the exquisite artistry of frost-work upon a windowpane. Architecture, it has been said, is frozen music. Be it so. Geometry is frozen architecture.
>
> No, the belief that mathematics, because it is abstract, because it is static and cold and gray, is detached from life, is a mistaken belief. Mathematics, even in its purest and most abstract estate, is not detached from life. It is just the ideal handling of the problems of life, as sculpture may idealize a human figure or as poetry or painting may idealize a figure or a scene. Mathematics is precisely the ideal handling of the problems of life, and the central ideas of the science, the great concepts about which its stately doctrines have been built up, are precisely the chief ideas with which life must always deal and which, as it tumbles and rolls about them through time and space, give it its interests and problems, and its order and rationality. That such is the case a few indications will suffice to show. The mathematical concepts of constant and variable are represented familiarly in life by the notions of fixedness and change. The concept of equation or that of an equational system, imposing restriction upon variability, is matched in life by the concept of natural and spiritual law, giving order to what were else chaotic change and providing partial freedom in lieu of none at

all. What is known in mathematics under the name of limit is everywhere present in life in the guise of some ideal, some excellence high-dwelling among the rocks, an "ever flying perfect" as Emerson calls it,[1] unto which we may approximate nearer and nearer, but which we can never quite attain, save in aspiration. The supreme concept of functionality finds its correlate in life in the all-pervasive sense of interdependence and mutual determination among the elements of the world. What is known in mathematics as transformation—that is, lawful transfer of attention, serving to match in orderly fashion the things of one system with those of another—is conceived in life as a process of transmutation by which, in the flux of the world, the content of the present has come out of the past and in its turn, in ceasing to be, gives birth to its successor, as the boy is father to the man and as things, in general, become what they are not. The mathematical concept of invariance and that of infinitude, especially the imposing doctrines that explain their meanings and bear their names—what are they but mathematicizations of that which has ever been the chief of life's hopes and dreams, of that which has ever been the object of its deepest passion and of its dominant enterprise, I mean the finding of worth that abides, the finding of permanence in the midst of change, and the discovery of the presence, in what has seemed to be a finite world, of being that is infinite? It is needless further to multiply examples of a correlation that is so abounding and complete as indeed to suggest a doubt whether it be juster to view mathematics as the abstract idealization of life than to regard life as the concrete realization of mathematics." [15, pp. 645–646].

The many texts he authored shows him as one constantly in self-debate as to the place and role of mathematics in education and a wider social context—a credible and insistent voice that would be heard on what is a just and worthy sense of its full human significance.

3 What Mathematics Isn't (Halmos and Keyser)

Relationships with mathematics vary from person to person according to situation, age, position, temperament, education, and so on, but one thing that binds us all together is the battle to explain ourselves to others in ways that are meaningful yet still faithful to the discipline. We mathematicians have been misunderstood for as long as there have been professions available to us—it goes with the territory, as they say. To partly counter this, Halmos gave an informal lecture (on 12th December, 1967, as part of the centennial celebrations of the University of Illinois of which he was an alumnus), speaking on the topic of mathematics as a creative art. Attempting to clarify what the subject entails, how it is prosecuted, and by whom, and why ("I am not here to proselyte but to enlighten—I seek not converts but friends. I do not want to teach you what mathematics is, but only *that* it is."), he opened with the words

"Do you know any mathematicians—and, if you do, do you know anything about what they do with their time? Most people don't. When I get into conversation with the man next to me in a plane, and he tells me that he is something respectable like a doctor, lawyer, merchant, or dean, I am tempted to say that I am in roofing and siding. If I tell him that I am a mathematician, his most likely reply will be that he himself could never balance his chequebook, and it must be fun to be a whiz at math. If my neighbor is an astronomer, a biologist, a chemist, or any other kind of natural or social scientist, I am, if anything, worse off—this man *thinks* he knows what a mathematician is, and he is probably wrong. . . .

C.P. Snow points to and deplores the existence of two cultures;[2] he worries about the physicist whose idea of modern literature is Dickens, and he chides the poet who cannot state the second law of thermodynamics. Mathematicians, in converse with well-meaning, intelligent, and educated laymen . . . are much worse off than physicists

[1] Ralph Waldo Emerson (1803–1882), an American essayist, lecturer, philosopher and poet who led the so called transcendentalist movement of the mid-19th century.

[2] *The Two Cultures and the Scientific Revolution* (1959), authored by British scientist and novelist Charles Percy Snow, controversially tendered that the division of science and the humanities into "two cultures"—coupled with an educational bias towards the latter at the expense of the former—had created a barrier to, and was unhealthy for, the success of any post war western society.

> in converse with poets. It saddens me that educated people don't even know that my subject exists. There is something that they call mathematics, but they neither know how the professionals use that word, nor can they conceive why anybody should do it. It is, to be sure, possible that an intelligent and otherwise educated person doesn't know that egyptology exists, or haematology, but all you have to tell him is that it does, and he will immediately understand in a rough general way why it should and he will have some empathy with the scholar of the subject who finds it interesting." [7, p. 375],

saying further,

> "As a first step toward telling you what mathematicians do, let me tell you some of the things they do not do. To begin with, mathematicians have very little to do with numbers. You can no more expect a mathematician to be able to add a column of figures rapidly and correctly than you can expect a painter to draw a straight line or a surgeon to carve a turkey—popular legend attributes such skills to these professions, but popular legend is wrong." (p. 376),

and (p. 377)

> "Not even those romantic objects of latter day science fiction, the giant brains, the computing machines that run our lives these days—not even they are of interest to the mathematician as such. Some mathematicians are interested in the logical problems involved in the reduction of difficult questions to the sort of moronic baby talk that machines understand: the logical design of computing machines is definitely mathematics. Their construction is not, that's engineering, and their product, be it a payroll, a batch of sorted mail, or a supersonic plane, is of no mathematical interest or value.
>
> Mathematics is not numbers or machines; it is also not the determination of the heights of mountains by trigonometry, or compound interest by algebra, or moments of inertia by calculus. Not today it isn't. At one point in history each of those things, and others like them, might have been an important and non-trivial research problem, but once the problem is solved, its repetitive application has as much to do with mathematics as the work of a Western Union messenger boy has to do with Marconi's genius."

It is quite possible that Halmos had taken at least part of his cue from Keyser, based on a lecture delivered by the latter some six decades earlier at Columbia University on 16th October, 1907 (as part of a series of invited talks by over twenty different speakers on a host of subjects in science, philosophy and art). Covering a number of aspects of mathematics, Keyser emphasised its utility as an indispensable companion to scientists, a natural language for philosophers, and a creative pursuit in its own right, feeling the need to state publicly that

> "Indeed the time is at hand when at least the academic mind should discharge its traditional fallacies regarding the nature of mathematics and thus in a measure promote the emancipation of criticism from inherited delusions respecting the kind of activity in which the life of the science consists. Mathematics is no more the art of reckoning and computation than architecture is the art of making bricks or hewing wood, no more than painting is the art of mixing colors on a palette, no more than the science of geology is the art of breaking rocks, or the science of anatomy the art of butchering.
>
> Did not Babbage or somebody invent an adding machine? And does it not follow, say Holmes and Schopenhauer, that mathematical thought is a merely mechanical process? Strange how such trash is occasionally found in the critical offering of thoughtful men and thus acquires circulation as [a] golden coin of wisdom. It would not be sillier to argue that, because Stanley Jevons constructed a machine for producing certain forms of logical inference, therefore all thought, even that of a philosopher like Schopenhauer or that of a poet like Holmes, is merely a thing of pulleys and levers and screws, or that the pianola serves to prove that a symphony by Beethoven or a drama by Wagner is reducible to a trick of mechanics." [14, pp. 29–30],

having paralleled Halmos in writing

> "..., such is the character of mathematics in its profounder depths and in its higher and remoter zones that it is well nigh impossible to convey to one who has not devoted years to its exploration a just impression of the scope and magnitude of the existing body of the science. An imagination formed by other disciplines and accustomed to the interests of another field may scarcely receive suddenly an apocalyptic vision of that infinite interior world." (p. 6),

and

> "... the modern developments of mathematics constitute not only one of the most impressive, but one of the most characteristic, phenomena of our age. It is a phenomenon, however, of which the boasted intelligence of a "universalized" daily press seems strangely unaware; and there is no other great human interest, whether of science or of art, regarding which the mind of the educated public is permitted to hold so many fallacious opinions and inferior estimates."(p. 8).

I think we all of us feel, just as keenly, the very same frustrations on display here. After all, the populace should at the very least know that what grammar is to literature and poetry, the colour wheel is to painting, and scales are to musical composition, then so is arithmetic to mathematics.

There have been many essays and articles scribed on the essence of mathematics—covering things such as its history, heart, soul, core, evolution—and it has proven to be a complex and sometimes emotive subject that possesses both subjective and objective aspects of scrutiny. The theoretical epistemology of the discipline (the origin, nature and scope of mathematical knowledge) and its philosophical ontology (existence, beliefs, categorisations of reality (discussed later)) have provided fertile ground for erudite discussion, for as a discipline it is *sui generis*.

4 More Cogitations

In this section I set down some more pronouncements from our two protagonists which fit well with the title of this essay, helping us further appreciate their mindsets and values. Firstly, the reader is shown in detail what Keyser felt about the way mathematical education—beyond those obvious earthly powers of rational thought, logic and general numeracy that it bestows—touches upon and should influence the very soul of humanity. Secondly, we see that Halmos spoke of the analogies between mathematics and other areas and disciplines, writing in a way I've not seen elsewhere in bringing them together well to inform his audience.

From Keyser, then,

> "To humanize the teaching of mathematics means so to present the subject, so to interpret its ideas and doctrines, that they shall appeal, not merely to the computatory faculty or to the logical faculty but to all the great powers and interests of the human mind. That mathematical ideas and doctrines, whether they be more elementary or more advanced, admit of such a manifold, liberal and stimulating interpretation, and that therefore the teaching of mathematics, whether in secondary schools or in colleges, may become, in the largest and best sense, human, I have no doubt. That mathematical ideas and doctrines do but seldom receive such interpretation and that accordingly the teaching of mathematics is but seldom, in the largest and best sense, human, I believe to be equally certain. That the indicated humanization of mathematical teaching, the bringing of the matter and the spirit of mathematics to bear, not merely upon certain fragmentary faculties of the mind, but upon the whole mind, that this is a great desideration is, I assume, beyond dispute.
>
> How can such humanization be brought about? The answer, I believe, is not far to seek. I do not mean that the answer is easy to discover or easy to communicate. I mean that the game is near at hand and that it is not difficult to locate it, though it may not be easy to capture it. The difficulty inheres, I believe, in our conception of mathematics itself; not so much in our conception of what mathematics, in a definitional sense, is, ..., but in our sense or want of sense of what mathematics, whatever it may be, humanly signifies. In order to humanize mathematical teaching it is necessary, and I believe it is sufficient, to come under the control of a right conception of the human significance

of mathematics. It is sufficient, I mean to say, and it is necessary, greatly to enlarge, to enrich and to vitalize our sense of what mathematics, regarded as human enterprise, signifies.

What does mathematics, regarded as an enterprise of the human spirit, signify? What is a just and worthy sense of the human significance of mathematics?

To the extent in which any of us really succeeds in answering that question worthily, his teaching will have the human quality, in so far as his teaching is, in point of external circumstance, free to be what it would. I believe it is important to put the question, and it is with the putting of it rather than with the proposing of an answer to it that I am here at the outset mainly concerned. For any one who is really to acquire possession of an answer that is worthy must win the answer for himself. I need not say to you that such an acquisition as a worthy answer to this kind of question does not belong to the category of things that may be lent or borrowed, sold or bought, donated or acquired by gift. No doubt the answers we may severally win will differ as our temperaments differ. Yet the matter is not solely a matter of temperament. It is much more a matter first of knowledge and then of the evaluation of the knowledge and of its subject. To the winning of a worthy sense of the human significance of mathematics two things are indispensable, knowledge and reflection: knowledge of mathematics and reflection upon it. To the winning of such a sense it is essential to have the kind of knowledge that none but serious students of mathematics can gain. Equally essential is another thing and this thing students of mathematics in our day do not, or do but seldom, gain. I mean the kind of insight and the liberality of view that are to be acquired only by prolonged contemplation of the nature of mathematics and by prolonged reflection upon its relations of contrast and similitude to the other great forms of spiritual activity.

The question, though it is a question about mathematics, is not a mathematical question, it is a philosophical question. And just because it is a philosophical question, mathematicians, despite the fact that one of the indispensable qualifications for considering it is possessed by them alone, have in general ignored it. . . . , and [this] may help to explain the curious paradox that whilst the world, whose mathematical knowledge varies from little to less, has always as if instinctively held mathematical science in high esteem, it has at the same time usually regarded mathematicians as eccentric and abnormal, as constituting a class apart, as being something more or something less than human. . . .

If you ask the world—represented, let us say, by the man in the street or in the market place or the field—to tell you its estimate of the human significance of mathematics, the answer of the world will be, that mathematics has given mankind a metrical and computatory art essential to the effective conduct of daily life, that mathematics admits of countless applications in engineering and the natural sciences, and finally that mathematics is a most excellent instrumentality for giving mental discipline. Such will be the answer of the world. The answer is intelligible, it is important, and it is good so far as it goes; but it is far from going far enough and it is not intelligent." [15, pp. 638–639].

From Halmos,

"A little feeling for the nature of mathematics and mathematical thinking can be got by the comparison with chess. The analogy, like all analogies, is imperfect, but it is illuminating just the same. The rules for chess are as arbitrary as the axioms of mathematics sometimes seem to be. The game of chess is as abstract as mathematics. (That chess is played with solid pieces, made of wood, or plastic, or glass, is not an intrinsic feature of the game. It can just as well be played with pencil and paper, as mathematics is, or blindfold, as mathematics can.) Chess also has its elaborate technical language, and chess is completely deterministic.

There is also some analogy between mathematics and music. The mathologist feels the need to justify pure mathematics exactly as little as the musician feels the need to justify music. Do practical men, the men who meet payrolls, demand only practical music—soothing jazz to make an assembly line worker turn nuts quicker, or stirring marches to make a soldier kill with more enthusiasm? No, surely none of us believes in that kind of justification; music, and mathematics, are of human value because human beings feel they are.

The analogy with music can be stretched a little further. Before a performer's artistic contribution is judged, it is taken for granted that he hits the right notes, but merely hitting the right notes doesn't make him a musician. We don't get the point of painting if we compliment the nude Maya on being a good likeness, and we don't get the point of a historian's work if all we can say is that he didn't tell lies. Mere accuracy in performance, resemblance in appearance, and truth in storytelling doesn't make good music, painting, history: in the same way, mere logical correctness doesn't make good mathematics.

