Palestine Journal of Mathematics

Vol. 11(3)(2022) , 700–707

Approximation by Stancu type Lupaş operators

Taqseer Khan and Shuzaat Ali Khan

Communicated by Siraj Uddin

MSC 2010 Classification: 41A10, 41A25, 41A36, 40A30

Keywords: Lupaş operators; modulus of continuity; Korovkin approximation theorem, Voronovskaja type theorems

The authors would like to pay thanks to the reviewer for the valuable comments for improving the quality of the paper.

Abstract This paper studies approximating properties of Stancu variant of Lupaş operators using Korovkin approximation theorem. Some direct theorems are proved and Voronovskaja type theorems for first and second order derivatives are established.

1 Introduction

Making use of the identity

$$\frac{1}{(1-t)^{\alpha}} = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} t^k, \quad |t| < 1,$$

for $(\alpha)_k = \alpha(\alpha+1)(\alpha+2)\dots(\alpha+k-1), k \geq 1$ and $(\alpha)_0 = 1$, A. Lupaş [16] introduced the following sequence of positive linear operators

$$T_n(f;x) = (1-a)^{nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{k!} f\left(\frac{k}{n}\right) a^k, \quad x \geq 0,$$

for the function $f : [0,\infty) \to \mathbb{R}$.
By considering $a = \frac{1}{2}$, O. Agratini [3] obtained the following operators

$$T_n(f;x) = 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{k!} f\left(\frac{k}{n}\right) a^k, \quad x \geq 0, \tag{1.1}$$

and studied their approximation properties. Many researchers have examined various generalizations of these operators (see [1, 2, 5, 7, 10, 11, 12]). Very recently, Mursaleen *et al.* constructed q-analogue of another generalization of these operators and investigated their approximation properties [20]. The present paper introduces Stancu type generalization of the operators defined in (1.1) and proves approximation results for them. Stancu type generalizations of various operators have been constructed and investigated, *e.g.*, in [9, 13, 18].

2 Construction of operators

We introduce Stancu type generalization of Lupaş operators in (1.1) as follows:

$$T_n^{\alpha,\beta}(f;x) = \frac{1}{2^{nx}} \sum_{k=0}^{\infty} \frac{(nx)_k}{2^k k!} f\left(\frac{k+\alpha}{n+\beta}\right), \tag{2.1}$$

where

$$f : [0,\infty) \to \mathbb{R},\ 0 \leq \alpha \leq \beta.$$

When we take $\alpha = \beta = 0$ in equation (2.1), operators (1.1) are immediate. Clearly, (2.1) is more general than (1.1).
We denote by $e_0 = 1, e_1 = x, e_2 = x^2$ and prove the following useful lemma.

Lemma 2.1. *For the operators in (2.1), the followings are obtained*

(i) $T_n^{\alpha,\beta}(e_0;x) = 1$;

(ii) $T_n^{\alpha,\beta}(e_1;x) = x$;

(iii) $T_n^{\alpha,\beta}(e_2;x) = \frac{n^2}{(n+\beta)^2}x^2 + \frac{2n(1+\alpha)}{(n+\beta)^2}x + \frac{\alpha}{(n+\beta)^2}$.

Proof. With easy calculations, (i) and (ii) are obtained.

(iii)