Goodness, high quality, are judged on grounds more important than validity, but less describable. A good piece of mathematics is connected with much other mathematics, it is new without being silly (think of a "new" western movie in which the names and the costumes are changed, but the plot isn't), and it is deep in an ineffable but inescapable sense—the sense in which Johann Sebastian is deep and Carl Philip Emmanuel is not.[3] The criterion for quality is beauty, intricacy, neatness, elegance, satisfaction, appropriateness—all subjective, but all somehow mysteriously shared by all.

Mathematics resembles literature also, differently from the way it resembles music. The writing and reading of literature are related to the writing and reading of newspapers, advertisements, and road signs the way mathematics is related to practical arithmetic. We all need to read and write and figure for daily life: but literature is more than reading and writing, and mathematics is more than figuring. The literature analogy can be used to help understand the role of teachers and the role of the pure-applied dualism.

Many whose interests are in language, in the structure, in the history, and in the aesthetics of it, earn their bread and butter by teaching the rudiments of language to its future practical users. Similarly many, perhaps most, whose interests are in the mathematics of today, earn their bread and butter by teaching arithmetic, trigonometry, or calculus. This is sound economics: society abstractly and impersonally is willing to subsidize pure language and pure mathematics, but not very far. Let the would-be purist pull his weight by teaching the next generation the applied aspects of his craft; then he is permitted to spend a fraction of his time doing what he prefers. From the point of view of what a good teacher must be, this is good. A teacher must know more than the bare minimum he must teach; he must know more in order to avoid more and more mistakes, to avoid the perpetuation of misunderstanding, to avoid catastrophic educational inefficiency. To keep him alive, to keep him from drying up, his interest in syntax, his burrowing in etymology, or his dabbling in poetry play a necessary role.

The pure-applied dualism exists in literature too. The source of literature is human life, but literature is not the life it comes from, and writing with a grim purpose is not literature. Sure there are borderline cases: is Upton Sinclair's "Jungle" literature or propaganda? (For that matter, is Chiquita Banana an advertising jingle or charming light opera?) But the fuzzy boundary doesn't alter the fact that in literature (as in mathematics) the pure and the applied are different in intent, in method, and in criterion of success.

Perhaps the closest analogy is between mathematics and painting. The origin of painting is physical reality, and so is the origin of mathematics—but the painter is not a camera and the mathematician is not an engineer. The painter of "Uncle Sam Wants You" got his reward from patriotism, from increased enlistments, from winning the war—which is probably different from the reward Rembrandt got from a finished work. How close to reality painting (and mathematics) should be is a delicate matter of judgment. Asking a painter to "tell a concrete story" is like asking a mathematician

[3]Halmos is comparing Johann Sebastian Bach (1685–1750), the great German composer and musician of the Baroque period, with his fifth child and second surviving son Carl Philipp Emanuel (1714–1788).

to "solve a real problem." Modern painting and modern mathematics are far out—too far in the judgment of some. Perhaps the ideal is to have a spice of reality always present, but not to crowd it the way descriptive geometry, say, does in mathematics, and medical illustration, say, does in painting." [7, pp. 387–389].

5 Some Related Thoughts (From Others)

5.1 On Human Aspects of Mathematics

I begin this section—which is designed to extend and embellish the preceding narrative—with some comments made by yet another American (and seemingly little known) mathematician, Alfred W. Adler (1930–). In an article that, somewhat unusually at the time, appeared in a 1972 issue of *The New Yorker* magazine (established as a pre-eminent forum for serious fiction, essays and journalism), he wrote thus:

> "Mathematics, like chess, requires too direct and personal a confrontation to allow graceful defeat. There is no element of luck; there are no partners to share the blame for mistakes; the nature of the discipline places it precisely at the center of the intellectual being, where true cerebral power waits to be tested. . . .
>
> There are two simple and ruthless ethical standards by which the purity of any discipline can be determined. Mathematics has at times seemed almost alone in the attainment of these standards. What is required is, first, an institutionalized indifference to men whose work has been completed—a disregard or contempt for those who have accomplished much but who have lost the will to create and whose major accomplishments are of the past. This applies perfectly to mathematics." [1, p. 39].[4]

I only include this because the piece seems to have attracted attention—perhaps more than it should—judging by the number of citations it has received (I put this down to the sales levels of *The New Yorker* over many years, and the unusually forthright nature of the essay in places). Overall, whilst he offered an entertaining and provocative monograph, this rather dismal portrayal of our beloved subject is brutal, and off the mark, as it were—for there is surely more respect around for the efforts of others than Adler suggests, and he belittles those incremental advances that occur most of the time in mathematics. Some of those who have gone before deserve to be recognised and not forgotten, and this principle was in part a motivation in writing about Halmos and Keyser as counterexamples to Adler's harsh assessment of legacy and bequest—things that I would say genuine mathematicians hold close to their hearts, and rightly so. When we ponder, in quiet moments, what the discipline is and isn't, and what it does and doesn't mean to us, we should give more than a mere nod of acknowledgement to men and women who have occupied an admirable space in history and who have done the same, particularly when—as in the case of Halmos and Keyser—they leave their deliberations behind so eloquently in print. We should look upon them and their ilk with propitious gratitude, for they remind us that the human aspects of mathematics—its connections to knowledge and our lives as a framework for rigour and excellence in thinking and being—form a phenomenon which is an abiding one that never fails to be relevant to us. Introspective contemplation on the topics raised here brings naturally to light the way we teach, conduct ourselves, and interact with others in academic research/teaching and in work outside education—in other words, we become aware of how we feel about mathematics at our very core, and how it shapes the behaviours and *raison d'être* that filter through our careers and day-to-day existence. I would without hesitation endorse it as an exercise to which everyone should expose themselves periodically, being simultaneously cathartic and regenerative.

Staying on this theme, American academic and researcher turned respected expositor Reuben Hersh mentioned the observation made among some of the mathematical community that the typical teacher or research operative is a philosophical split personality. Advocating a humanist philosophy of mathematics that continues to both challenge and complement mainstream systems of thought in the field, he wrote the following in 1998:

[4]Since the journal's launch in 1921 it has published both regular and occasional essays on mathematics featuring the results, skills and lives of, and interviews with, particular mathematicians that are appropriate for public consumption.

> "When he is working at his mathematics, he has no doubt that the objects he is studying have in some sense a real, objective existence. ... This is, so to speak, his week-day religion. It is a variety of "Platonism" (also often called "realism").
>
> However, if he should be challenged to explain where, how, in what sense any of these invisible, intangible, infinite entities is real or exists, he is likely to turn tail, and retreat hypocritically into some form of formalism. That is, he drops any claim that anything in pure math really exists; all we really are doing, he explains, is making logical deductions from meaningless axioms. This is "formalism", so to speak his Sunday religion.
>
> This hypocritical schizophrenia is widely practised, and it serves the purpose, ..., of getting the philosophers off our necks and letting us do math as usual.
>
> Nevertheless, hypocrisy and schizophrenia cannot really be good for the soul." [13, p. 13].

I have come to believe that lots of us move between, or else alter our strength of feeling about, these two fundamental philosophies (if not depending on age, then perhaps on mood or the particular cerebral activities taking place and the nature of cognitive labour involved), and that it isn't a problem, especially as no-one can prove that mathematics sits within either camp. The issue has, though, provoked much discussion and debate—often pitting one side against another—and its personal truth is (as mathematician R.G. Ayoub wrote) as much a matter of quasi-theological belief as it is a matter of intellectual certitude.[5] It is fair to say that neither Halmos nor Keyser, from what I have read, themselves drilled down into the meaning of mathematics in this way, but preferred to operate more at 'surface level' while indeed still fully aware of and acknowledging the sometimes equivocal relationship we professionals have with it and the deeper underlying issues this throws up for some people; remember us mathematicians constantly have to deal with, simply as human beings, things that stimulate our senses and can so readily invoke feelings of certainty and confusion through their perceived transience and perpetuity, solidity and ethereality, presence and latency, depth and superficiality. As I have discovered, taking the position of an unashamed undoctrinaire on this matter—embracing ambivalence and vacillation alike—makes life much easier.

5.2 On Mathematical Creativity

At the heart of the practice of mathematical research is creativity, which is a very individual concept with which mathematicians are on intimate terms. Any assessment of core features of mathematics cannot omit this notion, and here I restrict myself to the inclusion of a couple of quotes that appeal to me. John E. Littlewood (1885–1977)—long time collaborator with the great G.H. Hardy—produced a light-hearted essay (republished posthumously, in 1978) in which, musing on the mathematical life, he summarised the creative process as comprising four stages:

> "It is usual to distinguish four phases in creation: preparation, incubation, illumination, and verification, or working out. For myself I regard the last as within the range

[5] Put simply, the Platonic view (taken by, for example, G.H. Hardy and Charles Hermite) assents to the form of mathematical objects, and all deductions therefrom, as existing as abstractions in their own right, separate from intelligent agents and placed ready for revelation. Formalists (such as David Hilbert and physicist Percy (Peter) Bridgman) subscribe to the view that mathematics consists of essentially man made constructs and structures assigned human meaning at source, from which (formal) manipulations of symbols follow axiomatic premises and adhere to strictly prescribed rules; both tenets are alive and well among mathematicians, even if in a somewhat inchoate way most of the time (some people are comfortable upholding both arguments, Leopold Kronecker being one of them). Hersh—offering a subjective third (and alternative) existential 'home' for mathematics besides the mental or material that seem to cover other objects and constructs of society—goes on to propose "a way of thinking about the reality and [actuality] of mathematics which lets us keep our mathematical objects really existing, really meaningful, without resort to mysticism." In a very brief paper he locates the reality of mathematics in the social-cultural realm, which to him keeps it human (along with many things that are the substance of our lives). The interested reader is pointed towards an earlier lengthy and thought-provoking essay where he went into much greater detail centred around three core assertions designed to address the Platonism-formalism dichotomy whose gravity he chose to term a plight: "(1) The philosophical notions about mathematics commonly held by the working mathematician are incompatible with each other and with our actual experience and practice of mathematical work."; "(2) The present impasse in mathematical philosophy is the aftermath of the great period of foundationist controversies ..."; "(3) Many of the difficulties and stumbling blocks in the philosophy of mathematics are created by inherited philosophical prejudices which we are free to discard if we choose to do so." [12, p. 31]. We outline a resolution by L.A. White, which chimes with Hersh, later in this section.

> of any competent practitioner, given the illumination.
>
> Preparation is largely conscious, and anyhow *directed* by the conscious. The essential problem has to be stripped of accidentals and brought clearly into view; all relevant knowledge surveyed; possible analogues pondered. It should be kept constantly before the mind during intervals of other work. This last is advice from Newton.
>
> Incubation is the work of the subconscious during the waiting time, which may be several years. Illumination, which can happen in a fraction of a second, is the emergence of the creative idea into the conscious. This almost always occurs when the mind is in a state of relaxation, and engaged lightly with ordinary matters.[6] . . . Illumination implies some mysterious rapport between the subconscious and the conscious, otherwise emergence could not happen." [20, p. 114].

Littlewood describes the phenomenon well, I think, to which can be added another voice who plays up the hope and buoyancy on which a mathematician often calls. Deductive processes are powerful in themselves and the bedrock of mathematics, but in order for a protagonist in research to keep the candle of creativity aflame it is necessary to bring to bear other approaches that are important and at times crucial. American mathematician Maxime Bôcher spoke, in September 1904, to the Department of Mathematics of the International Congress of Arts and Science in St. Louis where he listed non-deductive methods that the creative mathematician deploys in practice—intuition, experiment, analogy (induction)—and supplemented them with the so called 'method of optimism' on which he elaborated as one that

> ". . . leads us either wilfully or instinctively to shut our eyes to the possibility of evil. Thus the optimist . . . will say, if he stops to reflect on what he is doing: "I know that I have no right to divide by zero; but there are so many other values which the expression by which I am dividing might have that I will assume that the Evil One has not thrown a zero in my denominator this time." This method, if a proceeding often unconscious can be called a method, has been of great service in the rapid development of many branches of mathematics, . . ." [4, pp. 134–135],

qualifying his thoughts accordingly:

> "While no one of these methods can in any way compare with that of rigorous deductive reasoning as a method upon which to base mathematical results, it would be merely shutting one's eyes to the facts to deny them their place in the life of the mathematical world, not merely of the past but of today. There is now, and there always will be room in the world for good mathematicians of every grade of logical precision. It is almost equally important that the small band whose chief interest lies in accuracy and rigor should not make the mistake of despising the broader though less accurate work of the great mass of their colleagues; as that the latter should not attempt to shake themselves wholly free from the restraint the former would put upon them."

It is perhaps the kind of optimism to which Bôcher alludes that keeps us going through much of our labours in research—manifest as a willingness to persist with an idea or technique with a view to filling in details at a later point, refusing to get bogged down in a moment of creativity as the drive to push through it prevails. He wrote

> "The fact is that what we call mathematical rigor is merely one of the foundation stones of the science; an important and essential one surely, yet not the only thing upon which we can rely. A science which has developed along such broad lines as mathematics, with such numerous relations of its parts both to one another and to other sciences, could not long contain serious error without detection. This explains how, again and again, it has come about, that the most important mathematical developments have taken place by methods which cannot be wholly justified by our present canons of mathematical rigor, the logical "foundation" having been supplied only long after the superstructure had been raised." (p. 134),

[6] Author's Note: This has been acknowledged and described by top mathematicians—among them Gauss and Poincaré—and is a familiar one to many of us; it is discussed further in the Appendix as a point of historical interest.

observing that insights are sometimes gained without the element of rigour which arrives subsequent to them.

5.3 On the Position of Mathematics

Moving on, Hungarian-American mathematician, physicist, and computer scientist John von Neumann (generally regarded as the foremost mathematician of his time) carved out a hugely successful career by integrating pure and applied sciences. He was of the opinion that most people, including mathematicians, agree that mathematics is not an empirical science—or at least that it is practiced in a manner which differs in several decisive respects from techniques associated with empirical sciences—and yet its development had, for him, been very closely linked to the natural sciences (and still is in important ways for many) or, more generally, to any science which interprets experience on a higher level than one which is solely descriptive. It therefore held a rather strange duplicity and 'double face' in its very nature which, once realised, had to be accepted and absorbed into one's thinking on the subject. Von Neumann did not believe that any simplified, unitarian delineation of mathematics was possible without sacrificing its essence, and admitted that expressing a subjectively coherent view on the discipline was challenging (the article from which his words below are taken was originally published in 1947):

> "A discussion of the nature of intellectual work is a difficult task in any field, even in fields which are not so far removed from the central area of our common human intellectual effort as mathematics still is. A discussion of the nature of any intellectual effort is difficult *per se*—at any rate, more difficult than the mere exercise of that particular intellectual effort. . . . It is exceptional that one should be able to acquire the understanding of a process without having previously acquired a deep familiarity with running it, with using it, before one has assimilated it in an instinctive and empirical way.
>
> Thus any discussion of the nature of intellectual effort in any field is difficult, unless it presupposes an easy, routine familiarity with that field. in mathematics this limitation becomes very severe, if the discussion is to be kept on a nonmathematical plane. The discussion will then necessarily show some very bad features; points which are made can never be properly documented, and a certain over-all superficiality of the discussion becomes unavoidable." [23, p. 172];

an aim of this essay was to circumvent a version of the shallowness alluded to by von Neumann and to which it is easy to fall prey if one is not careful.