$$
\begin{aligned}
T_n^{\alpha,\beta}(e_2;x) &= \frac{1}{2^{nx}}\sum_{k=0}^{\infty}\frac{(nx)_k}{2^k k!}\left(\frac{k+\alpha}{n+\beta}\right)^2 \\
&= \frac{1}{2^{nx}(n+\beta)^2}\sum_{k=0}^{\infty}\frac{(nx)_k}{2^k k!}(k^2+2k\alpha+\alpha^2) \\
&= \frac{1}{2^{nx}(n+\beta)^2}\sum_{k=0}^{\infty}\frac{(nx)_k}{2^k k!}k^2 + \frac{2\alpha}{2^{nx}(n+\beta)^2}\sum_{k=0}^{\infty}\frac{(nx)_k}{2^k k!}k + \frac{\alpha^2}{(n+\beta)^2} \\
&= \frac{nx}{2^{nx+1}(n+\beta)^2}\sum_{k=1}^{\infty}\frac{(nx+1)_{k-1}}{2^{k-1}(k-1)!}k + \frac{\alpha nx}{2^{nx}(n+\beta)^2}\sum_{k=1}^{\infty}\frac{(nx+1)_{k-1}}{2^{k-1}(k-1)!} \\
&+ \frac{\alpha^2}{(n+\beta)^2} \\
&= \frac{nx}{2^{nx+1}(n+\beta)^2}\sum_{k=1}^{\infty}\frac{(nx+1)_{k-1}}{2^{k-1}(k-1)!}((k-1)+1) + \frac{2\alpha nx}{2^{nx}(n+\beta)^2}2^{nx} \\
&+ \frac{\alpha^2}{(n+\beta)^2}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{nx}{2^{nx+1}(n+\beta)^2}\left[\sum_{k=1}^{\infty}\frac{(nx+1)_{k-1}}{2^{k-1}(k-1)!}(k-1) + \sum_{k=1}^{\infty}\frac{(nx+1)_{k-1}}{2^{k-1}(k-1)!}\right] + \frac{2\alpha nx}{(n+\beta)^2} \\
&+ \frac{\alpha^2}{(n+\beta)^2} \\
&= \frac{nx}{2^{nx+2}(n+\beta)^2}\left[\sum_{k=2}^{\infty}\frac{(nx+1)(nx+2)_{k-2}}{2^{k-2}(k-2)!} + 2^{nx+1}\right] + \frac{2\alpha nx}{(n+\beta)^2} + \frac{\alpha^2}{(n+\beta)^2} \\
&= \frac{nx(nx+1)}{2^{nx+2}(n+\beta)^2}\left[\sum_{k=2}^{\infty}\frac{(nx+2)_{k-2}}{2^{k-2}(k-2)!} + 2^{nx+1}\right] + \frac{2\alpha nx}{(n+\beta)^2} + \frac{\alpha^2}{(n+\beta)^2} \\
&= \frac{nx(nx+1)}{(n+\beta)^2} + \frac{nx}{(n+\beta)^2} + \frac{2\alpha nx}{(n+\beta)^2} + \frac{\alpha^2}{(n+\beta)^2} \\
&= \frac{n^2}{(n+\beta)^2}x^2 + \frac{2n(1+\alpha)}{(n+\beta)^2}x + \frac{\alpha^2}{(n+\beta)^2}. \square
\end{aligned}
$$

We compute moments in the following.

Lemma 2.2. *The first and the second moments for the operators (2.1) are given by*

(i) $T_n^{\alpha,\beta}((t-x);x) = 0$;

(ii) $T_n^{\alpha,\beta}((t-x)^2;x) = \left(\frac{n^2}{(n+\beta)^2}-1\right)x^2 + \frac{2n(1+\alpha)}{(n+\beta)^2}x + \frac{\alpha^2}{(n+\beta)^2}$.

Proof. By linearity of operators, (i) is obvious.

(ii) Making use of Lemma 2.1 and linearity, we have

$$\begin{aligned} T_n^{\alpha,\beta}((t-x)^2;x) &= T_n^{\alpha,\beta}(t^2;x) - 2xT_n^{\alpha,\beta}(t;x) + x^2T_n^{\alpha,\beta}(1;x) \\ &= \left(\frac{n^2}{(n+\beta)^2}\right)x^2 + \frac{2n(1+\alpha)}{(n+\beta)^2}x + \frac{\alpha^2}{(n+\beta)^2} - 2x^2 + x^2 \\ &= (\frac{n^2}{(n+\beta)^2} - 1)x^2 + \frac{2n(1+\alpha)}{(n+\beta)^2}x + \frac{\alpha^2}{(n+\beta)^2}. \square \end{aligned}$$

3 Korovkin approximation theorem

In this section, we prove the classical Korovkin theorem given in [14, 15, 17] for the operators defined in (2.1). Korovkin theorem is a succinct result which guarantees the uniform convergence of a sequence of linear positive operators on a certain function space provided that the sequence converges uniformly for some functions $e_r(x) = x^r \ (x \geq 0), \ r = 0, 1, 2.$

We denote by $C[0,\infty)$ the set of all real valued continuous functions defined on $[0,\infty)$ which is a Banach space under the norm

$$\|f\| = \sup_{x\in[0,\infty)} |f(x)|.$$

Theorem 3.1. *For each $f \in C[0,\infty)$, the operators $T_n^{\alpha,\beta}(.;x)$ converge uniformly to f on the compact domain $[0,a]$ $(a > 0)$ as $n \to \infty$.*

Proof. We have by Lemma 2.1,

$$\lim_{n\to\infty} T_n^{\alpha,\beta}(e_0;x) = 1,$$
$$\lim_{n\to\infty} T_n^{\alpha,\beta}(e_1;x) = x,$$
$$\lim_{n\to\infty} T_n^{\alpha,\beta}(e_2;x) = x^2.$$

Thus, the sequence of positive linear operators $T_n^{\alpha,\beta}(.;x)$ converges uniformly to f, where f is one of the functions $1, t, t^2$ on the compact interval $[0,a]$, $a > 0$. Therefore, by the Korovkin approximation theorem the result holds for every continuous function f on the compact interval $[0,a]$. This proves the theorem. $\square$

4 Convergence in weighted space

In this section, we investigate convergence of our operators in weighted space of functions. Let $\phi(x) = e_0 + e_2$ be a weight function. Let $B_{x^2}[0,\infty)$ be the linear space of all functions h satisfying the condition $|h(x)| \leq K_h(1+x^2)$, where K_h is a constant associated with the function h. We denote the subspace of all continuous functions of $B_{x^2}[0,\infty)$ by $C_{x^2}[0,\infty)$. Also, we denote by $C^*_{x^2}[0,\infty)$, the subclass of $C_{x^2}[0,\infty)$ of those functions h for which $\lim_{x\to\infty} \frac{h(x)}{1+x^2}$ is finite. The norm on the space $C^*_{x^2}[0,\infty)$ is defined by

$$\|h\|_{x^2} = \sup_{x\in[0,\infty)} \frac{|\ h(x)\ |}{1+x^2}.$$

We shall denote this space by $Q_\phi(\mathbb{R}^+)$.

Lemma 4.1. *Let $T_n^{\alpha,\beta}(.;x)$ be operators defined by (2.1). Then for the weight function $\phi(x)$ above, we obtain*

$$\|T_n^{\alpha,\beta}(\phi;x)\|_{x^2} \leq M,$$

where M is a positive constant greater than 1.

Proof. Using linearity and Lemma 2.1, we obtain

$$\begin{aligned} T_n^{\alpha,\beta}(\phi;x) &= T_n^{\alpha,\beta}(e_0+e_2;x) \\ &= T_n^{\alpha,\beta}(e_0;x)+T_n^{\alpha,\beta}(e_2;x) \\ &= 1+\frac{n^2}{(n+\beta)^2}x^2+\frac{2n(1+\alpha)}{(n+\beta)^2}x+\frac{\alpha^2}{(n+\beta)^2}. \end{aligned}$$

Then

$\|T_n^{\alpha,\beta}(\phi;x)\|_{x^2}$

$$\begin{aligned} &= \sup_{x\geq 0}\left\{\frac{1}{1+x^2}+\frac{n^2}{(n+\beta)^2}\frac{x^2}{1+x^2}+\frac{2n(1+\alpha)}{(n+\beta)^2}\frac{x}{1+x^2}+\frac{\alpha^2}{(n+\beta)^2}\frac{1}{1+x^2}\right\} \\ &< 1+\frac{n^2}{(n+\beta)^2}+\frac{2n(1+\alpha)}{(n+\beta)^2}+\frac{\alpha^2}{(n+\beta)^2}. \end{aligned}$$