In his powerful and penetrating book *Mathematics Without Apologies*, Michael Harris refers in his opening chapter to a historian's characterisation of 'modernism' in mathematics through the rise of professional autonomy which allowed the discipline to break away from the constraints of so called pre-modern conceptions about the relations between mathematics and philosophy or the physical sciences, being brought about by

> ". . . the incorporation of mathematics within the structure of the modern research university—the creation of an international community of professional mathematicians—and new attitudes to the subject matter and objectives of mathematics. The new form and the new content appeared at roughly the same time and have persisted with little change, in spite of the dramatic expansion of mathematics and of universities in general in the second half of the twentieth century." [11, p. 4].

We today are the beneficiaries of the emergence and consolidation of mathematics as a self-standing field with which our intimacies are as varied as the spectrum of our personalities and ever changing working practices as mathematicians. Those who wish to lift the veil on mathematics and offer its secrets to a wider audience fall into a particular kind of category, and have always existed—as Halmos and Keyser bear witness—regardless of how mathematics is situated within society at any given moment in time. They share a special kind of rapport with the subject that is laudable, allowing us into their world of perspectives and in turn to glimpse the emotional investments incurred and the unrecognised pathos of the mathematical life. In satisfying our existence to external ears Harris contends that it is all too easy to set aside our ethical compasses and to collude in the misrepresentation of our intentions and incentives by allowing our activities to be described in misleading terms (in his Preface to [11] he picks out those promises of "immutable truths", "ineffable beauty" and "*Golden Goose*" utilities that we offer, entering willingly into a kind of Faustian contract). Thus, put quite simply, authors such as Halmos and Keyser (along with others from the roster of like-minded men and women) can remind us of our place, our obligations to probity and credibility, and the need for self-examination, helping to prevent us becoming alienated from ourselves so that we may become more authentic and confident in what we do and why we do it—not with an air of elitism or arrogance, but with the pneuma of the unapologetic, carefree adventurer mapping a unique and wondrous landscape that we claim as our own.

5.4 On Mathematical Culture and Philosophy

It is worth flagging up that 'cultural' aspects of mathematics (such as the humanist posture taken by Hersh) have occupied a few minds for quite some time. American mathematician Raymond L. Wilder, for example—a man who acquired philosophical and anthropological interests, and was ever in search of the cultural basis of mathematics—orated on the theme at the 1950 International Congress of Mathematics (Cambridge, Massachusetts), underscoring the innate symbolic characterisation of mathematics that underpins the essence of mathematical learning:

> "... the so-called "international character" of mathematics is due in large measure to the standardization of symbols that it has achieved, thereby stimulating diffusion. Without a symbolic apparatus to convey our ideas to one another, and to pass on our results to future generations, there wouldn't be any such thing as mathematics—indeed, there would be essentially no culture at all, since, with the possible exception of a few simple tools, culture is based on the use of symbols. A good case can be made for the thesis that man is to be distinguished from other animals by the way in which he uses symbols. Man possesses what we might call *symbolic initiative*; that is, he assigns symbols to stand for objects or ideas, sets up relationships between them, and operates with them as though they were physical objects. So far as we can tell, no other animal has this faculty, although many animals do exhibit what we might call *symbolic reflex* behavior. Thus, a dog can be taught to lie down at the command "Lie down," and of course to Pavlov's dogs, the bells signified food. In a recent issue of a certain popular magazine a psychologist is portrayed teaching pigeons to procure food by pressing certain combinations of colored buttons. All of these are examples of symbolic reflex behavior—the animals do not create the symbols." [27, pp. 140–141];

this is but a tiny snapshot of his creditable attempt to paint a full picture of what 'mathematical culture'—multi-faceted as it is—really means, and his treatise [26] is worth a read for anyone wanting to know more about this topic.

We have mentioned Platonists and formalists—the two dominant philosophical clusters of people advocating mathematical reality who have any firm popularity (so called intuitionism and its daughter, constructivism, have long since had their day, along with extensions thereof and other minor varieties that appeal more to fringe aesthetes whose business is interpretative and contextual ambiguities)—20th century American anthropologist Leslie A. White (a close friend of Wilder) offering the names of Mary Somerville and Edward Everett for the first group, and P.W. Bridgman, Edward Kasner and James Newman for the second in a 1947 article [25] about the nature of mathematical existence that is premised on the essential questions "Do mathematical truths reside in the external world, there to be discovered by man, or are they man-made inventions? Does mathematical reality have an existence and a validity independent of the human species or is it merely a function of the human nervous system?".[7] He opened his sapient and authoritative piece with

[7]Hersh and Philip J. Davis, in their 1981 text *The Mathematical Experience* (Birkhäuser), put it thus (p. 406): "Do we really have to choose between a formalism that is falsified by our everyday experience, and a Platonism that postulates a mythical fairyland where the uncountable and the inaccessible lie waiting to be observed by the mathematician whom God blesses with a good enough intuition? It is reasonable to propose a different task for mathematical philosophy, not to seek indub[i]table truth, but to give an account of mathematical knowledge as it really is—fallible, corrigible, tentative, and evolving, as is every other kind of human knowledge." The theme is reprised in Hersh's later offering, *18 Unconventional Essays On the Nature of Mathematics* (Springer, 2006), where in his introductory notes as editor he outlines just how much work has been conducted on the philosophy of mathematics to make it a creditable academic topic.

> "From a psychological and anthropological point of view, this latter conception [formalism] is the only one that is scientifically sound and valid. There is no more reason to believe that mathematical realities have an existence independent of the human mind [Platonism] than to believe that mythological realities can have their being apart from man. The square root of minus one is real. So were Wotan and Osiris.[8] So are the gods and spirits that primitive peoples believe in today. The question at issue, however, is not, Are these things real?, but Where is the locus of their reality? It is a mistake to identify reality with the external world only. Nothing is more real than an hallucination." (pp. 290–291),

adding immediately

> "Our concern here, . . . , is not to establish one view of mathematical reality as sound, the other illusory. What we propose to do is to present the phenomenon of mathematical behavior in such a way as to make clear, on the one hand, why the belief in the independent existence of mathematical truths has seemed so plausible and convincing for so many centuries, and, on the other, to show that all of mathematics is nothing more than a particular kind of primate behavior." (p. 291)

and arguing his baseline assertion:

> "The following propositions, though apparently precisely opposed to each other, are equally valid; one is as true as the other: 1. "Mathematical truths have an existence and a validity independent of the human mind," and 2. "Mathematical truths have no existence or validity apart from the human mind." Actually, these propositions, phrased as they are, are misleading because the term "the human mind" is used in two different senses. In the first statement, [it] refers to the individual organism; in the second, to the human species. Thus both propositions can be, and actually are, true. Mathematical truths exist in the cultural tradition into which the individual is born, and so enter his mind from the outside. But apart from cultural tradition, mathematical concepts have neither existence nor meaning, and of course, cultural tradition has no existence apart from the human species. Mathematical realities thus have an existence independent of the individual mind, but are wholly dependent upon the mind of the species."

The reader is encouraged to access his words in full, as it places mathematics within a timeline of cultural inheritance that is meshed with the cultural present for any individual. He makes no distinction between mathematics and other realms of (un)civilised culture which possess us and succumb to modifications and re-synthesis as they are passed through generations, save for its own unique symbolism and constructs. Each individual is birthed into a pre-existing set of beliefs, tools, customs and institutions. These culture traits shape and mould each person's life, giving it content and direction. Mathematics is, for White, but one of many streams in the totality of culture, and the discipline exists as the organic behaviour response to a constant flow and revision of mathematical culture; they combine to act in varying degree upon individuals, who respond according to their constitutions. He summarises thus:

> ". . . we see that there is no mystery about mathematical reality. We need not search for mathematical "truths" in the divine mind or in the structure of the universe. Mathematics is a kind of primate behavior as languages, musical systems and penal codes are. Mathematical concepts are man-made just as ethical values, traffic rules, and bird cages are man-made. But this does not invalidate the belief that mathematical propositions lie outside us and have an objective reality. They do lie outside us. They existed before we were born. As we grow up we find them in the world about us. But this objectivity exists only for the individual. The locus of mathematical reality is cultural tradition, i.e., the continuum of symbolic behavior. This theory illuminates also the phenomena of novelty and progress in mathematics. Ideas interact with each other in the nervous systems of men and thus form new syntheses. If the owners of these nervous systems are aware of what has taken place they call it invention . . . , or "creation," . . . If they do not understand what has happened, they call it a "discovery" and believe

[8] Gods in (resp.) Norse and Egyptian mythology.

> they have found something in the external world. Mathematical concepts are independent of the individual mind but lie wholly within the mind of the species, i.e., culture. Mathematical invention and discovery are merely two aspects of an event that takes place simultaneously in the cultural tradition and in one or more nervous systems. Of these two factors, culture is the more significant; the determinants of mathematical evolution lie here. The human nervous system is merely the catalyst which makes the cultural process possible." (p. 303).

In previous centuries age old philosophical questions such as "What is Man?", "What is Mind?" and "What is Language?" were dealt with as re-cast scientific problems, with the result that anthropology, psychology and linguistics detached themselves from a broad domain of philosophy to become autonomous disciplines around which new theories could be built and tested. Hersh notes that activity in the philosophy of mathematics certainly lagged behind the philosophy of science for many years, but redress has been made in this respect and some commonly held ideas (slightly paraphrased here) seem to have been established (see 'Fresh Breezes in the Philosophy of Mathematics', *Amer. Math. Month.* **102**, 589–594 (1995)): 1) Mathematics is human; it is part of, and fits into, human culture. 2) Mathematical knowledge is fallible; like science, mathematics can advance itself by making mistakes and then correcting and re-correcting them. 3) There are different versions of proof or rigour, depending on time, place, and other things; the use of computers in proofs is a non-traditional version of rigour. 4) Empirical evidence, numerical experimentation, and probabilistic proof all help us choose what to believe in mathematics; Aristotelian logic isn't necessarily always the best way to decide. 5) Mathematical objects are a special variety of social-cultural-historical object; we can tell mathematics from literature or religion but, nevertheless, mathematical objects are shared ideas (like Moby Dick in literature, or the Immaculate Conception in religion).

5.5 On the Evolution of Mathematics

Von Neumann's take on the development of mathematical ideas—in particular the traction they gain and then have slip away—remains relevant today for its appreciation of their inception, growth and potential exhaustion over time. He believed, as already seen, that empirics has always laid a big claim on the inception of new mathematics, but conceded that natural forces inevitably take over and often cause fragmentation and dilution in topics that travel so far from their original source that they become depleted, peripheral, marginalised and ultimately incidental beyond existing for an interested few; at this point the mathematical chaff then outweighs the wheat, and the crop is a largely spent one though it can still linger and continue to yield a harvest of sorts:

> "I think that it is a relatively good approximation to truth—which is much too complicated to allow anything but approximations—that mathematical ideas originate in empirics, although the genealogy is sometimes long and obscure. But, once they are so conceived, the subject begins to live a peculiar life of its own and is better compared to a creative one, governed by almost entirely aesthetical motivations, than to anything else and, in particular, to an empirical science. There is, however, a further point which, I believe, needs stressing. As a mathematical discipline travels far from its empirical source, or still more, if it is a second and third generation only indirectly inspired by ideas coming from "reality," it is beset third with very grave dangers. It becomes more and more purely aestheticizing, more and more purely *l'art pour l'art*. This need not be bad, if the field is surrounded by correlated subjects, which still have closer empirical connections, or if the discipline is under the influence of men with an exceptionally well-developed taste. But there is a grave danger that the subject will develop along the line of least resistance, that the stream, so far from its source, will separate into a multitude of insignificant branches, and that the discipline will become a disorganized mass of details and complexities. In other words, at a great distance from its empirical source, or after much "abstract" inbreeding, a mathematical subject is in danger of degeneration. At the inception the style is usually classical; when it shows signs of becoming baroque, then the danger signal is up." [23, pp. 183–184].

This seems to be a habitual problem (not just confined to mathematics), and contributes to the ever expanding body of publications to the point where it is difficult to filter out key works from the steady flux of enervated ones in areas that have seen their youth and vigour fade. In response to a suggestion (by German-American Richard Courant, no less) that the river of mathematics might (if divorced from physics) "break up into many separate little rivulets and finally dry up altogether", Hersh and Davis—who also wrote of "diminution [of mathematical production] due to irrelevance or obsolescence" and "internal saturation"—argued in *The Mathematical Experience* that (p. 22) "What has happened is rather different. It is as if the various streams of mathematics have overflowed their banks, run together, and flooded a vast plain, so that we see countless currents, separating and merging, some of them quite shallow and aimless. Those channels that are still deep and swift-flowing are easy to lose in the general chaos."

In reality, mathematics grows exponentially, and a proliferation of research and knowledge has caused a slight unease for many years. As far back as 1890 the English mathematician and astronomer James W.L. Glaisher gave an address that was published in one of the weekly issues of the science journal *Nature* (dated 11th September, and rendered to its British Association under the Mathematics and Physics Section for which he acted as President). Touching on a variety of issues related to mathematical teaching and research, he offered his thoughts on the expansion of the field in both its scope and the volume of disseminated results even at that time:

> "Passing now to the consideration of pure mathematics itself, that is to say, of the abstract sciences, which can only be conquered and explored by mathematical methods, it is difficult not to feel somewhat appalled by the enormous development they have received in the last fifty years. The mass of investigation, as measured by pages in Transactions and Journals, which is annually added to the literature of the subject, is so great that it is fast becoming bewildering from its mere magnitude, and the extraordinary extent to which many special lines of study have been carried. To those who believe, if any such there are, that mathematics exists for the sake of its applications to the concrete sciences, it must indeed seem that it has long since run wild, and expanded itself into a thousand useless extravagances. Even the mathematician must sometimes ask himself the question—not infrequently put to him by his friends—"To what is it all tending? What will be the result of it all? Will there be any end?" The last question is readily answered. There certainly can be no end; so wide and so various are the subjects of investigation, so interesting and fascinating the results, so wonderful the fields of research laid open at each succeeding advance—no matter in what direction—that we may be sure that, while the love of learning and knowledge continue to exist in the human mind, there can be no relaxation of our efforts to penetrate still further into the mysterious worlds of abstract truth which lie so temptingly spread before us. The more that is accomplished, the more we see remaining to be done. Every real advance, every great discovery, suggests new fields of enquiry, displays new paths and highways, gives us new glimpses of distant scenery. . . . As for the other questions, it is very difficult to render intelligible, even to a mathematician, the kind of knowledge acquired by mathematical research in a new field until he has made himself acquainted with its processes and notation, and we cannot hope to find in the remote regions of an abstract science many results so simple and striking as to appeal forcibly to the imagination of those who are unfamiliar with its conceptions and ideas. It would seem, therefore, that the question, "To what is it all tending?" could never be answered in general terms. I do not think any mathematician could see his way to a reply, or even give definite meaning to the question. He might feel daring enough to predict the probable drift of his own subject, but he could scarcely get a broad-enough view to enable him to indulge his fancy with respect to more than a very minute portion of the field already open to mathematical investigation. To the outsider I am afraid that the subject will continue to present much the same appearance as it does now; it will always seem to be stretching out into limitless symbolic wastes, without producing any results at all commensurate with its expansion." [5, p. 465].

Among the numerous observations on this matter that I have read, a jocular one comes from American mathematician Ralph P. Boas who, writing about his early days working at the journal *Mathematical Reviews* (where from 1945 to 1950 he was Executive Editor at its Brown University offices in Rhode Island, having spent a few years prior to this as an enthusiastic and diligent reviewer himself), began with a poem he titled 'Inflation':

> *"The growth of mathematics is producing discontent:*
> *The inflation rate per annum is pushing 10 per cent.*
> *Faced with so much information, it's not easy to succeed*
> *In locating any theorem that you appear to need.*
> *Our plethora of indices can leave you in the lurch,*
> *For it takes less time to prove it than to go and make a search.*
> *I can offer a solution, but it's totally upsetting:*
> *We need to introduce a way of constantly forgetting*
> *The results that won't be needed for another 20 years,*
> *At the end of which they'll surface with appropriate loud cheers,*
> *While the ones that won't be needed till forever and a day,*
> *Once their authors get their tenure will be firmly thrown away."* [3, p. 79],

musing later that (p. 86)

> "At the beginning, it was fascinating to have the world's mathematical literature flowing across my desk, but by the time I left, there was already much too much. It has even been suggested, almost seriously, that we need to have another reviewing journal to review MR."[9]

5.6 What Are We As Mathematicians?

Mathematician and theoretical physicist Freeman J. Dyson (1923–2020) wrote, amusingly (p. 130 of 'Mathematics in the Physical Sciences', *Sci. Amer.* **211**, 128–146 (1964)), "The vast majority of working scientists, myself included, find comfort in the words of the French mathematician Henri Lebesgue: "In my opinion a mathematician, in so far as he is a mathematician, need not preoccupy himself with philosophy—an opinion, moreover, which has been expressed by many philosophers." " An overriding number of mathematicians work without ever trying to gain a philosophical understanding of what they do, finding that this is no obstacle to them. However, while neglect of philosophical issues does not impede mathematical research, any discussion about the possibility of a *centre fixe* for the occupation that is mathematics quickly becomes embroiled in questions about the plurality of foundational philosophies that are partly the result of a failed attempt to bind the discipline together with an unbreachable self-consistency (this had seen geometry replaced by arithmetic and set theory in an attempt to form, through mathematical logic, a unified perspective that had been dashed by the early 1930s), and this appeals to some mathematicians of a certain disposition. For myself, I feel that there are results which are particularly deep and revealing (within a branch of mathematics or as a connecting bridge between two areas, or else in explaining a mystery of nature/science) and have the Platonic stamp of power, surprise, or *a posteriori* inevitability. More routine mathematics—which, let's face it, occupies a lot of journal space in the many hundreds of titles in existence—bears the man-made mark of formalism and the finiteness of the human species, high quality as it can be. I can't settle on one side of the philosophical divide in permanent residence, and there are many like me. One thing of which I am sure, though, is that us mathematicians get a deep sense of personal meaning and self-orientation from our research that allows us to eclipse some everyday mundanities—and this, on a daily basis, is sufficient in and of itself. A great thing, yes, but it's a pity we can't convey our own feelings about the subject more easily to the layman. Surrounded as we are by the mathematically afraid, and the loud and proud mathematically illiterate (whose indifference to their lack of schooling is not repeated with regard to music, or literature, or poetry, or art), we are forced to exist in a vacuum where the shouts and cries of our insights and epiphanies are smothered and denied an external hearing. It's the worst of both worlds—our work is a foundational enigma to most of us, and our achievements are an enigma to most of everyone else.

Having earlier quoted Halmos and Keyser on flawed beliefs and basic ideas about mathematics, we reprise it as a finishing point for this section where we direct the reader to recent communications on stereotypical characterisations of the subject and members of its community—this

[9] *Mathematical Reviews* was launched by the American Mathematical Society at the start of 1940. From a first issue comprising 32 pages and 176 reviews, the publication has grown into the massive MathSciNet database (it began in 1996 as the first web based electronic presentation of *M.R.* information entries, preceded by release of a CD-ROM version in 1989) which contains over 3 million publications, around 9 million citations, and is supported by a reviewer community of researchers numbering over 17,000; it holds listings from more than 2,000 journals, and 100,000 items are added each year.

rounds off the discourse (if slightly briefly), highlighting some thoughtful essays on a not inconsequential matter. Mathematical historian Snezana Lawrence wrote, in 2016,

> "..., the medical, psychological, and anecdotal/apocryphal all tell us about the supposedly recognisable and easily identifiable common characteristics of mathematicians. Some of these are: physical frailty, mental illness, lack of empathy. Some of the characteristics give a glimpse of hope in terms of high (probably only related to academic) achievement: high systematizing and intellectual ability, ... and here we run out of things to say. In fact, that too—the high academic ability and achievement—can be seen as something that is highly undesirable if the correlation is established between high academic achievement and physically unattractive demeanour." [18, p. 113].

She suggested the reader draw a Venn diagram which places these commonly held judgments about mathematicians in one set, and overlap them with a set of traits that would be those referring to the standard (*i.e.*, entrenched) vision of a 'professor'. This, in combination with films depicting mathematicians—where correlation and causation are completely intermingled in representations of personality—would, she proffered, offer more conclusions about the high functioning levels and impressive intellectual capacities of mathematicians when set against their supposed difficulties in navigating social environments successfully and/or finding lasting friendship, love and happiness; what an odd and slightly distressed bunch we are, portrayed by image as inept human beings apparently fit for mathematics and little else. Of course the reality is somewhat different, and Lawrence voices that having spent over twenty years in the company of all types of mathematicians "..., these descriptions certainly form a picture that is rather surprisingly difficult to reconcile with the experience." Citing merely "absentmindedness" as a recurring theme of any real note, her point is that perceptions of mathematical protagonists play into the views of would-be teachers, students, and the general citizenry in different ways that are often unhelpful to the discipline itself and to those with a vested interest in the guardianship of its societal profile. The general climate of misinformation and false impressions is picked up in an accompanying article by Markus Pantsar ('The Great Gibberish—Mathematics in Western Popular Culture', *ibid.*, pp. 409–437) who sets out a most thorough treatment of mathematical customs in films and television series, which is augmented by Tony Mann's 'Mathematics and Mathematical Cultures in Fiction: The Case of Catherine Shaw' (*ibid.*, pp. 375–385) where we learn of the rich history of a nuanced topic that is addressed in all aspects. These articulate works reinforce the fact that what the public reads, listens to, and watches both sculpts, and is fashioned by, the prevailing mind maps of mathematics and mathematicians where symbiotic forces on show are questionable; when the input of professional stakeholders is minimal or absent, so too their accuracy and legitimacy are usually lacking and the impregnable loop of perplexity and ignorance is maintained. There is a body of evidence to link mathematical personality and talent to various shades of autism, but there is also plenty of research to counter these arguments and weaken some of the tired, overused clichés about mathematics and the people who partake in all that it has to offer for a living.

6 New and Recent Developments

It seems appropriate—having looked at some of the existential and related matters with which mathematicians have wrestled historically (Halmos and Keyser did so, in their own ways), and still do—to turn our gaze towards developments that have entered the orbit of mathematics as minor satellite planets whose density and pull are each yet to be properly assessed. Cultural, social and political influences most surely play into the formation and demise of archetypes, and they overspill strongly into the prominence and position of some academic areas. While mathematics remains largely untroubled by these compared with other fields (though its internal fabric is not immune from them), this is not to say that the subject does not evolve with time and its practitioners are always working in their particular here and now with an eye on what came before. It carries contemporary praxes that also embody selective imprints of past eras—actively retaining some, while relinquishing others—and these lie at the heart of mathematical philosophies and habits which, embracing new ideas, keep the discipline moving forward. On this latter point, related to both research and teaching, emerging themes point towards a future for mathematics having extra dimensions that may prove to be consequential if not transformative.

6.1 On-Line Research

It appears that unconcealed and refreshingly transparent teamworking on-line—a very recent *modus operandi* to emerge—will become established as a legitimate and increasingly accepted one in some circles (and in time, perhaps normative under certain circumstances), and the reader is referred to an excellent article which comprehensively sets out its current scope and potential. Pease *et al.* [21]—noting that records of on-line collaborative mathematical activity provide us with a novel, rich, searchable, accessible and sizeable source of data for empirical investigations into such mathematical activities—discuss how the resources of crowdsourced mathematics can be used to help formulate and answer questions about mathematical practice, and what their limitations might be. They describe quantitative approaches to studying crowdsourced mathematics, reviewing work from cognitive history (comparing individual and collaborative proofs), social psychology (on the prospects for a measure of collective intelligence), human-computer interaction (on the factors that have led to the success of one such project), network analysis (on the differences between collaborations on open research problems and known-but-hard problems), and argumentation theory (on modelling the argument structures of on-line collaborations). They also give an overview of qualitative approaches, examining work from empirical philosophy (on explanation in crowdsourced mathematics), the sociology of scientific knowledge (on conventions and conversations in on-line mathematics), and ethnography (on contrasting conceptions of collaboration), suggesting how these diverse methods can be applied to crowdsourced mathematics and when each might be appropriate; this is an informative and timely paper—especially as this very public exploit sits at an intriguing cusp in the general locus of mathematical industry—and we can expect more to come from this new area of study which is rather organic in the way it discloses things about the true nature of problem solving in mathematics (this can be haphazard, disjointed, quick, slightly chaotic and ferociously intense, or else involve a mixture of the directed, controlled, slow, organised and methodical—they all make up those familiar ebbs and flows in the non-linear progression and accretion of ideas related to any serious task in hand, and are markers of the reality of research for mathematicians).

6.2 On-Line Teaching

Another feature of change in the way mathematical experience is being shaped at undergraduate/postgraduate level is through the use of recorded lectures/tutorials (the move will inevitably filter into research, given enough weight and prominence). Post Covid-19 pandemic, this vehicle of conveyance has a newly raised profile that (in contrast to existing non-traditional formats such as M.O.O.Cs (Massive Open On-line Courses), webinars, and other distance education frameworks) is yet to be properly understood and analysed. Lindsay and Evans note an unprecedented global shift to on-line teaching and learning as a response to the shut down of universities during 2020 and 2021—resulting in an abundance of newly developed video resources signposting a potentially altered educational landscape going forward—predicting that the body of accumulated research in this area is poised to grow substantially from its relatively small current size. Accordingly, they have conducted a timely and revealing review with the aim of providing a synthesis of current literature on lecture capture in tertiary mathematics education, focusing on its consequences for student attendance and attainment. The work—which is an exhaustive one—allows suggestions to be made that might alleviate those unintended and unprofitable repercussions reported, confirming that it is by no means an educational panacea and that mathematics is certainly one subject where its introduction as an alternative to physical person-to-person classes should be made with both care and caution; the abstract to their write-up is clear, and informative:

> "Lecture capture (L.C.), the process of recording face-to-face lectures for future viewing, has become a common technology in Western universities in the twenty-first century, yet research on its effectiveness has lagged behind its implementation. Despite the rapid, widespread implementation, research regarding the impact L.C. has on pedagogy and student attainment is limited and not conclusive in its findings. It is still unclear if there is a causal or a correlated relationship between attainment and usage of L.C. This systematic review sought to collate and compare the current literature on the efficacy of L.C. in tertiary mathematics education and provide practical advice for institutions that use or plan to use L.C. The literature is consistent in the opinion that

students and administrators positively view L.C. for its utility and flexibility despite the moderately strong evidence that most institutions face attendance drops. However, most students do tend to see attending lectures/watching recordings as an "either-or." The literature predominantly reports a negative association between attainment and the use of L.C. as a substitute to live lectures. The proportion of students who choose to skip live lectures has steadily increased over the last decade as the student campus culture adjusts to L.C. Within this group, L.C. is used imperfectly, providing false benefits and promoting surface learning strategies. There is evidence that regular use of L.C. by this large group of students may diminish the quality of their learning. We offer research-informed, evidence-based recommendations to mitigate the unplanned and counterproductive impact of L.C. implementation." [19, p. 1].

The authors declare, quite rightly, that its emergence could signal a major revamp of higher education and state the paramount need that the sector be well supported by research which leads, rather than trails, any large scale L.C. deployment and secures stringent evaluations so as to best harness whatever potential it possesses.

6.3 'Auto' Mathematics

The above may well each become a *point d'appui* of 21st century mathematics across university campuses. The use of computers in generating new results and providing (or assisting with) proofs is—though more established in practice—an issue of debate that can still polarise mathematicians (and the jury remains out, if you will). This is best illustrated by selecting the words of academics holding opposing views that capture the main reasoning both for and against these roles. A vocal advocate is Rutgers University's Israeli-American Doron Zeilberger, a firm believer in experimental mathematics and the exploitation of symbolic computing capabilities—in combination with powerful mathematical algorithms—to precipitate insight and prove (or part prove) results. In Toufik Mansour's 4-page record of his 2021 'Interview with Doron Zeilberger' (Interview No. S3I3 in *Enum. Comb. Appl.* **1**) the interviewee said, of his goals,

> "Since, very soon, pure human-generated mathematics will be done better and faster by computers, I dedicate my research to teaching computers to do mathematics, that in my case is mostly combinatorics and number theory. In fifty years, computers will not need us, but until then, it is fun to act as "coaches"." (p. 1),

and, asked about his most influential results and why they have been so significant, replied

> "To use the cliche *the whole is larger than the sum of its parts*, I hope that my most significant contribution is in the **realization**, that the future of mathematics is in the direction of close collaboration with our sillicon friends, until they won't need us anymore." (p. 2).

In a previous 2015 interview (with Ron Aharoni, and available from Zeilberger's personal webpage), when quizzed about his "controversial" vision of the future of mathematics, and how it should be conducted, he answered

> "Current mathematics is the way it is, because, until recently, it was done by purely human, *paper-and-pencil*, means. Another factor that held it back was the *religious-fanatical* insistence that all mathematical knowledge should be proved *rigorously*. This is all going to be obsolete very soon. Using computers will expand our mathematical horizons (almost) infinitely, and we would be able to find so [much] fascinating new knowledge, and very soon would have to abandon our hang-up for (*alleged*) absolute certainty, and learn to live with facts that are "only" %99.99999999999999999 certain."