Noting that $\lim\limits_{n\to\infty}\frac{n^2}{(n+\beta)^2}=1$, $\lim\limits_{n\to\infty}\frac{2n(1+\alpha)}{(n+\beta)^2}=0=\lim\limits_{n\to\infty}\frac{\alpha^2}{(n+\beta)^2}$, we find that there exists a constant $M>1$ such that

$$\|T_n^{\alpha,\beta}(\phi;x)\|_{x^2}\leq M,$$

which proves the lemma. □

Remark: By the above lemma it is observed that the operators $T_n^{\alpha,\beta}(.;x)$ act from the space $C_{x^2[0,\infty)}$ to the space $B_{x^2[0,\infty)}$.

Theorem 4.2. *Let $T_n^{\alpha,\beta}(.;x)$ be operators defined by (2.1) and $\phi(x)=1+x^2$ be the weight function. Then for each $f\in C^*_{x^2}[0,\infty)$, we have*

$$\lim_{n\to\infty}\|T_n^{\alpha,\beta}(f;x)-f(x)\|_{x^2}=0.$$

Proof. In view of Korovkin theorem (see [4]), it is sufficient to show that

$$\lim_{n\to\infty}\|T_n^{\alpha,\beta}(t^j;x)-x^j\|_{x^2}=0,\quad j=0,1,2.$$

By Lemma 2.1 (i), obviously, we find

$$\lim_{n\to\infty}\|T_n^{\alpha,\beta}(1;x)-1\|_{x^2}=0.$$

By Lemma 2.1 (ii), we get

$$\lim_{n\to\infty}\|T_n^{\alpha,\beta}(e_1;x)-e_1\|_{x^2}=\sup_{x\geq 0}\left|\frac{x}{1+x^2}-\frac{x}{1+x^2}\right|=0.$$

In view of Lemma 2.1 (iii), one infers that

$\|T_n^{\alpha,\beta}(e_2;x)-e_2\|_{x^2}$

$$\begin{aligned} &= \sup_{x\geq 0}\left|\left(\frac{n^2}{(n+\beta)^2}-1\right)\frac{x^2}{1+x^2}+\frac{2n(1+\alpha)}{(n+\beta)^2}\frac{x}{1+x^2}+\frac{\alpha^2}{(n+\beta)^2}\frac{1}{1+x^2}\right| \\ &\leq \left(\frac{n^2}{(n+\beta)^2}-1\right)+\frac{2n(1+\alpha)}{(n+\beta)^2}+\frac{\alpha^2}{(n+\beta)^2}\to 0 \ \ as\ \ n\to\infty. \end{aligned}$$

Hence, we obtain

$$\lim_{n\to\infty}\|T_n^{\alpha,\beta}(e_2;x)-e_2\|_{x^2}=0$$

and this completes the proof. □

5 Rate of convergence

Here, we compute the rate of convergence in terms of moduli of continuity defined in [8]. Let $f \in C[a,b]$ and $\delta > 0$. The first and the second moduli of continuity of f are given by

$$\omega_1(f;\delta) = \sup\{|f(x+h) - f(x)|,\ \ 0 \leq h \leq \delta,\ \ x, y \in [a,b]\},$$

$$\omega_2(f;\delta) = \sup\{|f(x+h) + f(x-h) - 2f(x)|,\ \ 0 \leq h \leq \delta, x+h, x, x-h \in [a,b]\}$$

respectively. It is well known that $\lim\limits_{\delta \to 0^+} \omega_1(f;\delta) = 0$ for any $\delta > 0$ and

$$|f(y) - f(x)| \leq \omega_1(f;\delta)\left(\frac{|y-x|}{\delta} + 1\right). \tag{5.1}$$

Theorem 5.1. *Let $f \in C[0,a](a > 0)$ and $T_n^{\alpha,\beta}(.;x)$ be the operators given in (2.1). Then following holds*

$$\left|T_n^{\alpha,\beta}(f;x) - f(x)\right| \leq (1+a^2)\omega_1\left(f; \frac{1}{\sqrt{(n+\beta)}}\right). \tag{5.2}$$