Zeilberger freely admits to often overplaying his stance quite deliberately to gain the attention that he feels the topic deserves, but his faith in computer-based teaching and reseach centred around algebraic software is genuine and resolute. The late F.F. Bonsall, on the other hand, had this to say in 1982 (see pp. 13–14 of 'A Down-To-Earth View of Mathematics', *Amer. Math. Month.* **89**, 8–15), at a time when general purpose computer algebra packages were in relative infancy (having been developed almost exclusively as bespoke tools during the 1960s and 1970s) and their applications—real and conceptual—were necessarily limited:

> "It is natural and desirable that mathematicians should be attracted by famous unsolved problems and should make great efforts to solve them. Such efforts bring new and valuable methods into mathematics, but it is well to keep some sense of proportion. The problem itself may have very little intrinsic importance. If such a problem, . . . , is solved by some clever new idea, that is magnificent; but a solution by a cumulative application of existing methods may do nothing more than demonstrate the cleverness of the solver. It is worse still if the solution involves computer verification of special cases, and in my view such a solution does not belong to mathematical science at all.
>
> It is no better to accept without verification the word of a computer than the word of another mathematician. In fact the tedium of routine tasks makes programming errors exceedingly probable. We cannot possibly achieve what I regard as the essential element of a proof—our own personal understanding—if part of the argument is hidden away in a box. . . .
>
> Perhaps we are seeing the birth of a new kind of computer-assisted quasi-mathematics, but it has no place in the science of mathematics and if it is to survive must develop its own scientific method—perhaps more akin to the experimental sciences."

He finished his essay with a real strength of feeling, writing (p. 14)

> "I believe that mathematics is one of the most important of human activities. It is not just a game, though we enjoy playing it. It is not just an art, though at times it is superlative art. It is not the construction of chains of silly little deductions as philosophers seem to imagine. Mathematics is an enduring down-to-earth science concerned with the discovery of amazingly deep and unexpected theorems. It is not easy to convince our fellow citizens of this and so we cannot expect a lavish supply of public funds for our support. So let us avoid wasting those funds on pseudo mathematics with computers and use them instead to support a few real live mathematicians."

Acceptance of computerised proofs and the search for novel results, the automated checking of mathematical 'instances'—either to confirm or else refute a claim—whose magnitude as a task is beyond mortal provisions, and the day-to-day appeal to software as a driver of research and education, have marked an interesting change within the dynamics of the field of mathematics. The likes of Zeilberger are winning the day, but not without intermittent challenge from those who share some of the misgivings of someone such as Bonsall (Halmos also harboured his own).

7 Mathematics Teaching, Exposition and Research

7.1 Teaching

Much has been written and spoken about teaching, and here both Keyser and Halmos feature. In his Ann Arbor speech, Keyser acknowledged that we have a propensity to indulge ourselves in the discipline which is justified, but felt that our responsibilities are different—and somewhat wider, rooted in a social duty—when we become teachers:

> "A farmer was once asked why he raised so many hogs. "In order," he said, "to buy more land." Asked why he desired more land, his answer was, "in order to raise more corn." Being asked to say why he would raise more corn, he replied that he wished to raise more hogs. If you ask the normal mathematician why he explores the iceberg[10] so much, his answer will be, in effect at least, "in order to explore it more." In this exquisite circularity of motive, the farmer and the normal mathematician

[10]He had earlier mentioned being asked, by another author, "what is the human significance of "this majestic intellectual cosmos of yours, towering up like a million-lustered iceberg into the arctic night," seeing that, among mankind, none is permitted to behold its more resplendent wonders save the mathematician himself?" This, Keyser wrote, would be the question to pose in response to a mathematician's attempt "to indicate the human significance of [the discipline when] regarded as the science of the forms of thought as forms", constituting of themselves, it may be said, "an infinite and everlasting world whose beauty, though it is austere and cold, is pure, and in which is the secret and citadel of whatever order and harmony our concrete universe contains, . . ."

> are well within their rights … just as a musician would be within his rights if he chanced to be so exclusively interested in the work of composition as never to be concerned with having his creations rendered before the public and never to attempt a philosophic estimate of the human worth of music. … No one, I believe, may contest the normal mathematician's right as a mathematical student or investigator to be quite indifferent as to the social value or the human worth of his activity. Such activity is to be prized just as we prize any other natural agency or force that, however undesignedly, yet contributes, sooner or later, directly or indirectly, to the weal of mankind. The fact is that, among motives in research, scientific curiosity, which is neither moral nor immoral, is far more common and far more potent than charity or philanthropy or benevolence. But when the mathematician passes from the rôle of student or investigator to the rôle of teacher, that right of indifference ceases, for he has passed to an office whose functions are social and whose obligations are human. It is not his privilege to chill and depress with the encasing fogs of the iceberg. It is his privilege and his duty, in so far as he may, to disclose its "million-lustered" splendors in all their power to quicken and illuminate, to charm and edify, the whole mind." [15, p. 641],

stating—in answer to the question of why mathematics attracted such small numbers in advanced university courses compared to other subjects—that (p. 639)

> "… I do not accept the traditional and still current explanation, that the phenomenon is due to a well-nigh universal lack of mathematical faculty. I maintain, on the contrary, that a vast majority of mankind possess mathematical faculty in a very considerable degree. That the average pupil's interest in mathematics is but slight, is a matter of common knowledge. His lack of interest is, in my opinion, due, not to a lack of the appropriate faculty in him, but to the circumstance that he is a human being, whilst mathematics, though it teems with human interest, is not presented to him in its human guise."

By all accounts he was a dedicated teacher during his time as such, as was Halmos who took his teaching very seriously and was a firm believer in disciplined preparation, something he applied to all aspects of his professional duties. His starting point for instruction was (this is taken from a 1994 talk in Cincinnati)

> "Many students confuse education with memorization. They tend to think that if we know the boiling point of beer, the gestation period of elephants, the conjugation of French irregular verbs, and the population of Burma, together with many other such goodies about the moon, whales, protons, synapses, schizophrenia, and interest rates, then we are educated. A walking encyclopedia is, however, rarely an educated person." [9, pp. 849–850],

and in his honest and admired autobiographical work of 1985, *I Want To Be a Mathematician: An Automathography*, he wrote pragmatically that

> "A big part of my profession is to profess it—that is, to be a professor, a teacher—and I've done a considerable amount of professing, both in the classroom and out. Are there any tricks of the trade, and have I picked up any?
>
> No, not really. Some things are obvious: to teach a subject you must know it, you must love it, you should know your students and take them seriously, and you must whenever possible anticipate and avert their difficulties and misunderstandings. As a teacher you are *in loco parentis*, and you must be ready with realistic advice on more subjects than your … course prepared you for." [8, p. 269].

Halmos was a great advocate of, and reported much success in, what had become known as the "Moore method"—a way of creating a problem solving attitude in students (encouraging a healthy competition among them in a classic Socratic style environment of heuristic knowledge transfer and open discussion), and named after the American topologist Robert L. Moore who spent many years at the University of Texas (almost half a century, from 1920 to 1969; he has been described as one of the most charismatic and inspiring university teachers of mathematics the U.S.A. has ever seen). Halmos held its distinguishing characteristics to be a stimulation of "greater mathematical maturity", and the effecting of more "inclination and ability to ask penetrating questions." (this "research attitude" required a teacher to have training and be active in research, as Moore was), the role of the instructor being that of an attentive coach. He embraced the timelessness of basic aspects of mathematical education, and took a broad view of it:

> "... How can we teach abstract concepts and the relations among them, how can we teach intuition, recognition, understanding? How can we teach these things so that when we are done our ex-student can not only pass an examination by naming the concepts and listing the relations, but he can also get pleasure from his insight, share it with others, apply it to the "real world", and, if he is talented and lucky, be vouchsafed the discovery of a new one?
>
> The answer is that we cannot. The only way I know of for an individual to share in humanity's slowly acquired understanding is to retrace the steps. Some old ideas were in error, of course, and some might have become irrelevant to the world of today, and therefore no longer fashionable, but on balance every student must repeat all the steps—ontogeny must recapitulate phylogeny[11] every time." [8, p. 270].

We restrict ourselves here to a further quote, from Wilder, to complement those above; he continues with his thoughts on symbolic initiative and reflex that suits our purpose:

> "As an aspect of our culture that depends so exclusively on symbols, as well as the investigation of certain relationships between them, mathematics is probably the furthest from comprehension by the non-human animal. However, much of our mathematical behavior that was originally of the *symbolic initiative* type drops to the *symbolic reflex* level. This is apparently a kind of labor-saving device set up by our neural systems. It is largely due to this, I believe, that a considerable amount of what passes for "good" teaching in mathematics is of the symbolic reflex type, involving no use of symbolic initiative. I refer of course to the drill type of teaching which may enable stupid John to get a required credit in mathematics but bores the creative minded William to the extent that he comes to loathe the subject! What essential difference is there between teaching a human animal to take the square root of 2 and teaching a pigeon to punch certain combinations of colored buttons? Undoubtedly the *symbolic reflex* type of teaching is justified when the pupil is very young—closer to the so-called "animal" stage of his development, as we say. But as he approaches maturity, more emphasis should be placed on his symbolic initiative." [27, p. 141];

one can see why Halmos and Keyser took the views they did about teaching, and ways to approach it. It is pleasing to see that in a mathematical world that is always capable of revealing itself as occasionally territorial, secretive, misanthropic, emulous and even hostile (such is human nature when high performing people, with ambition and egos, are gathered), there too friendliness, thoughtfulness, beneficence and inclusiveness has always existed, exhibited by some of the best and brightest names in the business.

7.2 Exposition

Our theme now touches upon that delineated in Section 3, where we explore the public understanding of mathematics as appraised by Halmos and Keyser. In a relatively short (for him) 1918 essay on some fundamental aspects of mathematics on which he wished to write, Keyser opened with an interesting point where he pondered thus:

> "In a recent book Sir Oliver Lodge[12] has said that "the mathematical ignorance of the average educated person has always been complete and shameless." To those who

[11] This is a historical hypothesis that the development of the embryo of an animal, from fertilisation to gestation or hatching (ontogeny), goes through stages resembling or representing those of successive adult progression in the evolution of the animal's remote ancestors (phylogeny); there are analogies, too, between the notion and the transformation of a mathematician that occurs from fledgling researcher to mature operator. It can be interpreted in a fuller context still, as meaning that the physical, moral, cultural or intellectual development of any individual passes through phases similar to those of that individual's species, society or civilisation.

[12] Oliver Joseph Lodge (1851–1940) was a British physicist and writer involved in the development of, and the holder of key patents for, radio. I have been unable to source the text from which this quote was taken by Keyser.

> know how vast a body of human achievements the term mathematics has come to denote, to those who are aware of the immense development of the subject in modern times, and especially to such as understand and value its spiritual significance as manifest in its bearings upon the higher concerns of man, this indictment can not fail to seem a pretty terrible charge.
>
> The charge is a double one: complete ignorance and shamelessness. The two counts, however, are not independent and this fact is a mitigating consideration. If the first count be correct, the second must be so too. For complete ignorance is complete innocence and innocence is never ashamed. Is the first count correct? The answer depends. For what does Sir Oliver mean by an "educated person"? He has not told us. He might, of course, have so defined the term that his statement would be true by definition. He might, for example, have said that by "educated" he meant what the world means by it. In that case the indictment would be just. For the world has never deemed incompleteness of mathematical ignorance to be essential to education. If, however, Mr. Lodge wishes us to understand that by "educated" he means *liberally* educated, then the indictment is unjust, provided one conceives liberal education as the late Lord Kelvin conceived it;[13] for this great man used to tell his students that among the "essentials of a liberal education is a mastery of Newton's 'Principia' and Herschel's 'Astronomy.' "[14] But are there not educators who would deny Kelvin's contention? Undoubtedly there are and have been many of them. Such educators, . . . , widely divergent as are their outlooks upon the world, would yet unite in denying the contention impetuously or even with scorn." [17, pp. 481–482],

stating that it was evident that Lodge's "deliverance is debatable", and "worthy of consideration".

Halmos spoke more practically. From his 1967 speech we see that he said

> "Usually when a mathematician lectures, he is a missionary. Whether he is talking over a cup of coffee with a collaborator, lecturing to a graduate class of specialists, teaching a reluctant group of freshman engineers, or addressing a general audience of laymen—he is still preaching and seeking to make converts. He will state theorems and he will discuss proofs and he will hope that when he is done his audience will know more mathematics than they did before." [7, p. 375],

and he knew well, as Keyser had, that for the most part the public was mired in incertitudes, nescience, technical immaturity, obtuseness and lazy ideas when it comes to grasping the fundamentals of mathematics as it really is. His words are instructive, and will have helped those sat before him (p. 376):

> "I call my subject mathematics—that's what all my colleagues call it, all over the world—and there, quite possibly, is the beginning of confusion. The word covers two disciplines—many more, in reality, but two, at least two, . . . In order to have some words with which to refer to the ideas I want to discuss, I offer two temporary and ad hoc neologisms. Mathematics, as the word is customarily used, consists of at least two distinct subjects, and I propose to call them *mathology* and *mathophysics*. Roughly speaking, mathology is what is usually called pure mathematics, and mathophysics is called applied mathematics, but the qualifiers are not emotionally strong enough to disguise that they qualify the same noun. If the concatenation of syllables I chose here reminds you of other words, no great harm will be done; the rhymes alluded to are not completely accidental. I originally planned to entitle this lecture something like "Mathematics is an art," or "Mathematics is not a science," or "Mathematics is useless," but the more I thought about it the more I realized that I mean that "Mathology is an art," "Mathology is not a science," and "Mathology is useless." When I am through, I hope you will recognize that most of you have known about mathophysics before, only you were probably calling it mathematics; I hope that all of you will recognize

[13] William Thomson (1st Baron Kelvin, 1824–1907) was a Belfast born British mathematician, mathematical physicist and engineer of prominence.

[14] Sir Frederick William Herschel (1738–1822) was a German born British astronomer and composer. The great English mathematician and physicist Sir Isaac Newton (1642–1726/7) is widely recognised as one of the most influential scientists of all time.

> the distinction between mathology and mathophysics; and I hope that some of you will be ready to embrace, or at least applaud, or at the very least, recognize mathology as a respectable human endeavor."

He subsequently used analogies (drawn from literature, chess, music and painting, as seen in Section 4 earlier) to illustrate his views, to good effect. The point is that each of them were keen—one might say driven—to share the Gospel of Mathematics as far and wide as the word might reach, and not to do so would have been nothing short of an abrogation of duty.

7.3 Research

Mathematical research has different aspects to it with reference to its practice and philosophy, and we take a snapshot through the words of Halmos and Keyser which disclose contrasting (though not contradictory) academic stances. Halmos—an active researcher who enjoyed the challenges involved—doesn't appear to have written much at all, if anything, explicitly about his motivations (he may have thought them obvious), but did offer the following on imparting to students the abilities needed for research:

> "Why do mathematicians do research? There are several answers. The one I like best is that we are curious—we need to know. That is almost the same as "because we want to", and I accept that—that's a good answer too. There are, however, other answers, ones that are more practical.
>
> We teach mathematics to the engineers, physicists, biologists, psychologists, economists—and mathematicians of the future. If we teach them to solve only the problems in the book, their education will be out of date before they graduate. Even from the crude, mundane, industrial, commercial point of view, our students must be prepared to answer future questions that weren 't even asked when they were in our classes. It is not enough to teach them everything that 's known—they must know also how to find out what has not yet been found. They must, in other words, be trained to solve problems—-to do research. A teacher who is not always thinking about solving problems—ones he does not know the answer to—is psychologically simply not prepared to teach problem solving to his students." [8, p. 322],

He had, evidently from experience, also considered how a line of research often comes about, writing

> "Where do the good questions, the research problems, come from? They probably come from the same hidden cave where authors find their plots and composers their tunes—and no one knows where that is or can even remember where it was after luckily stumbling into it once or twice. One thing is sure: they do not come from a vague desire to generalize. Almost the opposite is true: the source of all great mathematics is the special case, the concrete example. It is frequent in mathematics that every instance of a concept of seemingly great generality is in essence the same as a small and concrete special case. Usually it was the special case that suggested the generalization in the first place." (p. 324),

and the role of problems was something he considered to be of very great importance. In the introductory remarks of a 1980 article 'The Heart of Mathematics' (*Amer. Math. Month.* **87**, 519–524), Halmos set down on the opening page what experience had led him to feel was true—that *problems* were at the heart of mathematics. He wrote

> "What does mathematics *really* consist of? Axioms (such as the parallel postulate)? Theorems (such as the fundamental theorem of algebra)? Proofs (such as Gödel's proof of undecidability)? Concepts (such as sets and classes)? Definitions (such as the Menger definition of dimension)? Theories (such as category theory)? Formulas (such as Cauchy's integral formula)? Methods (such as the method of successive approximations)?
>
> Mathematics could surely not exist without these ingredients; they are all essential. It is nevertheless a tenable point of view that none of them is at the heart of the subject,

> that the mathematician's main reason for existence is to solve problems, and that, therefore, what mathematics *really* consists of is problems and solutions.
>
> "Theorem" is a respected word in the vocabulary of most mathematicians, but "problem" is not always so. "Problems," as the professionals sometimes use the word, are lowly exercises that are assigned to students who will later learn how to prove theorems. These emotional overtones are, however, not always the right ones." (p. 519),

concluding his opening gambit—having offered examples of theorems whose proofs are simple and hard (resp., the commutativity of addition for natural numbers, and the solvability of polynomial equations over the complex field), and problems whose solutions are routine and incredibly evasive (finding an unbeatable strategy for tic-tac-toe, and locating all zeros of the Riemann zeta function)—as follows: "... Moral: theorems can be trivial and problems can be profound. Those who believe that the heart of mathematics consists of problems are not necessarily wrong." It is difficult to argue against this for at an early learning stage they give fun, insights, and exposure to creative thinking and technical flexibilty, while in time well chosen questions are central to challenging but achievable research aims—the lifeblood of huge numbers of practising mathematicians.

Keyser's angle was also rooted in functionalities. In a most readable 1923 paper ('Mathematics as a Career', *The Sci. Month.* **17**, 489–497) whose aim was to "help college and university students having a bent for mathematics to answer some of the questions they ought to consider before deciding whether or not to devote their lives to this field of science", he finished with a section on the rewards to be gained from the profession of mathematics—weighing the "material" against the "spiritual" (pp. 496–497):

> "... The material rewards of the mathematician are notably inferior to those of some of his university colleagues, in law, for example, in medicine, and in engineering, for these, in addition to their professorial salaries, often receive incomes, sometimes large incomes, from outside practice of their professions—professions whose service, though it is not superior to that rendered by the mathematician, is more obvious to the indiscriminating multitude, called the public. But the genuine devotee of science is not disheartened by the spectacle of such *iniquity*. He is content with such an income as enables him to support his family decently and to do the work to which he has been summoned by the inner call of his talents.
>
> The life-work of the mathematician is richly compensated; but the compensations are not material—they are spiritual. One of them is the joy of life-long contact and intimate association with the eager minds of the young. Another is life-long companionship with men devoted to science and other fields of scholarship. Another is the privilege of long summer vacations affording special opportunities for study, research, writing and travel. The mathematician's subject is an honored one and his life is a life of perpetual contact with fundamental thought. He knows that his science is the science of eternal verities and that its service is essential alike to the prosperous conduct of ordinary human affairs, to the advancement of science and to the support and progress of civilization. And, though he can not gain material wealth, his work, if he be a man of genius, may bring him fame—"the lofty lucre of renown." "

The above has the thread of research running through it, but earlier in the piece he addressed the notion in more concrete terms and with a frank unambiguity that retains its pertinence still:

> "I have thus far said nothing about research, having reserved it for special consideration because of its grave importance. There is hardly another term so often heard in university circles and no other is mentioned with quite so much respect. Indeed, one sometimes gains the impression that scientific men, or some of them, regard research as being, in comparison with all other activity, not only awe-inspiring but sacred or holy, almost divine. Not infrequently men speak of it with a solemnity like that of a sinner recommending virtue and righteousness, and doubtless they sometimes do it from similar motives, conscious or unconscious.
>
> What does the term mean? In current use it has two meanings, differing in respect to dignity, a minor meaning and a major one. In mathematics the minor meaning

of the term research covers a large variety of work which, though valuable, involving something of the spirit and art of discovery and adding somewhat to the body of mathematical knowledge, yet requires neither creative genius nor a *very* high order of talent. I refer to such work as that of effecting improvements in the exposition of classical doctrines; the discovery of new theorems of ordinary difficulty, interest and importance; new demonstrations of important old theorems; the invention of new but subordinate methods; the detection and correction of imperfections in established theories; and so on. Much of the matter found in journals devoted to what is called research is of the sort I have indicated. Of course, it does not represent research in its major sense.

The major meaning of the term, mathematical research, is clearly revealed and rightly represented by nothing save the great achievements of creative mathematicians. Such creative activity assumes various forms. It may show itself in the discovery of a powerful *method*, like the analytical geometry of Descartes and Fermat or the calculus of Newton and Leibnitz; it may show itself in the creation of a great *doctrine*, like the projective geometry of Desargues and Poncelet or the function-theory of the complex variable (Cauchy, Riemann, Weierstrass); it may show itself in the form of historical research, as in the monumental *Geschichte der Mathematik* of Moritz Cantor; it may show itself in the form of contributions to the logical foundations of the science, as in the *Principia* of Whitehead and Russell or the *Tractatus* of Wittgenstein; it may show itself in the *applications* of mathematics to empirical science, as in the Einstein Theory of Relativity or the Quantum Theory of Planck." (pp. 491–492).

8 Some Final Comments

8.1 On the Use of Quotations

The reader is directed to a rather interesting article [10] on the issue of text citation to emphasise or advance a viewpoint, in which Hanna and Larvor remind us to guard against its use without proper regard to source context. This is especially true when dealing with mathematicians who on evidence disagree about the nature, value and aesthetic of proof, for example, and more generally exhibit wildly different practices in, and approaches to, their research and teaching—the authors submit that the community holds diverse and sometimes conflicting testimonies (though that is not to say that there are no areas of consensus about features of the discipline that cultivate normative judgements and homogeneity of opinion) which makes *verbatim* inclusions potentially problematic if not handled with integrity. Thus, those writing about mathematical education and philosophy—wishing to draw on the documented experiences of past/present mathematicians in constructing and enhancing narrative pieces—must be careful not to distort the words of others in any instance, for to commit such an indiscretion compromises a longstanding and powerful mode of inference. In full adherence to this, I like to think that the quotations I have deployed have done no disservice to either Halmos or Keyser (or others who cannot speak for themselves) in the way their personal stances on a meaning for mathematics have been lifted from the pages of near and distant history. More than willing to rescind any inaccuracies if I have strayed from my intent of veracity and candour, I turn the essay back to them with final quotes from each which provide a fitting end to their selected contributions. I have then chosen, in closing, to outline the bothersome relationship that educationalists can have with mathematics, and the harmful effects that a new and brazen inner breed of fervent social activists and political campaigners will, from their ranks of sanctimonious animation, bring to bear on its roots if divisive waves of critical race theory are allowed to infect, and gain a foothold in, conventional mathematical education.

8.2 Back to Halmos and Keyser

First, we augment their previous words on what mathematics constitutes. Halmos described it concisely, writing

"Mathematics is abstract thought, mathematics is pure logic, mathematics is creative art. All these statements are wrong, but they are all a little right, and they are all nearer the mark than "mathematics is numbers" or "mathematics is geometric shapes." For the professional pure mathematician, mathematics is the logical dovetailing of a carefully selected sparse set of assumptions with their surprising conclusions via a conceptually elegant proof. Simplicity, intricacy, and above all, logical analysis are the hallmark of mathematics." [7, p. 380],

while Keyser, as was his style, framed it more grandly and with typical flamboyance:

"... pernicious, because more deeply imbedded and persistent, is the fallacy that the mathematician's mind is but a syllogistic mill ... That fallacy is the *Carthago delenda* of regnant methodology. Reasoning, indeed, in the sense of compounding propositions into formal arguments, is of great importance at every stage and turn, as in the deduction of consequences, in the testing of hypotheses, in the detection of error, in purging out the dross from crude material, in chastening the deliverances of intuition, and especially in the final stages of a growing doctrine, in welding together and concatenating the various parts into a compact and coherent whole. But, indispensable in all such ways as syllogistic undoubtedly is, it is of minor importance and minor difficulty compared with the supreme matters of Invention and Construction. *Begriffbildung*, the resolution of the nebula of consciousness into star-forms of definite ideas; discriminating sensibility to the logical significances, affinities and bearings of these; susceptibility to the delicate intimations of the subtle or the remote; sensitiveness to dim and fading tremors sent below by breezes striking the higher sails; the ability to grasp together and to hold in steady view at once a multitude of ideas, to transcend the individuals and, compounding their forces, to seize the resultant meaning of them all; the ability to summon not only concepts but doctrines, marshalling them and bringing them to bear upon a single point, like great armies converging to a critical centre on a battle field. These and such as these are the powers that mathematical activity in its higher rôles demands. The power of ratiocination, as already said, is of exceeding great importance but it is neither the base nor the crown of the faculties essential to "Mathematicised Man." " [14, pp. 30–31].

When asked where he thought mathematics was going, and where he felt it should go, Halmos said

"I have an instinctive, emotional reaction to both parts. I don't think it's going anywhere, and that's exactly where it should go. In other words, I'm giving the completely reactionary, classical, pure mathematicians's answer. Mathematics just is, we nibble away at it, I don't think we direct it worth a damn, and it seems to me as silly to ask where is it going as to ask where is the dawn going. You might say it's going to the morning and to noon, but it isn't going anywhere, it just is. Of course, I know some people would say, "Well, it's going to more and more applications, going towards more and more abstraction, and it should go that way or the other way." My emotional reaction to all of those things is that it's baloney." [2, p. 27].

The 'it is what it is' approach of Halmos rings true with Keyser's description—a rather poetic one—of the burgeoning, holistically natural expansion and maturation of mathematics which has not dated:

"It is not, ..., by ... comparisons nor by statistical methods nor by any external sign whatever, but only by continued dwelling within the subtle radiance of the discipline itself, that one at length may catch the spirit and learn to estimate the abounding life of modern mathesis: oldest of the sciences, yet flourishing to-day as never before, not merely as a giant tree throwing out and aloft myriad branching arms in the upper regions of clearer light and plunging deeper and deeper root in the darker soil beneath, but rather as an immense mighty forest of such oaks, which, however, literally grow into each other so that by the junction and intercrescence of limb with limb and root with root and trunk with trunk the manifold wood becomes a single living organic growing whole." [14, p. 9].

This is the way most of us understand our subject to be—almost an animate entity that continually regenerates and thrives under it own terms, selectively retaining vital organs and body parts while allowing others to perish in a flowing wave of autonomous transformation and renewal.

Finally, Keyser wrote on what mathematics gives to a person in general terms, noting the assertions of its depreciators—that the subject is all too easily interpreted as being [14, p. 21] "... only a logical grind, suited only to narrow and straitened intellects content to tramp in treadmill fashion the weary grounds of deduction?", and further, that "... continued pursuit of the study leaves the mind narrow and dry, meagre and lean, disqualifying it both for practical affairs and for those large and liberal studies where moral questions intervene and judgment depends, not on nice calculation by rule, but on a wide survey and a balancing of probabilities?"—before rejecting them resoundingly (pp. 21–22):

> "The answer is, no. Those things not only do not follow but they are not true. Every count in the indictment, whether explicit or only implied, is false. Not only that, but the opposite in each case is true. On that point there can be no doubt; authority, reason and fact, history and theory, are here in perfect accord. Let me say once for all that I am conscious of no desire to exaggerate the virtues of mathematics. I am willing to admit that mathematicians do constitute an important part of the salt of the earth. But the science is no catholicon for mental disease. There is in it no power for transforming mediocrity into genius. It cannot enrich where nature has impoverished. It makes no pretense of creating faculty where none exists, of opening springs in desert minds. *"Du bist am Ende—was du Bist."* The great mathematician, like the great poet or great naturalist or great administrator, is born. My contention shall be that where the mathetic endowment is found, there will usually be found associated with it, as essential implications in it, other endowments in generous measure, and that the appeal of the science is to the whole mind, direct no doubt to the central powers of thought, but indirectly through sympathy of all, rousing, enlarging, developing, emancipating all, so that the faculties of will, of intellect and feeling learn to respond, each in its appropriate order and degree, like the parts of an orchestra to the "urge and ardor" of its leader and lord."

Halmos, in his autobiography, stated at the outset his own specific and dominant gain from the subject:

> "... that I like to understand mathematics, and to clarify it for myself and for the world, more even than to discover it. The joy of suddenly learning a former secret and the joy of suddenly discovering a hitherto unknown truth are the same to me—both have the flash of enlightenment, the almost incredibly enhanced vision, and the ecstacy and euphoria of released tension. At the same time, discovering a new truth, similar in subjective pleasure to understanding an old one, is in one way quite different. The difference is the pride, the feeling of victory, the almost malicious satisfaction that comes from being first." [8, p. 3].

His interest in words was, he admitted, even greater (p. 5), writing

> "... I wish I could prove ... that the [mathematicians] who do like words become famous and beloved for their clear explanations, whereas the ghosts of the others are muttered and sworn at by exasperated students every day."