If f is continuously differentiable on $[0,a]$ then

$$\left|T_n^{\alpha,\beta}(f;x) - f(x)\right| \leq \frac{(1+a)^2}{\sqrt{(n+\beta)}}\omega_1\left(f'; \frac{1}{\sqrt{(n+\beta)}}\right). \tag{5.3}$$

Proof. Using Inequality (5.1) together with the Cauchy-Schwarz inequality, one can obtain

$$\begin{aligned}\left|T_n^{\alpha,\beta}(f;x) - f(x)\right| &\leq \left(1 + \delta\, T_n^{\alpha,\beta}(|t-x|;x)\right)\omega_1(f;\delta)\\ &\leq 1 + \delta\left(T_n^{\alpha,\beta}((t-x)^2;x)\right)^{\frac{1}{2}}\omega_1(f;\delta).\end{aligned}$$

Let us choose $\delta = \frac{1}{\sqrt{(n+\beta)}}$. Then in view of Lemma 2.2 (ii), (5.2) is obtained.
Next, suppose f has a continuous derivative on $[0,a]$. Then by the Mean Value Theorem, we can write

$$f(x) - f(y) = (x-y)f'(x) + (x-y)(f'(t) - f'(x),$$

where t is a point such that $x < t < y$. Writing (5.1) for f' and using the same argument as exercised in the previous proof, one can write

$$\begin{aligned}\left|T_n^{\alpha,\beta}(f;x) - f(x)\right| &\leq \left\{T_n^{\alpha,\beta}(|t-x|;x) + \frac{1}{\delta}\, T_n^{\alpha,\beta}(|t-x|^2;x)\right\}\omega_1(f';\delta)\\ &\leq \left(T_n^{\alpha,\beta}((t-x)^2;x)\right)^{\frac{1}{2}}\left(1 + \frac{1}{\delta}\left(T_n^{\alpha,\beta}((t-x)^2;x)\right)^{\frac{1}{2}}\right)\omega_1(f';\delta).\end{aligned}$$

Noting that $x \in [0,a]$ and $\delta = \frac{1}{\sqrt{(n+\beta)}}$, we conclude that Lemma 2.1 implies (5.3). □
In the sequel, we obtain an estimate involving the second order modulus of smoothness.

Theorem 5.2. *Let $T_n^{\alpha,\beta}(.;x)$ be the operators defined by (2.1) and $f \in C[0,a](a > 0)$. Then the following holds*

$$\left|T_n^{\alpha,\beta}(f;x) - f(x)\right| \leq \left(3 + 2a\, max\left\{1, \frac{a}{n+\beta}\right\}\right)\omega_2\left(f; \frac{1}{\sqrt{(n+\beta)}}\right).$$

Proof. This is easily obtained by using Lemma 2.1 and a theorem due to O. Agratini ([3], Theorem 3, page 44). □

6 Voronovskaja type theorems

In this section, we obtain couple of asymptotic formulas for our operators based on ([19], Theorem 1, p. 2423).

Theorem 6.1. *Let $T_n^{\alpha,\beta}(.;x)$ be the operators in (2.1) and $g \in C[a,b],\ x \in [a,b]$. Suppose that there exists a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ such that $\mu_i \in (0,\infty)$ for each $1 \leq i < \infty$ with $\lim_{n\to\infty} \mu_n = \infty$ and there exists a $1 < p < \infty$ such that the sequence $\left\{\mu_n^p T_n^{\alpha,\beta}(|. - x|^p; x)\right\}_{n\in\mathbb{N}}$ is bounded. Then for any continuous function $f : [a,b] \to \mathbb{R}$ which is differentiable at $x \in [a,b]$, we have*

$$\lim_{n\to\infty} \mu_n \Big[T_n^{\alpha,\beta}(f;x) - f(x)T_n^{\alpha,\beta}(1;x) - f'(x)T_n^{\alpha,\beta}((. - x);x)\Big] = 0. \tag{6.1}$$

Further, if

$$\lim_{n\to\infty} \mu_n \Big[T_n^{\alpha,\beta}(1;x) - 1\Big] = P(x), \lim_{n\to\infty} \mu_n T_n^{\alpha,\beta}((. - x);x) = Q(x), \tag{6.2}$$

then

$$\lim_{n\to\infty} \mu_n \Big[T_n^{\alpha,\beta}(f;x) - f(x)\Big] = P(x)f(x) + Q(x)f'(x).$$

Proof. We follow the technique given in ([6], Theorem 6.3.6, p. 117). By differentiability of f, we have

$$f'(x) = \lim_{t\to x} \frac{f(t) - f(x)}{t - x}.$$

We define the function $g : [a,b] \to \mathbb{R}$ by

$$g(t) = \begin{cases} \frac{f(t)-f(x)-f'(x)(t-x)}{t-x}, & t \neq x, \\ 0, & t = x. \end{cases}$$

Then g is continuous on $[a,b]$ and $f(t) - f(x) - f'(x)(t-x) = (t-x)g(t),\ \forall t \in [a,b]$. In $C[a,b]$, one can write

$$f - f(x).1 - f'(x).(. - x) = (. - x)g.$$

Operating by $T_n^{\alpha,\beta}$ on both sides and using linearity we get

$$T_n^{\alpha,\beta}(f;x) - f(x)\, T_n^{\alpha,\beta}(1;x) - f'(x)\, T_n^{\alpha,\beta}((. - x);x) = T_n^{\alpha,\beta}((. - x);x)g(x).$$

Next, by the hypothesis, there exists a constant $K_p > 0$ such that $\mu_n^p T_n^{\alpha,\beta}(|. - x|^p; x) \leq K_p, \forall n \in \mathbb{N}$. For every $h \in C[a,b]$, using Holder's inequality, we write

$$\begin{aligned} \left|T_n^{\alpha,\beta}((. - x);x)g\right| &\leq T_n^{\alpha,\beta}(|(. - x);x)g|) \\ &= T_n^{\alpha,\beta}\big(|(. - x);x)||g|\big) \\ &\leq \left[T_n^{\alpha,\beta}\big(|(. - x)|^p;x\big)\right]^{\frac{1}{p}} \left[T_n^{\alpha,\beta}\big(|g|^q;x\big)\right]^{\frac{1}{q}} \\ &\leq \frac{K_p^{\frac{1}{p}}}{\mu_n} \left[T_n^{\alpha,\beta}\big(|g|^q;x\big)\right]^{\frac{1}{q}} \end{aligned}$$

and then we obtain

$$\left|\mu_n\left[T_n^{\alpha,\beta}(f;x) - f(x)\, T_n^{\alpha,\beta}(1;x) - f'(x)\, T_n^{\alpha,\beta}((. - x);x)\right]\right| \quad \leq \quad K_p^{\frac{1}{p}} \left[T_n^{\alpha,\beta}\big(|g|^q;x\big)\right]^{\frac{1}{q}}.$$

By Korovkin theorem we know that $T_n^{\alpha,\beta}(.;x)$ converges uniformly on $C[a,b]$, so that for $g \in C[a,b]$, we have

$$\lim_{n\to\infty} T_n^{\alpha,\beta}\big(|g|^q;x\big) = |g(x)|^q = 0.$$

Employing the Squeeze theorem, (6.1) is obtained.
Next, we write

$$\begin{aligned}\mu_n[T_n^{\alpha,\beta}(f;x)-f(x)] &= \mu_n\big[T_n^{\alpha,\beta}(f;x)-f(x)\,T_n^{\alpha,\beta}(1;x)-f'(x)\,T_n^{\alpha,\beta}((.-x);x)\big]\\ &+ f(x)\,\mu_n\,[T_n^{\alpha,\beta}(1;x)-1]+f'(x)\,\mu_n\,T_n^{\alpha,\beta}((.-x);x).\end{aligned}$$