It produced in him an impetus to write about mathematics as a profession—what a mathematician does, and what it means to be one—his love for the subject married with his love of language; this was something he shared with Keyser in wishing to interlace the two throughout the course of his life. Each answered, from his own position, the unremitting inner call to act as an evangelist for mathematics with the written word expressing a commonality of purpose—bravo to them both.[15]

On August 3rd, 1915, at Berkeley, California, Keyser stood before a joint meeting of the American Mathematical Society, the American Astronomical Society, and Section A of the American Association for the Advancement of Science, with remit to enunciate on a topic familiar to him and dear to his heart—the human significance of mathematics. Speaking with meaning

[15] A point of clarification here, adding to the Introduction. Although Keyser was almost exclusively an educationalist (Halmos had his feet firmly planted in education and research barracks equally), it should be noted that he did supervise two Ph.D. students who each in their own ways rose to a position of some eminence in the field—Scottish born mathematician and science fiction writer Eric Temple Bell, and the Polish born American mathematician and logician Emil Post.

and authority to affirm the deep connection mathematics has with the wider human species, and deeply aware of its secrets unlocked by those who are able, he began by posing a basic question:

> "What is the human significance—what is the significance for humanity—of "the mother of the sciences"? ... The material is superabundant. What part of it or aspect of it is most available for the end in view? "In abundant matter to speak a little with elegance," says [ancient Greek poet] Pindar, "is a thing for the wise to listen to." It is not, however, a question of elegance. It is a question of emphasis, of clarity, of effectiveness. What shall be our major theme?" [16, p. 664],

suggesting

> "Shall it be the history of the subject? Shall it be the modern developments of mathematics, its present status and its future outlook? Shall it be the utilities of the science, its so-called applications, its service in practical affairs, in engineering and in what it is customary to call the sciences of nature? Shall it be the logical foundations of mathematics, its basic principles, its inner nature, its characteristic processes and structure, the differences and similitudes that come to light in comparing it with other forms of scientific and philosophic activity? Shall it be the bearings of the science as distinguished from its applications—the bearings of it as a spiritual enterprise upon the higher concerns of man as man? It might be any one of these things. They are all of them great and inspiring themes."

They are indeed, forming the faces of a diamond called Mathematics that we each fashion for ourselves, shaping it from a rough stone into a faceted gem of personalised form, sheen and beauty. Maybe this is what mathematics is—a precious jewel that we hew, polish, and carry with us every day of our professional lives; each is different, but all catch the light and glitter in their own way with an effulgent brilliance that never wanes.

8.3 The Malevolent Role of (Some) Educationalists as New Social Warriors

In this essay the overwhelming majority of cited personnel are real mathematicians—working, or having worked, at research level and with myriad well crafted opinions about the discipline distilled over their careers into a rich percipience that spans a number of epochs—and it is because they have something of merit and substance to say that I have sought to integrate their autonomous pronouncements and authoritative utterances into a structured narrative. Most of them have or had a strong in-built interest in education and instruction, but these imposing characters of calibre and stature (mathematical homiletics in some cases) are quite different from the modern archetypal mathematical educationalist who is given to weaving (and forcing) tenuous links between models, theories, actualities and praxical embodiments from (dare I say) a position of lower academic achievement in the subject and a weakened grasp of it, doomed to remain on its periphery whether realised or not (let's be honest—some of the literature in this area is, has been, and will continue to be, lightweight or pseudo 'research' drenched in obfuscating educational parlance and beclouding terminology wherein depth is subservient to proposals and designs that are niche, transient, or merely fashionable, and where a conflation of questionable perceptions of mathematics with its realities may stray into unfortunate delusion, deliberate academic distortion, or accidental sophism). This is why the voices of those with legitimate gravitas in mathematics—fully paid up stakeholders invested in its mechanics, its meaning, its heart, its soul, its major themes and smaller fringe ecosystems, its internal synergies, its customs, habits and rituals, its (a)symmetries, (ir)regularities and (un)predictability, its grand gestures and subtle resonances, its simultaneous inclusiveness and exclusivity, its revelations and its secrets, its timeless canons and its new horizons, its inductive and deductive modalities, its soothing tranquility and febrile turbulence, *etc.* (beyond its globally acknowledged functionality, applicability and serviceability)—deserve to be given a fresh airing and not lost to time.

On a related matter which cannot pass without comment, mathematics has—under the banner of reform—been hijacked by some educationalists who are motivated by notions of social justice and wish to reconfigure this sacred subject inside a skewed political and moral framework which has gained purchase over the past couple of years or so. Currently, mathematics has found itself a casualty in America where it is being used as part of a larger push nationwide to seemingly promulgate across school, college and university campuses the mantras of critical race theory which argues—with an almost religious fervour and the zeal of righteous indignation—that racism against minorities is embedded in almost every aspect of life. A million dollar 'Dismantling Racism in Mathematics Instruction' program (funded by the Bill and Melinda Gates Foundation) examines the curriculum offered to children and its delivery. It is predicated on the (absurd) assumption that seeking correct answers to mathematical questions actually sustains white supremacy, with teachers being asked to identify and challenge the ways that mathematics is (somehow) being used to uphold capitalist, imperialist and racist views, and to reflect on their own (latent, but ingrained) biases, so that instructional practice may be transformed by shifting classroom activites away from the objective nature of basic instruction to a 'modernised' quasi subjective one in which (imagined) power structures are eradicated and a pathway to "equitable math instruction" cleared (an 80+ page May 2021 work booklet has been produced to guide them, together with other 'support' documents); an interesting passage reads "Administrators should examine programs and policies and how white supremacy impacts student outcomes (e.g., tracking, course selection, intervention rosters). In addition, they can hold teachers accountable for completing the [tasks described]." A good teacher will by default offer different ways to approach mathematical concepts that might incorporate such things as project-based learning, experiential learning and explorational learning, and may well include historical and contemporary influences covering a range of ethnicities and cultural/native legacies, but this document—peddled in many states as a force for good—seeks to bring flagrant propaganda to mandated status.[16] Personally, as an educator I fail to see any connection between mathematics and the issues raised, and there are plenty of likeminded dissenters who think the initiative goes way beyond mere virtue signalling but points to a dangerous liberal agenda that aims to change education in general and now mathematics in particular, with cries to "decolonise the curriculum" of many subjects in the U.K., America, and elsewhere in the western world. It is also more evidence that mathematics suffers a lack of conceptual understanding in ordinary society, causing a potential vulnerability—if left unchecked this exploitative enterprise could prove to be a tragedy for generations to come as it will impose a destructive oppressor versus oppressed orthodoxy that can only ever erode, undermine and blunt the very foundations of mathematical education, and so mathematics, wherever it might slip through the safety net of reason and be adopted. To be clear, critical race theory—as currently presented to us all—has spawned a fissiparous movement and has no place in the institution of mathematics at any level.

> *"It must not be imagined that the sole function of mathematics—"the handmaiden of the sciences"—is to serve science. Mathematics has also been called "the Queen of the Sciences." If occasionally the Queen has seemed to beg from the sciences she has been a very proud sort of beggar, neither asking nor accepting favors from any of her more affluent sister sciences. What she gets she pays for. Mathematics has a light and wisdom of its own, above any possible application to science, and it will richly reward any intelligent human being to catch a glimpse of what mathematics means to itself. This is not the old doctrine of art for art's sake; it is art for humanity's sake."*
>
> Eric T. Bell (mathematician and expositor)

Appendix: On Mathematical Creativity (Historically)

This appendix outlines the early work on mathematical creativity which, as an integral element of all research mathematicians, has been discussed briefly in the main body of the narrative but is felt by the author to be deserving of its own separate treatment in historical context.

Picking up on Littlewood's reference to it in Section 5.2, we see his words

> "With a good deal of diffidence I will try to give some practical advice about research and the strategy it calls for. In the first place research work is of a different order from the "learning" process of pre-research education (essential as that is). The latter can easily be rote-memory, with little associative power: on the other hand, after a month's

[16] There is some push back though. In Ontario, Canada, for instance, text that stood alongside curriculum details was re-vamped and watered down in 2021 after public outcry, having in part previously read "Mathematics has been used to normalize racism and marginalization of non-Eurocentric mathematical knowledges, and a decolonial, anti-racist approach to mathematics education makes visible its historical roots and social constructions." Really?

> immersion in research the mind knows its problem much as one's tongue knows the inside of one's mouth. You must also acquire the art of "thinking vaguely," an elusive idea I can't elaborate in short form." [20, p. 116],

to which he added " . . . , it is inevitable that I should stress the importance of giving the subconscious every chance."

Let us ponder further the route to mathematical creativity, first recounting words of J.H. Poincaré (1854–1912) who—being very much observant of the existence of, and circumstances surrounding, subconscious work (he had a lifelong interest in the philosophy of mathematics and science)—wrote that

> ". . . it is not possible, and in any case, that it is not fruitful, unless it is, on the one hand preceded, and on the other hand followed, by a period of conscious work. Never . . . are these sudden inspirations produced except after several days of voluntary effort, which have appeared fruitless and which one believed were of no value, where it appeared that we have taken a totally false path. These efforts were, therefore, not as sterile as we had thought: they put our subconscious into motion, and without these, it would not have been set in motion and would not have produced anything.
>
> The necessity of the second period of conscious work, after the inspiration, can be better understood. It is necessary to put to work the result of this inspiration, to deduce immediate consequences, order them, and write the proofs. But above all it is essential to verify them." [22, p. 26].[17]

He is here referring to what he terms the "subliminal me", with a hypothesis suggesting itself: ". . . : the *subliminal me* is not inferior to my conscious self. It is not purely automatic; it is capable of discernment. It has a tactile sense; it has sensitivity. It knows how to choose; it knows how to guess. It knows how to guess better than the conscious me since it has succeeded where the conscious has failed." (pp. 26–27). Poincaré regarded the "subliminal me" as a facilitator of the aesthetic sensibilities owned by any true discoverer of mathematics, acting as a "delicate sieve" for the formation and transference of ideas and insights that are useful and harmonious; once set in motion by the brain, it selects with efficiency those ready for revelation to the conscious mind where they can be processed and placed on a sound footing (Poincaré used the term in the sense that Freud did, and pre-dated him). He had clearly experienced this more than once after returning to an irresolvable problem, and others have spoken of being similarly struck by such a mathematical epiphany while enjoying leisure divorced from any kind of study—the common element is the ability of the unconscious to work seemingly by itself in a profound and effective manner to stimulate enlightenment, something which many mathematicians recognise and regularly rely on for breakthrough moments in their work. Of course there are some who do not accept the notion of a symbiotically exquisite co-operation between unconscious and conscious modes of thought (disregarding the active unconscious as a matter of identity loss and forcibly demystifying it even when experienced), but they can be dismissed as irrelevant outliers in denial of something real.

A little historical context is in order. French mathematician Jacques S. Hadamard (1865–1963) had previously announced the four stages of mathematical creativity quoted from Littlewood, but the model is said to have been conceived in a 1926 text *The Art of Thought* by Graham Wallas (English socialist, social psychologist, and educationalist). This line of enquiry had been anticipated already by the likes of German physicist and physician Hermann von Helmholtz (1821–1894) along with Poincaré, on which Wallas drew. According to Hadamard, in a book first released in 1945 [6], compatriot M.E. Maillet seems to have been the first person to begin thinking seriously about some of the mechanisms involved in mathematical creation and invention, and the conditions necessary for them; at that time this theme, and its link to things such as axiomatisation, rigour and intuition, were debated hotly in the mathematical community. One particular interest of his lay in "mathematical dreams"—events where a solution to a problem manifests itself during sleep—but evidence from a tentative questionnaire of some mathematicians by Maillet suggested that they had no serious significance (one interesting incident of this

[17] 'L'Invention Mathématique'—from which this (translated) quote has been taken—was a now famous May 1908 lecture which Poincaré gave at the L'Institute Général Psychologique in Paris; it was published in Volume 10 of *L'Enseignement Mathématique* that same year (pp. 357–371), and elsewhere.

type was, however, reported reliably by the prominent abstract algebraist and number theorist L.E. Dickson involving his mother and her sister who had been working together—the former, whilst in a state of slumber, developed and articulated "in a loud and clear voice" the solution to a geometrical school problem which was transcribed by the latter (roused by the noise) and found to be correct in class next morning; there is (from a later investigation of similar type detailed below) also a description of Italian mathematician, philosopher, theologian and humanitarian Maria Agnesi (she devoted years of arduous and uninterrupted labour to the writing of *Instituzioni Analitiche* that was regarded, upon publication in 1748, as the best introduction extant to the works of Euler), who often found she had sleepwalked to her study during the night and written down the solution to a problem that had been the subject of her private meditations in the day and of her dreams hours later, before returning to bed oblivious to the episode until she arose the following morning[18]). Hadamard himself testified to having once received insight (into a matrix determinant evaluation, and in a quite different direction from any of those which he had previously tried to follow) on being awoken abruptly, but had no real explanation for it.

Inspired by a letter by Maillet that appeared in the 1901 volume of *L'Enseignement Mathématique*—and overseen by Swiss mathematician Henri Fehr together with psychologists Claparède and Flournoy—a different, and more considered, subsequent survey of the working practices of mathematicians proved useful in moving the topic forward once returns were scrutinised and conclusions published[19] (it comprised over thirty questions that were on view in the 1902 & 1904 volumes of *L'Enseignement Mathématique* of which Fehr was co-founder and an editor-in-chief; copies were distributed among mathematicians gathered at international congresses in Heidelberg and St. Louis, and some were sent out to savants across the globe). It served to provoke the important deliberations of Poincaré where we see the role of the unconscious—surreptitiously working on a problem previously pondered, before releasing a spontaneous 'conducting rod' or 'dazzling light' moment of realisation—treated with all due attentiveness (though Helmholtz had reported on it before him, in an 1896 speech). Poincaré mentioned Gauss as receiving the proof of an arithmetical theorem that had eluded him for some years ". . . , not on account of my painful efforts, but by the grace of God" like a sudden "flash of lightning". Marcus du Sautoy—present day Charles Simonyi Professor for the Public Understanding of Science and Professor of Mathematics at Oxford—has interpreted the exciting moment we experience when all of the strands of our mathematical thoughts seem to come together in consummate fusion and perfect alignment as being like a delicious harmonic resolution in a piece of music or an indefectible revelation of a whodunnit crime mystery, and these analogies can be readily extended into art and poetry. Hadamard, too, tells of the phenomenon occurring in other scientific fields (for instance, to physicist Langevin and chemist Ostwald), and elsewhere (citing Mozart, and poets Valéry, Lamartine and Housman) in his wide ranging treatment of the genesis and actuation of ideas. On a broader note, many creatives (including mathematicians) speak of sensations that arrive, over a short period, as a series of points of light that blind rather than illuminate, after which the exposed film paper of conscious thought is then ready to enter the proverbial dark room where the image is nursed to full materialisation. We should also not ignore here a different—but no less dynamic—relation between the two, referenced by Hadamard as being described quite beautifully by Victorian era polymath Francis Galton in a way with which mathematicians can identify:

> "When I am engaged in trying to think anything out, the process of doing so appears to me to be this: The ideas that lie at any moment within my full consciousness seem to attract of their own accord the most appropriate out of a number of other ideas that are lying close at hand, but imperfectly within the range of my consciousness. There seems to be a presence-chamber in my mind where full consciousness holds court, and where two or three ideas are at the same time in audience, and an ante-chamber full of more or less allied ideas, which is situated just beyond the full ken of consciousness. Out of this ante-chamber the ideas most nearly allied to those in the presence-chamber appear to be summoned in a mechanically logical way, and to have their turn of audience." [6, p. 25].