Then

$$\begin{aligned}&\lim_{n\to\infty}\mu_n[T_n^{\alpha,\beta}(f;x)-f(x)]\\ &= \lim_{n\to\infty}\mu_n\big[T_n^{\alpha,\beta}(f;x)-f(x)\,T_n^{\alpha,\beta}(1;x)-f'(x)\,T_n^{\alpha,\beta}((.-x);x)\big]\\ &+ \lim_{n\to\infty}f(x)\,\mu_n\,[T_n^{\alpha,\beta}(1;x)-1]+f'(x)\,\mu_n\,T_n^{\alpha,\beta}((.-x);x)\\ &= \lim_{n\to\infty}f(x)\,\mu_n\,[T_n^{\alpha,\beta}(1;x)-1]+f'(x)\,\mu_n\,T_n^{\alpha,\beta}((.-x);x)\\ &= P(x)\,f(x)+Q(x)f'(x)\end{aligned}$$

and this completes the proof. □

Theorem 6.2. *Let $T_n^{\alpha,\beta}(.;x)$ be the operators in (2.1) and $g\in C[0,\infty)$ and $x\in[0,\infty)$ be a point at which g' is continuously differentiable. Further, assume that $g(t)=\mathbf{O}(t^2)$ as $t\to\infty$. Then the following holds*

$$\lim_{n\to\infty}n(T_n^{\alpha,\beta}(g;x)-g(x))=\frac{\Psi(x)^2}{2}g''(x)$$

where $\Psi(x)=\big(\frac{n^2}{(n+\beta)^2}-1\big)x^2+\frac{2n(1+\alpha)}{(n+\beta)^2}x+\frac{\alpha^2}{(n+\beta)^2}$.

Proof. We write Taylor's expansion

$$g(k/n)-g(x)=(k/n-x)g'(x)+(k/n-x)^2\left(\frac{1}{2}\,g''(x)+\epsilon(k/n-x)\right), \tag{6.3}$$

where ϵ is bounded and $\lim_{t\to0}\epsilon(t)=0$. Operating by $T_n^{\alpha,\beta}$ on both sides of (6.3), one obtains

$$T_n^{\alpha,\beta}(g;x)-g(x)=T_n^{\alpha,\beta}((t-x);x)g'(x)+\frac{1}{2}\,T_n^{\alpha,\beta}((t-x)^2;x)g''(x)+T_n^{\alpha,\beta}(\ell_x;x), \tag{6.4}$$

where $\ell_x(t)=(t-x)^2\epsilon(t-x)$. It is clear that $T_n^{\alpha,\beta}((t-x);x)=0$. By Cauchy-Schwarz inequality and Lemma 2.2 (ii), we get

$$\begin{aligned}[T_n^{\alpha,\beta}(\ell_x;x)] &\le [T_n^{\alpha,\beta}((t-x)^2;x)][T_n^{\alpha,\beta}(\epsilon^2(t-x)^2;x)]\\ &\le \|\epsilon^2\|_\infty[T_n^{\alpha,\beta}((t-x)^2;x)]^2\\ &= \|\epsilon^2\|_\infty\left[\left(\frac{n^2}{(n+\beta)^2}-1\right)x^2+\frac{2n(1+\alpha)}{(n+\beta)^2}x+\frac{\alpha^2}{(n+\beta)^2}\right]^2.\end{aligned}$$

Then it is easily seen that $\lim_{n\to\infty}n(T_n^{\alpha,\beta}(\ell_x;x))=0$ and, therefore, on using Lemma 2.2 (ii), by (6.4), we arrive at the desired result.

7 Conclusion

In this paper Stancu variant of Lupaş operators has been constructed and its approximating properties have been investigated. The rate of convergence is studied and Voronovskaja type theorems are established. These operators exhibit interesting and more flexible character with regard to approximation of functions.

References

[1] U. Abel and M. Ivan, *On a generalization of an approximation operator defined by A. Lupaş*, Gen. Math., **15**(1) (2007) 21-34.