[18]I have sourced this from an article on Poincaré by R.C. Archibald in the December 1915 issue of the *Bull. Amer. Math. Soc.* **22**, 125–136.

[19]Results were disseminated, in suitable synoptic form, over the period 1905–1908 (in *L'Enseignement Mathématique*). A report of the exercise and collected findings (from more than one hundred respondents) was first issued in 1909 as a pamphlet.

Hadamard's text, discussing previous work as outlined here, is replete with critical analysis and review; while relatively short, the composition is a masterpiece of erudition and readability that brought the topic of creativity into a laudable sharp relief. It stands the test of time, and it is worth noting its background and scope, being a forerunner to subsequent modern day investigations in the field; from his Foreword of 21st August, 1944, we have

> "This study, like everything which could be written on mathematical invention, was first inspired by Henri Poincaré's famous [1908] lecture before the Société de Psychologie in Paris. I first came back to the subject in a meeting at the Centre de Synthèse in Paris (1937). But a more thorough treatment of it has been given in an extensive course of lectures delivered (1943) at the Ecole Libre des Hautes Etudes, New York City."

In his Introduction he wrote (p. xi)

> "Concerning the title of this study, two remarks are useful. We speak of invention: it would be more correct to speak of discovery. The distinction between these two words is well known: discovery concerns a phenomenon, a law, a being which already existed, but had not been perceived. Columbus discovered America: it existed before him; on the contrary, Franklin invented the lightning rod: before him there had never been any lightning rod.
>
> Such a distinction has proved less evident than appears at first glance. Toricelli[20] has observed that when one inverts a closed tube on the mercury trough, the mercury ascends to a certain determinate height: this is a discovery; but, in doing this, he has invented the barometer; and there are plenty of examples of scientific results which are just as much discoveries as inventions. Franklin's invention of the lightning rod is hardly different from his discovery of the electric nature of thunder. This is a reason why the aforesaid distinction does not truly concern us; and, as a matter of fact, psychological conditions are quite the same for both cases.
>
> On the other hand, our title is "Psychology of Invention in the Mathematical Field," and not "Psychology of Mathematical Invention." It may be useful to keep in mind that mathematical invention is but a case of invention in general, a process which can take place in several domains, whether it be in science, literature, in art or also technology."

This sets the scene at the start of a most impressive monograph which is now over seventy years old but still speaks to us in the way it deals with complex issues at the heart of creative endeavour in mathematics. It is to be hoped that I have articulated these in sufficient detail for any reader—together with other elements of mathematics germane to any treatise on its primary features elsewhere in the essay—despite, in the words of Keyser (attempting to identify the "great concepts" of mathematics), the fact that [17, p. 483] "There is room here merely to glance at a few of them, to call their names and to indicate some of their more obvious aspects somewhat as a traveler in the foothills may note the peaks of a great mountain range above and beyond."

And so, finally, we return to Poincaré (whose originality in works across pure and applied mathematics, coupled with his technical/philosophical expositions, formed an astonishing legacy of material left to us) for more words on the phenomenon of mathematical invention in which he made a clear and general distinction between those who could be creative and those who could not. Emphasising the importance of so called 'intuition'—in seeing the totality of an argument to provide a framework for the logical placing of its steps (syllogisms) in order to bridge any lapses in memorising them—he regarded this immanent ability as core to the actively creative mathematician over the mere passive participant, writing

> "On conçoit que ce sentiment, cette intuition de l'ordre mathématique, qui nous fait deviner des harmonies et des relations cachées, ne puisse appartenir à tout le monde. Les uns ne posséderont ni ce sentiment délicat et difficile à définir, ni une force de

[20]Evangelista Torricelli (1608–1647) was a 17th century Italian physicist and mathematician, and a student of Galileo. He is known for the discovery of Torricelli's trumpet (also—and perhaps more often—known as Gabriel's Horn in Christian tradition which holds that the archangel Gabriel will blow the horn to announce Judgement Day) whose surface area is infinite, but whose volume is finite; this was seen as an incredible paradox by many at the time, including Torricelli himself, and prompted a fierce controversy about the nature of infinity.

> mémoire et d'attention au-dessus de l'ordinaire, et alors ils seront absolument incapables de comprendre les Mathématiques un peu élevées; c'est le plus grand nombre. D'autres n'auront ce sentiment qu'à un faible degré, mais ils seront doués d'une mémoire peu commune et d'une grande capacité d'attention. Ils apprendront par coeur les détails les uns après les autres; ils pourront comprendre les Mathématiques et quelquefois les appliquer, mais ils seront hors d'état de créer. Les autres, enfin, posséderont à un plus ou moins haut degré l'intuition spéciale dont je viens de parler, et alors non seulement ils pourront comprendre les Mathématiques, quand même leur mémoire n'aurait rien d'extraordinaire, mais ils pourront devenir créateurs et chercher à inventer avec plus ou moins de succès, suivant que cette intuition est chez eux plus ou moins développée." ["We perceive that this sense, this intuition of mathematical order, which enables us to perceive harmonies and hidden relations, cannot be present in everyone. Some would not have either this delicate sense, which is difficult to define, nor the power of memory and attention above the average, and thus they will be incapable of understanding somewhat advanced mathematics: these are in the majority. Others will have this sense to a small degree, but they will be endowed with an uncommon memory and with a large capacity for concentration. They will learn by heart details one after another: they will be able to learn mathematics and sometimes learn to apply mathematics but they will not be able to create mathematics. The others finally, will possess, to a greater or lesser degree, a special intuition of which I have spoken, and thus not only can they understand mathematics, even though their memories are not unusually good, but they will be able to become creators and seek to create with more or less success, according to the extent that this intuition is developed in them."][21]

For those able to avail themselves, it is the exhilarating and intoxicating qualities of creative mathematics—and the intellectual wellbeing it provides—that makes it so precious, however it may appear or be summoned. French mathematician André Weil, looking back at parts of mathematical history and forward to possibilities anew, finished a reflective piece penned not too long after the end of World War 2 as follows, using words with which most would concur:[22]

> ". . . if, as Panurge, we ask the oracle questions which are too indiscreet, the oracle will answer us as it did Panurge: Trinck![23] This advice the mathematician follows gladly, pleased as he is to believe that he will be able to slake his thirst at the very sources of

[21] The French version is the original one (pp. 360–361) in the monograph mentioned in Footnote 17 of this appendix. Its English translation is taken from Ayoub's interpretation of the essay (see p. 22 of [22]).

[22] Weil—a founding member and *de facto* early leader of the celebrated mathematical Bourbaki group, whose life spanned most of the 20th century—set mathematics in the moment, under the full glare of history, as part of his consideration of what the future might hold for it. He began his essay with very astute words which resound loudly today given the problems nations have (both individually and collectively) across the planet [24, p. 295]:

> "If the mathematician is asked to express himself as to the future of his science, he has a right to raise the preliminary question: what kind of future is mankind preparing for itself? Are our modes of thought, fruits of the sustained efforts of the last four or five millennia, anything more than a vanishing flash? If, unwilling to stumble into metaphysics, one should prefer to remain on the hardly more solid ground of history, the same questions reappear, although in different guise: are we witnessing the beginning of a new eclipse of civilization? Rather than to abandon ourselves to the selfish joys of creative work, is it not our duty to put the essential elements of our culture in order, for the mere purpose of preserving it, so that at the dawn of a new Renaissance, our descendants may one day find them intact?"

This throws a sobering light on our creative work (whose "joys" are "selfish")—however much we may deem it to be a right and pour over its finer details as the apogee of our labours—as mathematicians who exist as part of a much larger world, and Weil justified his angles of thought well, continuing

> "These questions are not purely rhetorical; upon each man's answer, or rather (for such questions do not have answers), upon the attitude which he takes in front of them, depends in large measure the trend of his intellectual efforts. It was necessary, before writing about the future of mathematics, to formulate these questions, just as the faithful cleansed themselves before consulting the oracle. Let us now interrogate destiny."

What followed was an excellent assessment of where mathematics had been, where it was after the terrible ravages of war had taken their toll throughout large parts of a Europe yet to recover, and how it might emerge to form a new heritage for those to come.

[23] This is a reference to *The Life of Gargantua and of Pantagruel*, a pentalogy of novels written in the 16th century by François Rabelais. In the final tract, Panurge consults the oracle of Bacbuc on the question of marriage. The Pontiff Bacbuc (Hebrew for "bottle") leads Panurge into a side chapel, where the Holy Bottle is sitting in the middle of a fountain. Bacbuc casts something into it, whereupon the waters begin to bubble and the vessel makes a cracking sound ("trink" is German for "drink"). Bacbuc declares this to be a perfect answer and gives Panurge what appears to be a book but is in fact a flask full of wine, so that he may interpret the oracle. After drinking 'liquid text', Panurge announces that the potion inspires him to right actions, and he forthwith vows to marry as quickly and as often as possible, departing joyously.

knowledge, convinced as he is that they will always continue to pour forth, pure and abundant, while others have to have recourse to the muddy streams of a sordid reality. If he be reproached with the haughtiness of his attitude, if he be summoned to do his part, if he be asked why he persists on the high glaciers whither no one but his own kind can follow him, he will answer, with Jacobi:[24] For the honor of the human spirit." [24, p. 306],

having earlier written, rather poetically (p. 297), ". . . if logic is the hygiene of the mathematician, it is not his source of food; the great problems furnish the daily bread on which he thrives." While "great problems" occupy but the very finest of minds, the free rein of creativity is the thing held most dear to all research mathematicians who, expecting little recognition, wallow in an independence of mind that is summed up in two lines from a lengthy 1880 poem of R.F. Burton:

"Do what thy manhood bids thee do, from none but self expect applause;
He noblest lives and noblest dies who makes and keeps his self-made laws." . . .[25]

I can, sticking to the ever apposite adage *ne supra crepidam sutor iudicaret*, only say nicely put.

References

[1] A. Adler, Reflections: mathematics and creativity, *The New Yorker* (Feb. 19th Issue), 39–45 (1972).

[2] D. J. Albers, Paul Halmos by parts, *in* J. H. Ewing and F. W. Gehring (Eds.), *Paul Halmos: Celebrating 50 Years of Mathematics*, Springer Science, New York, 3–32 (1991).

[3] R. P. Boas, Early days at Mathematical Reviews, *in* J. H. Ewing and F. W. Gehring (Eds.), *Paul Halmos: Celebrating 50 Years of Mathematics*, Springer Science, New York, 79–86 (1991).

[4] M. Bôcher, The fundamental conceptions and methods of mathematics, *Bull. Amer. Math. Soc.* **11**, 115–135 (1904).

[5] J. W. L. Glaisher, Opening address (Section A—Mathematics and Physics), *Nature* **XLII**, Issue No. 1089, 464–468 (1890).

[6] J. Hadamard, *The Psychology of Invention in the Mathematical Field*, Dover Pubs, Inc., London (1954).

[7] P. R. Halmos, Mathematics as a creative art, *Amer. Sci.* **56**, 375–389 (1968).

[8] P. R. Halmos, *I Want To Be a Mathematician: An Automathography*, Springer Verlag, New York (1985).

[9] P. R. Halmos, What is teaching?, *Amer. Math. Month.* **101**, 848–854 (1994).

[10] G. Hanna and B. Larvor, As Thurston says? On using quotations from famous mathematicians to make points about philosophy and education, *Zent. Didak. Math.* **52**, 1137–1147 (2020).

[11] M. Harris, *Mathematics Without Apologies: Portrait of a Problematic Vocation*, Princeton University Press, Princeton (2015).

[12] R. Hersh, Some proposals for reviving the philosophy of mathematics, *Adv. Math.* **31**, 31–50 (1979).

[13] R. Hersh, What is mathematics, really?, *Mitt. Deutsch. Math.-Ver.* **6** (Issue 2), 13–14 (1998).

[14] C. J. Keyser, Mathematics, *in Lectures on Science, Philosophy and Art, 1907-1908*, Columbia University Press, New York, 5–44 (1908).

[15] C. J. Keyser, The humanization of the teaching of mathematics, *Science* (New Series) **35**, 637–647 (1912).

[16] C. J. Keyser, The human significance of mathematics, *Science* (New Series) **42**, 663–680 (1915).

[17] C. J. Keyser, A glance at some fundamental aspects of mathematics, *The Sci. Month.* **6**, 481–495 (1918).

[18] S. Lawrence, What are we like . . . , *in* B. Larvor (Ed.), *Mathematical Cultures: The London Meetings 2012–2014*, Birkhäuser, Cham, 111–125 (2016).

[19] E. Lindsay and T. Evans, The use of lecture capture in university mathematics education: a systematic review of the research literature, *Math. Edu. Res. J.* 21 pp. (2021).

[20] J. E. Littlewood, The mathematician's art of work, *The Math. Intell.* **1**, 112–118 (1978).

[24] German (Prussian) mathematician Carl Gustav Jacob Jacobi (1804–1851), who formed a triad with Wilhelm Bessel and Franz Neumann that is recognised as the nucleus of a revival of mathematics at German universities in the first half of the 19th century (brought about by the use of seminars in student instruction and the introduction of a research oriented environment for learning).

[25] These are taken from v.VIII of the poem *The Kasîdah of Hâjî Abdû El-Yezdî*, penned (under the pseudonym Hâjî Abdû El-Yezdî) by 19th century British explorer, writer, scholar and soldier, Sir Richard Francis Burton. The work translates as *Lay of the Higher Law*, and it is essentially a distillation of thought in the poetic idiom of that mystical tradition of Islam (so called Sufism) whose ideas Burton had hoped to bring to Western culture.

[21] A. Pease, U. Martin, F. S. Tanswell and A. Aberdein, Using crowdsourced mathematics to understand mathematical practice, *Zent. Didak. Math.* **52**, 1087–1098 (2020).

[22] J. H. Poincaré, Mathematical invention, *in* R. G. Ayoub (Ed.), *Musings of the Masters: An Anthology of Mathematical Reflections*, Mathematical Association of America, Washington, 17–30 (2004).

[23] J. von Neumann, The mathematician, *in* R. G. Ayoub (Ed.), *Musings of the Masters: An Anthology of Mathematical Reflections*, Mathematical Association of America, Washington, 169–184 (2004).

[24] A. Weil, The future of mathematics, *Amer. Math. Month.* **57**, 295–306 (1950).

[25] L. A. White, The locus of mathematical reality: an anthropological footnote, *Phil. Sci.* **14**, 289–303 (1947).

[26] R. L. Wilder, *Mathematics as a Cultural System*, Pergamon Press, Oxford (1981).

[27] R. L. Wilder, The cultural basis of mathematics, *in* R. G. Ayoub (Ed.), *Musings of the Masters: An Anthology of Mathematical Reflections*, Mathematical Association of America, Washington, 129–148 (2004).

"Mathematicians are asked how they have succeeded. Now, there are not only successes but also failures, and the reasons for failures would be at least as important to know."

Jacques S. Hadamard

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