[2] T. Acar and A. Aral, *Approximation properties of two dimensional Bernstein-Stancu Chlodowsky operators*, Matematiche (Catania), **68**(2) (2013) 15-31.

[3] O. Agratini, *On a sequence of linear positive operators*, Facta Univ. Ser. Math. Inform., **14** (1999) 41-48.

[4] F. Altomare, M. Campiti, *Korovkin type approximation theory and its applications*, de Gruyter Stud. Math. **17**, Berlin, 1994.

[5] B. M. Brown, D. Elliott and D. F. Paget, *Lipschitz constants for the Bernstein polynomials of a Lipschitz continuous function*, J. Approx. Theory, **49**(2) (1987) 196-199.

[6] P. J. Davis, *Interpolation and Approximation*, Dover Publications, Inc., New York (1975) (Republication, with minor corrections, of the 1963 original, with a new preface and bibliography)

[7] F. Dirik, *Statistical convergence and rate of convergence of a sequence of positive linear operators,* Math. Commun., **12**(2) (2007) 147-153.

[8] Z. Ditzian and V. Totik, *Moduli of Smoothness*, Springer-Verlag, New York, 1987.

[9] A. Erençin, G. B. Tunca and F. Taşdelen, *Some preservation properties of MKZ-Stancu type operators*, Sarajevo J. Math., **10**(22) (2014) 93-102.

[10] A. Erençin, G. B. Tunca and F. Taşdelen, *Some properties of the operators defined by Lupaş*, Rev. Anal. Numer. Theor. Approx., **43**(2) (2014) 168-174.

[11] G. C. Jain and S. Pethe, *On the generalizations of Bernstein and Szasz-Mirakjan operators*, Nanta Math., **10**(2) (1977) 185-193.

[12] M. K. Khan and M. A. Peters, *Lipschitz constants for some approximation operators of a Lipschitz continuous function*, J. Approx. Theory, **59**(3) (1989) 307-315.

[13] A. Kilicman, M. A. Mursaleen and A. A. H. A. Al-Abied, *Stancu type Baskakov-Durrmeyer operators and approximation properties*, Mathematics, **8** (2020): 1164; doi: 10.3390/math8071164.

[14] P. P. Korovkin, *On convergence of linear positive operators in the space of continuous functions*, Doklady Akademii Nauk, **90** (1953) 961-964.

[15] P. P. Korovkin, *Linear operators and approximation theory*, Hindustan Publishing Corporation, Delhi, 1960.

[16] A. Lupaş, *The approximation by some positive linear operators, In: Proceedings of the International Dortmund Meeting on Approximation Theory (M. W. Müller et al., eds.)*, Akademie Verlag, Berlin, (1995) 201-229.

[17] M. Mursaleen, V. Karakaya, M. Ertürk and F. Gürsoy, *Weighted statistical convergence and its application to Korovkin type approximation theorem*, Appl. Math. Comput., **218** (2012) 9132-9137.

[18] M. Mursaleen and Taqseer Khan, *On approximation by Stancu type Jakimovski-Leviatan-Durrmeyer operators*, Azerb. J. Math., **7**(1) (2017) 16-26.

[19] D. Popa, *An intermediate Voronovskaja type theorem*, RACSM, (2019), **113** 2421-2429.

[20] M. Qasim, M. Mursaleen, A. Khan and Z. Abbas, *Approximation by generalized Lupaş operators based on q-integers*, Mathematics, **8**(1) (2020): 68; doi: 10.3390/math8010068.

Author information

Taqseer Khan, Department of Mathematics, Faculty of Natural Sciences, Jamia Millia Islamia (a central university), New Delhi-110025, India.
E-mail: `tkhan4@jmi.ac.in`

Shuzaat Ali Khan, Department of Mathematics, Faculty of Natural Sciences, Jamia Millia Islamia (a central university), New Delhi-110025, India.
E-mail: `shuzaatkhan786@gmail.com`

Received: September 30, 2021.
Accepted: December 29, 2021.